Testing Dimensional Reduction in SU(2) Gauge Theory

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At high temperature, every $(d+1)$-dimensional theory can be reformulated as an effective theory in d dimensions. We test the numerical accuracy of this Dimensional Reduction for $(3+1)$-dimensional $SU(2)$ by comparing perturbatively determined effective couplings with lattice results as the temperature is progressively lowered. We observe an increasing disagreement between numerical and perturbative values from $T = 4T_c$ downwards, which may however be due to somewhat different implementations of dimensional reduction in the two cases.

1. INTRODUCTION

One of the major thrusts of nuclear physics in the next decade will be the effort to study the quark-gluon plasma. At the high temperatures reached, one can consider applying Dimensional Reduction. To test the accuracy of Dimensional Reduction, a purely gluonic system provides a simple, but relevant benchmark. Therefore, here we study $SU(2)$ gauge theory at finite temperature. Up to now, only gauge invariant correlation functions of colourless states have been measured. This approach gives no information about the effective action. Here, we measure directly the effective coupling constants. The analytics of Dimensional Reduction gives us perturbative expressions for these coupling constants. A numerical, non-perturbative determination allows us to monitor the two ways in which the perturbative calculation can fail: (i) As the temperature is lowered, the couplings increasingly depart from their perturbative values. (ii) The effective action becomes less local - couplings of higher-dimension operators, e.g. A_0^6, completely neglected in the perturbative approach, become sizeable.
The non-perturbatively determined coupling constants allow us, with certainty in principle, to extend the validity of Dimensional Reduction below temperatures achieved perturbatively. In that sense, we attempt a non-perturbative improvement of the dimensionally reduced action.

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2. DIMENSIONAL REDUCTION

The leading infrared behavior of $4d$-$SU(2)$ gauge theory at high temperatures is governed by its static (zero Matsubara frequency) sector, obtained by integrating out its non-static modes to leave behind an effective three-dimensional theory. The non-static modes are suppressed in the infrared. At high temperature, $g(T)$ becomes small due to asymptotic freedom. A perturbative treatment of the non-static modes is justified. This is called Dimensional Reduction.
The Euclidean action is

$$S^{4d,E} = \int \mathrm{d}^3\mathbf{x} \int_0^\beta \mathrm{d}\tau \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a}. \tag{1}$$

Expanding the A_μ-fields into Fourier series in the temporal direction, keeping only the static modes and integrating out all the others, we end up with a $3d$-$SU(2)$ gauge theory minimally coupled to a self-interacting adjoint scalar field [1]:

$$S^{3d,eff} = \int \mathrm{d}^3\mathbf{x}\{\frac{1}{4} F_{ij,0}^a F_0^{ij,a} + \mathrm{Tr}[D_j, A_{0,0}^a][D_j, A_{0,0}^a] \\ + m_D^2 \mathrm{Tr}(A_{0,0}^a)^2 + \lambda_A (\mathrm{Tr}(A_{0,0}^a)^2)^2\} \tag{2}$$

where the first term is the $3d$-$SU(2)$ gauge theory and the second term is the kinetic term of the static adjoint Higgs Field $A_{0,0}$. The coefficients of this $3d$ effective action, the gauge coupling g_3, the Debye mass m_D and the quartic coupling λ_A, can be determined from the perturbative expansion, or numerically as follows.

3. CANONICAL DEMON METHOD

Any action can be parameterized by

$$S = -\sum_\alpha \beta_\alpha S_\alpha, \tag{3}$$

where S_α's are interaction terms, and the β_α's are unknown effective coupling constants. The lattice action corresponding to the continuum theory (2) is after some reformulation [2]:

$$S = \beta_G \sum_{x,i<j} \left[1 - \frac{1}{2}\mathrm{Tr}U_i(x)U_j(x+\hat{i})U_i^\dagger(x+\hat{j})U_j^\dagger(x) \right]$$
$$+ \beta_A \sum_{x,i} \frac{1}{2}\mathrm{Tr}\tilde{A}_0(x)U_i(x)\tilde{A}_0(x+\hat{i})U_i^\dagger(x)$$
$$+ \sum_x \left[-\beta_2 \frac{1}{2}\mathrm{Tr}\tilde{A}_0^2 + \beta_4 (\frac{1}{2}\mathrm{Tr}\tilde{A}_0^2)^2 \right] \tag{4}$$

where a is the lattice spacing, $\beta_G = 4/(ag_3^2)$, $\beta_2 = \left[3 + (m_D a)^2/2\right]\beta_A$ and $\beta_4 = a\lambda_A\beta_A^2/4$. β_A is arbitrary under a rescaling of $\tilde{A}_0 \propto aA_{0,0_k}\sigma_k$. In addition we introduce an auxiliary system, called demon system, given by the action

$$S_D = -\sum_{\alpha=G,A,2,4} \beta_\alpha d_\alpha, \tag{5}$$

here the β_α's are the same as in S, and the d_α's are the "demon energies", constrained to lie in the interval $[-d_{max}, d_{max}]$. The total partition function factorizes into the one of the original system and that of the single demons. We can compute the distribution of each demon energy d_α. Solving this equation with respect to β_α allows us to determine the effective coupling constants:

$$\langle d_\alpha \rangle = \frac{1}{Z}\int d_\alpha e^{-\beta_\alpha d_\alpha}\mathrm{d}d_\alpha = \frac{1}{\beta_\alpha} - \frac{d_{max}}{\tanh(\beta_\alpha d_{max})}. \tag{6}$$

We follow closely the method of [3]. First, we generate a statistically independent configuration C with a canonical update of the $4d$-$SU(2)$-System only. On this given configuration C, we perform one microcanonical update of the ($4d$-$SU(2)$+Demon)-System, as follows: The change of a link, under multiplication by a randomly generated matrix close to the identity, causes a shift of the effective energy of the lattice, ΔS_α, which must be compensated with the demon energies d_α. The trial is accepted if all the new demon energies remain in the allowed region. After visiting all links, the demons "move" to another gauge configuration C' while keeping their energies. Because our ansatz for the $3d$-reduced action is correct in the presence of small fluctuations only, we restrict the microcanonical update to a small step size, and perform only one microcanonical sweep on each gauge configuration.

4. RESULTS

The $\beta_{G,A,2,4}$ are related to the physical coupling constants $g_3(\mu), m_D^2(\mu)$ and $\lambda_A(\mu)$, as defined after (4). For $m_D^2(\mu)$ we must add counterterms, calculated in [4], to remove UV-divergencies and compare with continuum $\overline{MS}$-scheme theoretical calculations. We measure the effective couplings as a function of temperature T/T_c. To monitor effects of the lattice discretization, we study, at each temperature, several systems of identical physical size and increasingly fine lattice spacing ($N_t = 2, 3, 4$ and 5).

4.1. $g_3^2(\mu)$-results

As shown in Fig. 1, we observe a clear, large disagreement between the measurements and the perturbative prediction. The measured values of the coupling appear to depend only mildly on the lattice spacing. Unfortunately, the disagreement with the perturbative prediction worsens as the continuum limit is approached. The origin of the problem, we believe, lies in our lattice-based approach. The perturbative calculation is performed in the continuum theory, and the Green's functions are computed for small spatial momenta $|\vec{p}| \ll T$. In contrast, the spatial momenta on our lattices are cut-off by the lattice spacing, not by the temperature. As we reduce the lattice spacing to approach the continuum limit, we allow for larger momenta and depart more and more from the infrared, perturbative result.

As a first attempt to mimic the effect of our lattice approach, we tried to replace $\Lambda_{\overline{MS}} \to \Lambda_{\overline{MS}}*f$ in the continuum perturbative predictions. As shown in Fig. 1, a factor $f \sim 300$ is necessary to

reach reasonable agreement with our data, to be compared with the value 19.82 relating the $\overline{MS}$ and the lattice renormalization schemes [5]. In Figs.2 and 3, the dashed line shows the effect of this rescaling $f = 300$ on the other couplings.
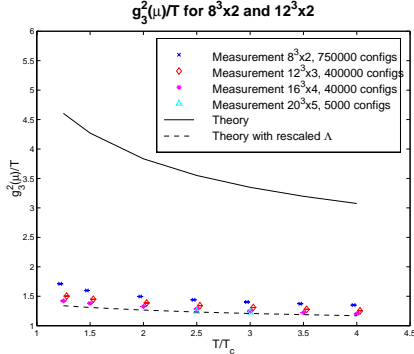

Figure 1. $g_3^2(\mu)/T$ vs T/T_c. Measurements and perturbative predictions (solid line) clearly disagree. Rescaling $\Lambda_{\overline{MS}}$ by a factor 300 restores agreement (dashed line).

4.2. $m_D^2(\mu)$-results

$m_D^2(\mu)$ stays somewhat below the theoretical predictions for all temperatures studied (up to $4T_c$). Because the Debye mass is only obtained after subtraction of UV-divergent counterterms, statistical errors quickly increase as the continuum limit is approached. There is a tendency for better agreement with perturbation theory at smaller a. As explained above, agreement should not be expected anyway since we keep larger spatial momenta than in the continuum approach. Nevertheless, the disagreement for m_D^2 is much smaller than for g_3^2, and could be accounted for by a milder rescaling of the Λ parameter.
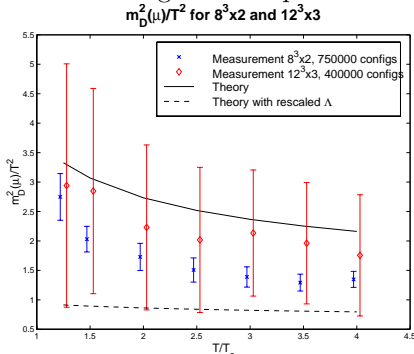

Figure 2. $m_D^2(\mu)/T^2$ vs T/T_c.

4.3. $\lambda_A(\mu)$-results

Like the other couplings, $\lambda_A(\mu)$ stays below the perturbative prediction at all temperatures. Scaling violations seem reasonably small.
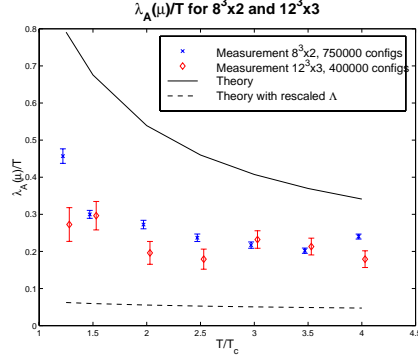

Figure 3. $\lambda_A(\mu)/T$ vs T/T_c.

5. CONCLUSION

In perturbative Dimensional Reduction, one considers only two and four-point functions, to one- or at most two-loop level [2], and neglects higher-dimension operators. Then, Dimensional Reduction appears to be valid down to $\sim 2T_c$ [6]. Numerically determined effective coupling constants allow a non-perturbative improvement of the theory. Higher-dimension operators, with couplings determined by additional demons, are easily included. We have added an A_0^6-term in the effective action, but the effect on the other couplings is small. Our results still disagree with continuum perturbative predictions even at $4T_c$. We intend to compare them with the perturbative results of [7], obtained directly on the lattice.

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