# The $U_A(1)$ Problem on the Lattice *

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If the expression of the topological charge density operator, suggested by fermions obeying the Ginsparg–Wilson relation, is employed, it is possible to prove on the lattice the validity of the Witten–Veneziano formula for the $\eta'$ mass. Recent numerical results from simulations with overlap fermions in 2 (abelian Schwinger model) and 4 (QCD) dimensions give values for the mass of the lightest pseudo-scalar flavour-singlet state that agree with theoretical expectations and/or experimental data.

## 1. Introduction

In a nut-shell the $U_A(1)$ problem lies in the fact that, in the absence of $U_A(1)$ anomaly contributions, the unphysical bound $m_{\mathrm{octet},I=0} < \sqrt{3}m_\pi$ holds [1]. In 1976 't Hooft [2] pointed out that the resolution of this problem had to be related to the existence of topologically non-trivial gauge field configurations in Euclidean QCD. Dwelling on this observation, it is possible to obtain an explicit formula for the $\eta'$ mass either in the 't Hooft limit ($N_c \to \infty$, with $g^2 N_c$ and $N_f$ held fixed [3]), as was done in [4], or by assuming that anomalous flavour-singlet axial WTIs retain their validity order by order in an expansion in $u \equiv N_f/N_c$ around $u = 0$, as argued in [5]. In both cases one can derive the "leading-order" Witten–Veneziano (WV) relation

$$m^2_{\eta'} = \frac{2N_f}{F^2_\pi} A\,, \quad A = \int d^4x \, \langle Q(x)Q(0)\rangle\Big|_{\mathrm{YM}}, \quad (1)$$

where $F_\pi \simeq 94\,\mathrm{MeV}$ is the pion decay constant, $A$ the "topological susceptibility" and $Q = \frac{g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F_{\mu\nu}F_{\rho\sigma}]$ the topological charge density. The notation $\langle ...\rangle|_{\mathrm{YM}}$ means that the $QQ$-correlator is to be computed in the pure Yang–Mills (YM) theory, i.e. in the absence of quarks.

The formal relations in eq. (1), when translated in any regularized version of QCD, such as lattice QCD, become more involved and quite a number of subtleties have to be dealt with in order to be able to determine the correct field theoretical expression of $A$, which should be used in the above equations. Two problems need to be solved to make the formulae in (1) precise and of practical use. One has to i) find a properly normalized lattice definition of the topological charge density, $Q_L$; ii) subtract from the $Q_L(x)Q_L(0)$ product appropriate contact terms, as required to make it an integrable (operator-valued) distribution.

A rigorous derivation of a formula for the $\eta'$ mass, which could be unambiguously employed in numerical simulations, can be given [6], if use is made of the (lattice) regularized anomalous flavour-singlet axial WTIs of full QCD in order to construct a properly defined $\langle Q_L Q_L\rangle$ correlation function. The remarkable result of this analysis is that no subtraction is needed to this end in the chiral limit, if the definition of lattice topological charge density suggested by fermions obeying the Ginsparg–Wilson (GW) relation [7] (e.g. overlap fermions [8]) is employed.

## 2. GW fermions and the $\eta'$ mass formula

Regularizing the fermionic part of the QCD action using GW fermions [7,8] offers the great advantage over the standard Wilson discretization [9] that global chiral transformations can be defined, which are an exact symmetry of the massless theory, as in the formal continuum limit.

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This follows from the relation $\gamma_5 D + D\gamma_5 = D\gamma_5 D$ [7], obeyed by the GW Dirac operator, $D$ [3]. In this regularization the $U_A(1)$ anomaly is recovered à la Fujikawa [10], because of the non-trivial Jacobian that accompanies the change of fermionic integration variables induced by a $U_A(1)$ lattice transformation. As a consequence the (anomalous) flavour-singlet WTIs in the presence of $N_f$ massless fermions take the form

$$\nabla_\mu \langle A^0_\mu(x)\hat{O}\rangle = \frac{2N_f}{2}\langle \mathrm{Tr}[\gamma_5 D(x,x)]\hat{O}\rangle + \langle \delta^x_A \hat{O}\rangle\,,$$

where $\hat{O}$ is any finite (multi)local operator, $\delta^x_A\hat{O}$ is its local variation and $\nabla_\mu A^0_\mu(x)$ is the divergence of the singlet axial current.

Neither $\mathrm{Tr}[\gamma_5 D(x,x)]$ nor $A^0_\mu(x)$ are finite operators, but finite linear combinations, $\hat{Q}$ and $\hat{A}^0_\mu$, can be easily constructed [11] by writing

$$\hat{Q}(x) = \frac{1}{2}\mathrm{Tr}[\gamma_5 D(x,x)] - \frac{Z}{2N_f}\nabla_\mu A^0_\mu(x) \quad (2)$$

$$\hat{A}^0_\mu(x) = (1-Z)A^0_\mu(x)\,. \quad (3)$$

where $Z$ is logarithmically divergent to lowest order in perturbation theory and vanishes as $u \to 0$.

With the above definitions the renormalized singlet axial WTI becomes

$$\nabla_\mu \langle \hat{A}^0_\mu(x)\hat{O}\rangle = 2N_f\langle \hat{Q}(x)\hat{O}\rangle + \langle \delta^x_A \hat{O}\rangle\,. \quad (4)$$

This equation shows that $\hat{Q}$ and $\hat{A}^0_\mu$ are correctly normalized and that there is no power-divergent mixing of $\mathrm{Tr}[\gamma_5 D(x,x)]$ with the pseudo-scalar quark density, which would bring in a dangerous lineraly divergent mixing coefficient.

A formula for the $\eta'$ mass can be obtained from the above WTIs, by observing that the $\eta'$ mass must vanish as $u \to 0$ for massless QCD to have no $\theta$-dependence. For that one starts by defining the lattice Green function [4]

$$\chi_{tL}(p) = \frac{1}{2N_f}\int d^4x\, \mathrm{e}^{-ipx}\nabla_\mu\langle \hat{A}^0_\mu(x)\hat{Q}(0)\rangle\,, \quad (5)$$

with $\hat{Q}$ given by eq. (2). In the full theory, where the $\eta'$ is massive, one has $\chi_{tL}(0) = 0$. On the other hand, since, as we obseved above, at the chiral point, the $\eta'$ mass vanishes as $u \to 0$, only the $\eta'$ pole will contribute to $\chi_{tL}(p)$ in this limit, leading to the relation

$$\lim_{p\to 0}\lim_{u\to 0}\chi_{tL}(p) = \frac{F_\pi^2}{2N_f}m^2_{\eta'}\Big|_{u=0}\,. \quad (6)$$

The l.h.s. of eq. (6) can be evaluated by using the WTI (4) with $\hat{O} = \hat{Q}$. Recalling that $Z$ vanishes when $u \to 0$ and $\delta^x_A\hat{Q} = 0$, we obtain

$$\frac{F_\pi^2}{2N_f}m^2_{\eta'}\Big|_{u=0} = \lim_{p\to 0}\lim_{u\to 0}\int d^4x\, \mathrm{e}^{-ipx} \langle \frac{1}{2}\mathrm{Tr}[\gamma_5 D(x,x)]\frac{1}{2}\mathrm{Tr}[\gamma_5 D(0,0)]\rangle\,. \quad (7)$$

The limits $u \to 0$ and $p \to 0$ in this expression can be readily performed in the order indicated, if one can assume that taking the first limit simply amounts to setting the fermion determinant equal to unity. One gets in this way

$$\frac{F_\pi^2}{2N_f}m^2_{\eta'}\Big|_{u=0} = = \int d^4x \langle \frac{1}{2}\mathrm{Tr}[\gamma_5 D(x,x)]\frac{1}{2}\mathrm{Tr}[\gamma_5 D(0,0)]\rangle\Big|_{\mathrm{YM}} \quad (8)$$

The restriction to pure YM theory, indicated in eq. (8), is an obvious consequence of the fact that, for a Green function of only gluonic operators, neglecting the fermion determinant is tantamount to limiting the functional integral to the pure gauge sector of the theory.

A relevant question at this point is to ask at which value of $N_c$ eq. (8) is supposed to be valid. The answer depends on the behaviour of QCD with $N_f$. Various scenarios are envisegeable.

1) In the most favorable situation, in which the limit $u \to 0$ of $\chi_{tL}(p)$ exists and is equal to the value it takes at $N_f = 0$ (fermion determinant equal to 1), formula (8) is valid for any value of $N_c$ and for each $N_c$ it yields the mass of the $\eta'$ meson (at leading order in $u = 0$) in the world with the corresponding number of colours [5].

2) If quenching can be attained only in the limit in which the number of colours goes to infinity, then eq. (8) will yield a formula for the $m^2_{\eta'}$ valid up to $\mathrm{O}(1/N_c)$ corrections [4].

[3] We set the lattice spacing, $a$, equal to 1. In all lattice formulae the continuum limit is, however, understood.

[4] A contact term, $\mathrm{CT}(p)$, should be added to the r.h.s. of eq. (5) to make it finite at $p \neq 0$. $\mathrm{CT}(p)$ is a polynomial of degree 4 in $p$, which vanishes at $p = 0$, because the r.h.s. of eq. (5) is certainly finite (actually zero) at $p = 0$. $\mathrm{CT}(p)$ plays no rôle in the argument below, since we will be finally only interested in the value of $\chi_{tL}$ at $p = 0$. For brevity we will not indicate it explicitly in the following.

3) Finally it may happen that taking the limit $u \to 0$ does not correspond to quenching. In this case one cannot pass from the fairly complicated eq. (7) to the more useful formula (8).

Remembering that $\frac{1}{2}\mathrm{Tr}[\gamma_5 D(x,x)]$ should be identified with the topological charge density [8], we conclude that eq. (8) can be rewritten in the very suggestive form

$$\frac{F_\pi^2}{2N_f} m_{\eta'}^2 \Big|_{u=0} = \lim_{V\to\infty} \frac{\langle (n_R - n_L)^2 \rangle}{V}\,, \qquad (9)$$

where $\langle (n_R - n_L)^2 \rangle$ is the expectation value of the square of the index of the GW fermion operator, $D$, and $V$ is the physical volume of the lattice.

In the form (9) the $\eta'$ mass formula can be directly compared with overlap fermion simulation data. Existing numerical results in 2 (abelian Schwinger model [12]) and in 4 (QCD [13]) dimensions agree quite well with theoretical expectations and/or experimental numbers [6].

## 3. Conclusions

We have shown that a formula for the $\eta'$ mass can be rigorously derived in lattice QCD, exploiting the anomalous flavour-singlet axial WTIs of the theory. If fermions obeying the GW relation are used, there exists a natural definition of topological density which is correctly normalized and for which the naive form of the WV formula holds without the need of introducing any subtraction.

In the literature two sorts of approaches have been proposed to deal with the problem of computing $A$ on the lattice, which have led to numerical values as good as those obtained with the present method [6]. The first one is based on a direct field theoretical definition of $A$ [14] that takes into account the need for the renormalization of $Q_L$ and the subtraction of the operators $F^2$ and $\mathbb{1}$ in the short distance expansion of $Q_L Q_L$. The second one makes use of the notion of "cooling" [15] to carry out the necessary operations of renormalization and subtraction. Both methods are reliable to the extent that they are able to capture the topological properties of the gauge field configurations that determine the number of zero modes of the Dirac operator. Simulations based on the geometrical definition of $Q_L$ of ref. [16] have not yet led to comparably good results [17].

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