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Finite-size scaling of interface free energies in the $3d$ Ising model

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We perform a study of the universality of the finite size scaling functions of interface free energies in the $3d$ Ising model. Close to the hot/cold phase transition, we observe very good agreement with the same scaling functions of the $4d$ $SU(2)$ Yang–Mills theory at the deconfinement phase transition.

1. Introduction

Finite Size Scaling (FSS) is a very powerful tool, particularly useful in extracting numerical estimates from Monte Carlo simulations. Numerical simulations give results on systems of finite size, where phase transitions cannot occur. Thus, a fundamental task in Monte Carlo studies of phase transitions is the extraction of results holding in the thermodynamic limit from data obtained in systems of finite size. FSS describes how a system of finite size approaches criticality in the thermodynamic limit. One of its most far–reaching aspects concerns the universality of the FSS functions. If universality holds, complex systems share the same critical behaviour as simpler systems. This opens the possibility to use simple models to obtain accurate information on much more complicated theories.

In a recent paper[1], a study of the FSS of spatial 't Hooft loops of maximal size across the deconfinement phase transition has been performed in the $SU(2)$ Yang–Mills theory. By this analysis, a numerical estimate of the dual string tension - for the first time observed in [2] - has been obtained. Moreover, also the FSS of the electric free energies of the theory has been investigated.

According to the Svetitsky–Yaffe conjecture[3], the $4d$ $SU(2)$ Yang–Mills theory at the deconfinement transition is believed to be in the same universality class as the $3d$ Ising model at the hot/cold phase transition. Several numerical studies have confirmed the validity of this conjecture. Thus, also the universal FSS functions of the electric and magnetic free energies of the $4d$ $SU(2)$ theory measured in [1], should match with FSS functions observable in the $3d$ Ising model. We investigate the universality of the FSS functions in the $3d$ Ising model at the hot/cold phase transition. In particular we focus on the FSS of interface free energies.

2. The observables

Our aim is to compare FSS functions measured in the Ising model with those observed in $SU(2)$ for the magnetic and the electric free energies. In order to set this connection, it is useful to recall the idea behind the Svetitsky–Yaffe conjecture for the case we are considering now. If we could integrate the degrees of freedom of the $4d$ $SU(2)$ gauge theory to write an effective action for the Polyakov loop $P(\vec{x})$, this effective action would be a $3d$ spin system with symmetry Z_2. Then Svetitsky and Yaffe give arguments according to which the (complicated) effective action for the Polyakov loop reduces - close to criticality - to the nearest–neighbour Ising interaction. If we now consider an $SU(2)$ Yang–Mills theory on the lattice, a spatial 't Hooft loop in the xy plane can be obtained by inverting the sign of the coupling of a set of time–like zt plaquettes. In [1] it is shown that $P(\vec{x})P^{\dagger}(\vec{x}+L\hat{z}) = -1$, where L is the extension of the system in direction z and $\hat{z}$

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is the unit vector along the z axis. This condition can be accomplished in the effective Ising model for the Polyakov loop, by changing the boundary conditions from periodic (p.b.c.) to antiperiodic (a.p.b.c.) in direction z. This generates a frustration in the spin system, which, in the cold phase, creates a surface of defects. One can set a.p.b.c. in one, two or in all three directions. In the $SU(2)$ theory, this corresponds to switching on 1, 2 or 3 orthogonal spatial 't Hooft loops of maximal size.

3. The computation

We study the $3d$ Ising model on a cubic lattice of size L^3, with the usual ferromagnetic nearest-neighbour interaction $\mathcal{S} = -\beta \sum_{\vec{x}\mu} \sigma_{\vec{x}} \sigma_{\vec{x}+\hat{\mu}}$. $\sigma_{\vec{x}}$ is the spin at the lattice site $\vec{x}$, $\hat{\mu}$ is the unit vector in direction μ and $\beta > 0$ is the coupling. The data presented in this paper have been collected on lattices of size up to $L = 32$. More details about the simulation parameters will be given in a forthcoming paper. We consider p.b.c. and a.p.b.c. in 1, 2 and 3 directions. $\mathcal{Z}_k(i)$, with respectively $i = 0, 1, 2, 3$ are the partition functions for these 4 different choices. The observable we consider is the free energy cost $F_k(i)$ to create such interfaces. It is given by

$$F_k(i) \equiv -\log \frac{\mathcal{Z}_k(i)}{\mathcal{Z}_k(0)} \quad i = 1, 2, 3 \tag{1}$$

The counterparts of the electric free energies of the $SU(2)$ theory are obtained by performing the Z_2 Fourier transform [4] in (1). In terms of the $\mathcal{Z}_k(i)$, their expressions are

$$Z_e(1) = \frac{1}{\mathcal{N}_e}(\mathcal{Z}_k(0) + \mathcal{Z}_k(1) - \mathcal{Z}_k(2) - \mathcal{Z}_k(3)) \tag{2}$$

$$Z_e(2) = \frac{1}{\mathcal{N}_e}(\mathcal{Z}_k(0) - \mathcal{Z}_k(1) - \mathcal{Z}_k(2) + \mathcal{Z}_k(3)) \tag{3}$$

$$Z_e(3) = \frac{1}{\mathcal{N}_e}(\mathcal{Z}_k(0) - 3\mathcal{Z}_k(1) + 3\mathcal{Z}_k(2) - \mathcal{Z}_k(3)) \tag{4}$$

where $\mathcal{N}_e = \mathcal{Z}_k(0) + 3\mathcal{Z}_k(1) + 3\mathcal{Z}_k(2) + \mathcal{Z}_k(3)$. The correlation length ξ sets the scale of the distances. It has a critical behaviour with exponent $\nu \approx 0.63$: $\xi \sim |t|^{-\nu}$, where $t = 1 - \beta_c/\beta$. We express the FSS functions in terms of the scaling variable $x = \mathrm{sign}(t)L|t|^{\nu} \propto L/\xi$. In Figure 1, we compare the FSS functions of $F_k(1)$

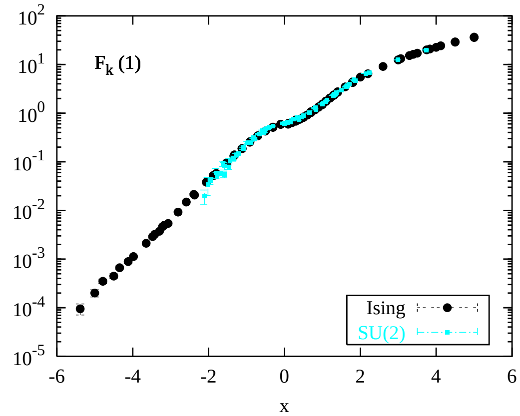

Figure 1. Comparison of the FSS functions of $F_k(1)$ between the Ising model and $SU(2)$.

between the Ising model and $SU(2)$. We show results both in the cold/deconfined phase ($x > 0$) and the hot/confined one ($x < 0$). On one hand, we observe that, according to FSS expectations, the data collected at various values of L and β lie on a single curve, depending only on x. On the other hand, it turns out to agree excellently with the FSS function of one spatial 't Hooft loop in $SU(2)$. Note that the scaling functions for $SU(2)$ and the Ising model do not directly match from the raw data. A rescaling of the correlation length by a factor α is necessary: $\xi_{\text{Ising}} = \xi_{SU(2)}/\alpha$. This implies that the variable x defined for the Ising model and the one used in $SU(2)$ [1] are related by: $x_{\text{Ising}} = \alpha x_{SU(2)}$. We have estimated the value $\alpha = 1.88(2)$ by rescaling the $x > 0$ Ising data in such a way that they overlap with the $SU(2)$ ones. We stress that, once α has been estimated, its value remains fixed and it is no more a fitting parameter. In Figures 2 and 3 the x variable for the $SU(2)$ data has been rescaled by α. In Figure 2, we display the comparison Ising–$SU(2)$ for $F_k(i)$ $i = 1, 2, 3$ at $x > 0$. The interface tension σ_i is a fitting parameter only for $F_k(1)$: we estimate the value $\sigma_1 = 1.495(15)$. For a full comparison, we have used the same fitting ansatz as in [1]. This result is consistent with – but more precise than – the value measured in [5]. For $F_k(i)$ $i = 2, 3$, we have set $\sigma_i = \sigma_1\sqrt{i}$, as follows from the expectations of the minimal interface area. This hypothesis is well satisfied, as apparent from the good quality of the fits. The very good match-

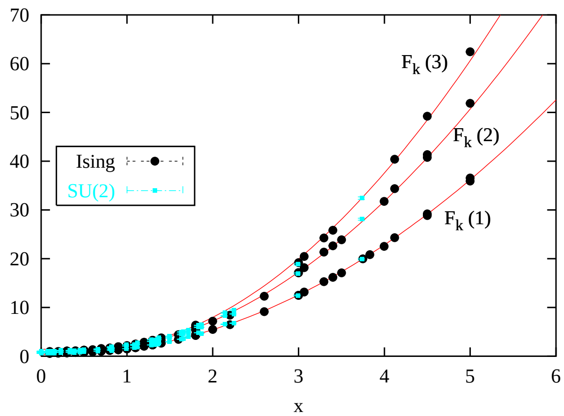

Figure 2. Comparison Ising-$SU(2)$ of the FSS functions of $F_k(i)$, $i = 1, 2, 3$. The continuous lines are fits according to the ansatz used in [1].

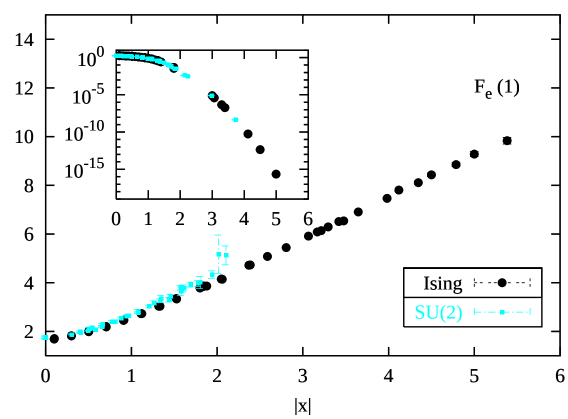

Figure 3. Comparison Ising-$SU(2)$ of the FSS functions of $F_e(1)$. The data in the main figure refer to the hot/confined phase. The insert displays the results in the cold/deconfined phase.

ing Ising–$SU(2)$ of the FSS functions for the free energy interface holds also for the cases of a.p.b.c. in 2 and 3 directions. Figure 3 concerns the FSS for $F_e(1)$; similar results hold for $F_e(2)$ and $F_e(3)$. Thus, also for the electric free energies, we observe very good agreement between the FSS functions of the two theories, with much increased accuracy in the Ising case. Finally, we have measured the product $\sigma_1\xi_+^2$, where ξ_+ is the correlation length in the hot phase. At large x and close to the critical point, the FSS functions of $F_k(1)$ and $F_e(1)$ have the simple asymptotic forms: $F_k(1) \approx \sigma_1 x^2 + C_k$ and $F_e(1) \approx |x|/\xi_+ + C_e$, where C_k and C_e are two constants. Then at large $|x|$, we have $F_k(1)/F_e(1)^2 \to \sigma_1\xi_+^2$. In order for this ratio to reach its asymptotic behaviour sooner, we have subtracted a constant C to the interface free energy $F_k(1)$. In Figure 4 we plot $A \equiv (F_k(1) - C)/F_e(1)^2$ as function of $|x|$. The data clearly show a flattening to a constant at large $|x|$. Fitting the last 5 points to the right with a constant, we obtain $(\sigma_1\xi_+^2) = 0.44(1)$. Extended discussions and details about the shown results will be presented in a forthcoming paper.

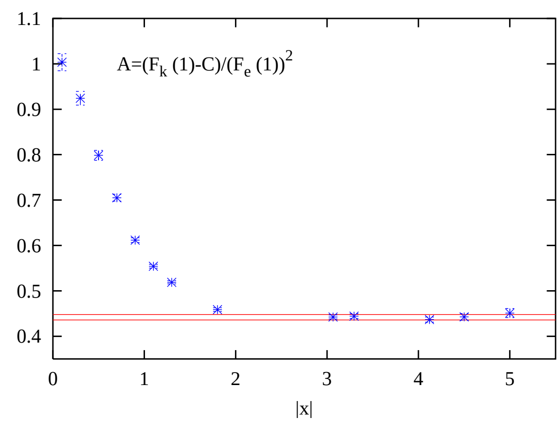

Figure 4. Approach of A to a constant. The strip is the estimated asymptotic value $\sigma_1\xi_+^2$.

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REFERENCES

1. P. de Forcrand and L. von Smekal, hep-lat/0107018 and hep-lat/0110135.
2. P. de Forcrand, M. D'Elia and M. Pepe, Phys. Rev. Lett. **86** (2001) 1438.
3. B. Svetitsky and L. G. Yaffe, Nucl. Phys. B **210** (1982) 423.
4. G. 't Hooft, Nucl. Phys. B **138** (1978) 1.
5. M. Hasenbusch and K. Pinn, Physica A **245** (1997) 366.