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Patterns of Spontaneous Chiral Symmetry Breaking in Vectorlike Gauge Theories

P. H. Damgaard*
TH-Division, CERN
CH-1211 Geneva 23, Switzerland

U.M. Heller
CSIT, Florida State University
Tallahassee FL 32306-4120, U.S.A.

R. Niclasen
Physics Department, University of Colorado
Boulder CO 80309, U.S.A.

B. Svetitsky
Center for Theoretical Physics
Laboratory of Nuclear Science and Department of Physics
Massachusetts Institute of Technology
Cambridge, MA 02139, USA
and
School of Physics and Astronomy
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University
69978 Tel Aviv, Israel†

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Abstract

It has been conjectured that spontaneous chiral symmetry breaking in strongly coupled vectorlike gauge theories falls into only three different classes, depending on the gauge group and the representations carried by the fermions. We test this proposal by studying SU(2), SU(3) and SU(4) lattice gauge theories with staggered fermions in different irreducible representations. Staggered fermions away from the continuum limit should, for all complex representations, still belong to the continuum class of spontaneous symmetry breaking. But for all real and pseudo-real representations we show that staggered fermions should belong to incorrect symmetry breaking classes away from the continuum, thus generalizing previous results. As an unambiguous signal for whether chiral symmetry breaks, and which breaking pattern it follows, we look at the smallest Dirac eigenvalue distributions. We find that the patterns of symmetry breaking are precisely those conjectured.

*On leave from: the Niels Bohr institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark.
†Permanent address.

1 Introduction

Consider an asymptotically free gauge theory of gauge group $\mathcal{G}$ coupled vectorially to N_f massless Dirac fermions that all transform according to some irreducible representation r of the gauge group. This theory will have a global chiral-flavor symmetry, whose group we denote by G. An obvious general question to ask is whether this group G will be spontaneously broken down to some subgroup H, and if so, what is the precise symmetry breaking pattern. This question is relevant for the construction of technicolor theories, but it is also of interest in its own right. Conventional folklore says that there are, for four-dimensional field theories, just three classes of breaking to consider [1]:

- The fermion representation r is pseudo-real: Chiral symmetries are enhanced from $SU(N_f) \times SU(N_f)$ to $SU(2N_f)$, and the expected symmetry breaking pattern is $SU(2N_f) \to Sp(2N_f)$.
- The fermion representation r is complex: The expected symmetry breaking pattern is $SU(N_f) \times SU(N_f) \to SU(N_f)$.
- The fermion representation r is real: Chiral symmetries are again enhanced to $SU(2N_f)$, and the expected symmetry breaking pattern is $SU(2N_f) \to SO(2N_f)$.

This remarkably simple classification of the possible symmetry breaking patterns is what follows from the assumption of maximal breaking of chiral symmetry with simultaneous preservation of maximal flavor symmetry. Indeed, spontaneous flavor symmetry breaking in vectorlike gauge theories is prohibited by the Vafa-Witten theorem [2], and the above symmetry breaking patterns are thus consistent with this theorem. Very recently it has been checked that the most promising candidate for a c-theorem in four dimensions is consistent with these breaking schemes for all simple gauge groups $\mathcal{G}$ and fermions transforming according to arbitrary irreducible representations under those gauge groups [3]. At a large number of colors, the Coleman-Witten argument [4] and a simple generalization of it to pseudo-real gauge groups [3] shows that indeed the symmetry breaking patterns are precisely those expected. But a general proof for a finite number of colors is still lacking.

One way to test whether the dynamics really is such these three types of symmetry breaking do occur is to compute the appropriate chiral condensates by numerical means in lattice gauge theory. This is unfortunately a highly non-trivial task. Consider the definition of a chiral condensate

$$\Sigma \;=\; \lim_{m\to 0}\lim_{V\to 0}\langle\bar{\Psi}\Psi\rangle \tag{1}$$

which requires the limit of infinite four-volume $V \to \infty$ to be taken prior to the massless limit $m \to 0$. While it is already increasingly time-consuming to simulate lighter and lighter fermions by Monte Carlo methods, the infinite volume limit obviously cannot be taken under any circumstances. Even in the quenched case, where the massless limit can be taken trivially, the above order parameter for spontaneous chiral symmetry breaking can only be used in conjunction with an often quite uncertain extrapolation to the infinite volume limit.

Instead, we shall here make use of a much more efficient technique. It is based on the fact that the distributions of the smallest Dirac operator eigenvalues, when measured over different volumes, contain the combined information of 1) whether spontaneous chiral symmetry breaking occurs, and 2) which symmetry breaking pattern is realized. So far, this has been confirmed for the fundamental and adjoint representations of SU(2) and SU(3) gauge groups [5]. Tests have also been performed on strongly coupled U(1) theories, in both 2 and 4 dimensions [6]. Here, we shall be a little more systematic, and probe also more exotic irreducible representations. If the above conjecture about spontaneous symmetry breaking is correct, we know precisely which patterns the different cases should follow. Because the distributions of the smallest Dirac operator eigenvalues are exact finite-volume scaling functions, this turns the finite volumes used in lattice gauge theory simulations into a distinct advantage.

We shall here briefly review how the distribution of the smallest Dirac operator eigenvalues can be used to show both that spontaneous chiral symmetry breaking occurs *and* specify uniquely to which symmetry breaking class it belongs. In fact, it is obvious that we need only consider the smallest Dirac operator eigenvalues because by the Banks-Casher relation

$$\Sigma = \pi\rho(0) \tag{2}$$

it is precisely the density of Dirac operator eigenvalues around the origin which determines whether or not chiral symmetry is spontaneously broken. At finite volume V this means that the eigenvalue spacing near the origin must go like $1/(V\rho(0))$ [7]. This leads to the definition of the microsopically rescaled spectral density [8],

$$\rho_s(\zeta) \equiv \frac{1}{V}\rho\left(\frac{\zeta}{V\Sigma}\right) , \tag{3}$$

which should have a finite limit as $V \to \infty$. Effectively, one is blowing up the very smallest Dirac operator eigenvalues so that the average eigenvalue spacing is of order unity, rather than $1/(V\rho(0))$. This is finite-volume scaling: the eigenvalue density should fall on one universal curve if the eigenvalues λ are rescaled according to $\zeta \equiv V\Sigma\lambda$. So to demonstrate spontaneous chiral symmetry breaking it suffices to check that the smallest Dirac operator eigenvalues indeed do satisfy this simple finite-volume scaling, even if the functional form of (3) were not known. Actually, the microscopic spectral densities and higher spectral correlation functions *are* in fact known analytically from universal Random Matrix Theory results (see [9, 10, 11, 12, 13] for the $\beta = 2$ universality class, and [14, 15, 16, 17, 18, 19] for the two other universality classes). This holds in arbitrary sectors of fixed topological charge ν. It is now also known how these results can be derived directly from field theory [20], although the method based on Random Matrix Theory remains the simplest.

The same argument holds for the distribution of individual Dirac operator eigenvalues, again rescaled to microscopic variables $\zeta = V\Sigma\lambda$. Analytical formulas exist for an infinite sequence of such eigenvalue distributions for all three universality classes [21]. The only exception is the $\beta = 1$ case, where so far only the case of odd topological charge ν has been computed in all generality, but here at least the lowest eigenvalue distribution is known for the most interesting case, namely $\nu = 0$ [22]. As both the spectral correlation functions, and the individual eigenvalue distributions differ markedly for the three different universality classes, all of these observables are excellent candidates for determining the chiral symmetry breaking pattern. The analytical functions with which to compare are generally rather simple combinations of Bessel functions.

2 Symmetries of the Dirac Operator and Chiral Symmetry Breaking

To make the connection between the three symmetry breaking classes and the Random Matrix Theory results, it is instructive to first work out the connection between (anti-unitary) symmetries of the Dirac operator, and the type of representation the fermions carry (*i.e.*, whether they are pseudo-real, complex, or real). In fact, this argument is only to make the connection to Random Matrix Theory.

We know already from the chiral Lagrangian alone that the microscopic Dirac operator spectrum always matches one-to-one the three symmetry breaking classes above. If Random Matrix Theory is to give equivalent results as those that can be derived from field theory, *i.e.*, from the relation between the fundamental Lagrangians and those of the corresponding effective chiral Lagrangians, it has to be determined by just these three symmetry breaking classes. What we learn from this is that there must exist a direct relation between the anti-unitary symmetries of the Dirac operator and the pattern of chiral symmetry breaking, in all generality. This is not too surprising, since the pattern of symmetry breaking is just given by the color representation of the fermions, and we shall return to this connection below.

The relevant question is whether there is an anti-unitary symmetry VK of the Dirac operator,

$$[VK, D(A)] \ = \ 0 \ , \tag{4}$$

where K is the operation of complex conjugation and V is unitary. The generic case is that there is no such symmetry, and then the matrix elements of the Dirac operator in a chiral basis are simply complex. The corresponding Random Matrix Theory ensemble is (chiral) unitary, *ie.,* $\beta = 2$ [23, 24].

If such a symmetry exists, then either $(VK)^2 = +1$ or $(VK)^2 = -1$, and both ψ and $VK\psi$ have the same eigenvalues of $D(A)$. When $(VK)^2 = +1$ the two eigenvectors ψ and $VK\psi$ are, however, not linearly independent ($VK\psi = \pm\psi$). But if $(VK)^2 = -1$, the two eigenvectors *are* linearly independent, and we have a genuine double degeneracy of eigenvalues. In the former case a basis exists so that the matrix elements of the Dirac operator are real, and in Random Matrix Theory language this corresponds to the (chiral) orthogonal ($\beta = 1$) ensemble [23]. In the latter case the matrix elements of the Dirac operator can be diagonalized by a symplectic transformation, and the corresponding Random Matrix Theory is the (chiral) symplectic ($\beta = 4$) one [23].

Now, the complex conjugation K affects both the gauge fields and the gamma matrices in the Dirac operator. Let Γ be the γ-matrix for which $\Gamma\gamma_\mu^*\Gamma^\dagger = \gamma_\mu$. Then Γ is the part of V, if V exists, which acts on the Dirac indices. For both the Euclidean Dirac and chiral representation of the γ-matrices we have $\Gamma = \gamma_1\gamma_3$. So we write $V = \Gamma S$, where S acts entirely in color space. Then $(VK)^2 = VKVK = VV^*KK = VV^*$, since $K^2 = 1$, and with $V = \Gamma S$ we have $(VK)^2 = SS^*\Gamma\Gamma^*$. For $\Gamma = \gamma_1\gamma_3$, $\Gamma^* = \Gamma$ and $\Gamma^2 = -1$. Therefore we find $(VK)^2 = -SS^*$.

For the gauge field part, let us remind ourselves of some basic properties of complex, real and pseudo-real representations. A *complex* representation is one which is not equivalent to its complex conjugate. A *real* representation is equivalent to its complex conjugate in the following sense. Let T^a be the (hermitian) generators in the given representation. It is real if there exists a unitary S so that

$$(T^a)^* \ = \ (T^a)^T \ = \ -\,S^{-1}T^aS \quad , \qquad SS^* \ = \ 1 \tag{5}$$

For such a representation, $U = \exp[i\omega^aT^a]$ is equivalent to its complex conjugate (equal after a similarity transformation). It is then possible to find a basis in which all T^a are purely imaginary (but of course still Hermitian). A *pseudo-real* representation is one for which there exists a unitary S so that

$$(T^a)^* \ = \ (T^a)^T \ = \ -\,S^{-1}T^aS \quad , \qquad SS^* \ = \ -\,1 \tag{6}$$

It is then no longer possible to find a basis in which all generators T^a are purely imaginary. The fundamental representation of SU(2) with its conventional Pauli matrices $\tau^a/2$ is a good example. Since (5) implies $S = S^T$, and (6) implies $S = -S^T$, the one major distinction between real and pseudo-real is whether S is *symmetric* or *antisymmetric.*

For a *real* representation we thus find $(VK)^2 = -1$ and the Dirac operator will belong to the *symplectic* ensemble, $\beta = 4$, while for a *pseudo-real* representation $(VK)^2 = 1$ and the Dirac operator belongs to the *orthogonal* ensemble, $\beta = 1$. This is a little counter-intuitive, compared to the symmetry breaking patterns. The reason is the behavior of the γ-matrices in the Dirac operator under complex conjugation.

So far we have only been concerned with the symmetries of the (massless) Dirac operator. In fact there is an intimate connection between the symmetries of the Dirac operator and the pattern of spontaneous symmetry breaking. This may seem surprising, since the conjectured symmetry breaking patterns are based on the symmetries of the condensate $\langle\bar{\Psi}\Psi\rangle$ rather than those of the Dirac operator. In order to elucidate the relation between the two, let us first quickly recall how the appearance of the three different patterns of spontaneous chiral symmetry breaking can be understood [1]. For this purpose it is convenient to introduce the two-component van der Waerden notation of dotted and undotted spinor indices, so familiar from *e.g.* supersymmetry. We consider 4-component massless Dirac spinors $\Psi(x)$, and choose to work in a chiral basis of γ-matrices. Then upper and lower parts of the 4-spinors simply correspond to the left-handed and right-handed components:

$$\Psi \;=\; \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} \;\equiv\; \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot\beta} \end{pmatrix} \;=\; \begin{pmatrix} \psi_\alpha \\ (\chi^\beta)^* \end{pmatrix} , \tag{7}$$

and the charge conjugate spinor is then given by

$$\Psi^C \;=\; C\bar{\Psi}^T \;=\; -\,i\gamma^0\gamma^2\bar{\Psi}^T \;=\; \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot\beta} \end{pmatrix} . \tag{8}$$

In other words, instead of working with one 4-component Dirac spinor Ψ, we can equally well work with two left-handed spinors from Ψ and its charge conjugate Ψ^C,

$$\psi_\alpha \quad \text{and} \quad \chi_\beta \;=\; \epsilon_{\beta\gamma}\chi^\gamma , \tag{9}$$

where in the last equation we have made use of the fact that the two-component spinor indices are raised and lowered by means of the antisymmetric ϵ-tensor. Consider now a well-known example of spontaneous chiral symmetry breaking, that of a QCD-like theory with N_f flavors of massless quarks transforming according to the fundamental representation of gauge group $\mathcal{G} = \mathrm{SU}(3)$. An order parameter is the well-known condensate of $\bar{\Psi}\Psi$, which includes a summation of both color and flavor indices. Let us make this explicit, and at the same time write the fermion bilinear in terms of the two-component spinors:

$$\bar{\Psi}\Psi \;=\; \epsilon^{\alpha\beta}\chi^{ia}_\beta\psi_{\alpha ia} \;\;+\;\; \text{h.c.} \tag{10}$$

Here α and β are the two-component spinor indices, while i and a denote flavor and color indices, respectively. Now, $\psi_{\alpha ia}$ transforms like a 3-representation under color while $\chi^{\beta ia}$ transforms like a $\bar{3}$ (see eq. (7)). It is thus convenient to relabel the latter spinor as a ψ-spinor with the color transformation property made explicit:

$$\bar{\Psi}\Psi \;=\; \epsilon^{\alpha\beta}\psi^{i(\bar{3})}_\beta\psi^{(3)}_{\alpha i} \;\;+\;\; \text{h.c.} \tag{11}$$

Then it is immediately clear that the generalization to an arbitrary complex representation r of gauge group $\mathcal{G}$ is

$$\epsilon^{\alpha\beta}\psi^{i(\bar{r})}_\beta\psi^{(r)}_{\alpha i} \;\;+\;\; \text{h.c.}$$

Note how the left-handed and right-handed pieces trivially are invariant under the same symmetries, since the right-handed part is just the Hermitian conjugate of the left-handed part. The term above is in fact the $\mathcal{G}$-invariant fermion bilinear of maximum vectorlike flavor symmetry, and if it attains a non-vanishing expectation value it is thus consistent with the Vafa-Witten theorem. The flavor symmetry remaining of the above expression is only $SU(N_f)$, and the symmetry breaking pattern, if realized, thus corresponds to

$$SU(N_f) \times SU(N_f) \;\rightarrow\; SU(N_f) \tag{12}$$

for all complex representations.

For real representations r of the gauge group $\mathcal{G}$ the representation r is equivalent to its complex conjugate $\bar{r}$. The initial symmetry is then bigger, enlarged to $SU(2N_f)$ because ψ and $S\chi$ (with S symmetric, as discussed above) transform in the same way under color, and thus can mix. The $\mathcal{G}$-invariant fermion bilinear of maximal flavor symmetry is then

$$\epsilon^{\alpha\beta}\psi^{ia}_{\beta}\psi^{b}_{\alpha i}S^{-1}_{ab}$$

where S is the symmetric matrix described above. Because of fermi statistics this bilinear can have non-vanishing expectation value. The continuous flavor symmetries remaining are only those of orthogonal transformations, so the symmetry breaking in that case should be

$$SU(2N_f) \;\rightarrow\; SO(2N_f) \tag{13}$$

Because of the doubling in symmetries it is in this case possible to consider also the breaking pattern of Majorana fermions, which effectively corresponds to replacing $2N_f$ by N_f (real, Majorana fermions).

Finally there is the pseudo-real case. Although the representation r in that case is not equivalent to $\bar{r}$, it is possible to arrange for a fermion transforming according to, say, $\bar{r}$ to transform according to r by multiplying by the antisymmetric matrix S of eq. (6). We can thus again, by this relabelling, work with fields that only transform according to the representation r. Because of its antisymmetry, the only way to form a non-vanishing bilinear out of anticommuting fermion fields is by multiplying with a matrix antisymmetric in flavor indices. The result is

$$\epsilon^{\alpha\beta}\psi^{ia}_{\beta}\psi^{jb}_{\alpha}S^{-1}_{ab}E_{ij}$$

where now $S = -S^T$, and also $E = -E^T$. The group of continuous flavor transformations leaving this quadratic form invariant is $Sp(2N_f)$, and the expected symmetry breaking pattern is thus

$$SU(2N_f) \;\rightarrow\; Sp(2N_f)\ . \tag{14}$$

All of this is standard. What is perhaps puzzling is that these considerations in no way involve the symmetries of the Dirac operator, the starting point for the analysis in terms of Random Matrix Theory. The conjectured symmetry breaking patterns were originally based on the intuitive idea of maximally breaking chiral symmetry without breaking flavor symmetries, an idea which subsequently found its justification in the Vafa-Witten theorem. This suggests that the Random Matrix Theory approach, and its associated three chiral matrix ensembles [23], in some way should contain the same ingredients that enter in the proof of the Vafa-Witten theorem. This idea is not totally far-fetched because in fact the main assumption on which the Vafa-Witten theorem rests [2] is positivity of the measure, which for the fermionic part can be traced back to the fact that Dirac eigenvalue density is even in $\lambda:\ \rho(\lambda) = \rho(-\lambda)$. This property is automatically built into the chiral Random Matrix Theory, with $\rho(\lambda)$ now being replaced by the eigenvalue density of the random matrices. As for the precise symmetry breaking patterns, we have seen that the classification in terms of Random Matrix Theory goes parallel with the classification based on the assumption of maximal chiral symmetry breaking (without breaking flavor symmetries) in that it depends on the color representation only. Without any reference to Random Matrix Theory, in a chiral basis the Dirac operator matrix elements are complex for complex representations, can be chosen real for pseudo-real representations, and can be chosen quaternion-real for real representations [23]. In the latter case the Dirac operator eigenvalues are doubly degenerate. In this sense the classification according to the Dyson indices β can be done independently of the specific chiral Random Matrix Theories. The fact that the chiral Random Matrix Theories in the microscopic limit can be mapped to precisely the zero-momentum mode effective chiral Largrangian corresponding to just the right cosets of symmetry breaking [25] is a remarkable fact for which there is clearly no simple explanation based only on group theoretic arguments.

3 Staggered Fermions

For staggered fermions the situation is both simpler and more complicated. More complicated is the pattern of symmetry breaking. Simpler are the symmetries of the Dirac operator. Since the staggered Dirac operator does not have any γ-matrices, but only sign factors, the (real) Kogut-Susskind phases, $\eta_\mu(x) = \pm 1$, the potential anti-unitary symmetry operators are just SK, with S as before acting in color space,

$$[SK, D(U)] \;=\; 0\;. \tag{15}$$

Now $(SK)^2 = SS^*$, and hence *real* representation staggered fermions should be associated with the chiral *orthogonal* ensemble, $\beta = 1$, and *pseudo-real* representation staggered fermions should correspondingly be associated with the chiral *symplectic* ensemble. As compared to the continuum case the two classes $\beta = 1$ and $\beta = 4$ are always swapped.

The case of complex representations, when no anti-unitary symmetry operator exists, is the same for continuum and for staggered fermions. The corresponding Random Matrix Theory ensemble is chiral *unitary*, $\beta = 2$.

Staggered fermions in the fundamental representation of SU(2) are an example of the $\beta = 4$, symplectic case. The anti-unitary transformation is then given by $SK = i\sigma_2 K$, since for elements of the gauge group SU(2) $(i\sigma_2)U^*(-i\sigma_2) = U$ and since for staggered fermions $D(U)^* = D(U^*)$. Thus $[i\sigma_2 K, D(U)] = 0$, and, since $i\sigma_2$ is real, $(i\sigma_2 K)^2 = (i\sigma_2)^2 = -1$, so we have the symplectic case. We see that it crucially hinges on the relation $D(U)^* = D(U^*)$.

Another example of staggered fermions in the symplectic ensemble are staggered fermions in the $j = 3/2$ representation of SU(2). The analogous transformation is then given by

$$U_{(3/2)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} U^*_{(3/2)} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\;. \tag{16}$$

This can be seen by noting that, in terms of u_{11} and u_{12}, the elements in the first row of the SU(2) matrix in the fundamental representation we have

$$U_{(3/2)} = \begin{pmatrix} u_{11}^3 & \sqrt{3}u_{11}^2 u_{12} & \sqrt{3}u_{11}u_{12}^2 & u_{12}^3 \\ -\sqrt{3}u_{11}^2 u_{12}^* & u_{11}(|u_{11}|^2 - 2|u_{12}|^2) & u_{12}(2|u_{11}|^2 + |u_{12}|^2) & \sqrt{3}u_{11}^* u_{12}^2 \\ \sqrt{3}u_{11}(u_{12}^*)^2 & -u_{12}^*(2|u_{11}|^2 + |u_{12}|^2) & -u_{11}^*(2|u_{11}|^2 + |u_{12}|^2) & \sqrt{3}(u_{11}^*)^2 u_{12} \\ -(u_{12}^*)^3 & \sqrt{3}u_{11}^*(u_{12}^*)^2 & -\sqrt{3}u_{12}^*(u_{11}^*)^2 & (u_{11}^*)^3 \end{pmatrix}\;. \tag{17}$$

Again it is instructive to see the expected pattern of chiral symmetry breaking also from the possible fermion bilinears that can be formed out of staggered fermions carrying different representations. To start, we generalize the argument of ref. [27] and rewrite the massless staggered fermion action for arbitrary real and pseudo-real representations, *i.e.*, taking

$$U_\mu^* = S^\dagger U_\mu S \qquad \text{with } SS^* = \pm 1 \; . \tag{18}$$

Using $\eta_\mu(x \pm \mu) = \eta_\mu(x)$ we can write

$$S_{st} = \frac{1}{2} \sum_{x=even,\mu} \eta_\mu(x)[A(x) + B(x)] \; , \tag{19}$$

with

$$\begin{aligned} A(x) &= \bar{\chi}_e(x)U_\mu(x)\chi_o(x+\mu) - \bar{\chi}_e(x)U_\mu^\dagger(x-\mu)\chi_o(x-\mu) \\ B(x) &= \bar{\chi}_o(x-\mu)U_\mu(x-\mu)\chi_e(x) - \bar{\chi}_o(x+\mu)U_\mu^\dagger(x)\chi_e(x) \; . \end{aligned} \tag{20}$$

Using the Grassmann nature of the χ's we can also express $B(x)$ as

$$\begin{aligned} B(x) &= -\chi_e^T(x)\left(U_\mu^\dagger(x-\mu)\right)^* \bar{\chi}_o^T(x-\mu) + \chi_e^T(x)U_\mu^*(x)\bar{\chi}_o^T(x+\mu) \\ &= \chi_e^T(x)S^\dagger U_\mu(x)S\bar{\chi}_o^T(x+\mu) - \chi_e^T(x)S^\dagger U_\mu^\dagger(x-\mu)S\bar{\chi}_o^T(x-\mu) \; . \end{aligned} \tag{21}$$

In the second line we have made use of the fact that the representation is either real or pseudo-real. Introducing (the signs are for later convenience, the upper one for real and the lower one for pseudo-real representations)

$$\bar{X}_e = (\, \bar{\chi}_e, \quad \pm\chi_e^T S^\dagger \,) \; , \qquad X_o = \begin{pmatrix} \chi_o \\ \pm S\bar{\chi}_o^T \end{pmatrix} \tag{22}$$

the massless staggered fermion action for real or pseudo-real representations can be written as

$$S_{st} = \frac{1}{2} \sum_{x=even,\mu} \eta_\mu(x) \left[\bar{X}_e(x)U_\mu(x)X_o(x+\mu) - \bar{X}_e(x)U_\mu^\dagger(x-\mu)X_o(x-\mu)\right] \; . \tag{23}$$

For N staggered fermions, we see that the usual $U(N)_e \times U(N)_o$ symmetry

$$\begin{array}{llll} \chi_e \longmapsto W_e\chi_e \;\; , & \bar{\chi}_o \longmapsto \bar{\chi}_o W_e^\dagger & & W_e \in U(N) \\ \chi_o \longmapsto W_o\chi_o \;\; , & \bar{\chi}_e \longmapsto \bar{\chi}_e W_o^\dagger & & W_o \in U(N) \end{array} \tag{24}$$

is enlarged to $U(2N)$:

$$X_o \longmapsto WX_o \; , \quad \bar{X}_e \longmapsto \bar{X}_e W^\dagger \qquad W \in U(2N) \; . \tag{25}$$

Next we recast the condensate in terms of X_o and $\bar{X}_e$:

$$\begin{aligned} \bar{\chi}\chi &= \frac{1}{4}\left[\bar{\chi}_e\chi_e - \chi_e^T\bar{\chi}_e^T + \bar{\chi}_o\chi_o - \chi_o^T\bar{\chi}_o^T\right] \\ &= \frac{1}{4}\left[\bar{X}_e\begin{pmatrix} 0 & 1\!\!1 \\ \mp 1\!\!1 & 0 \end{pmatrix} S\bar{X}_e^T \mp X_o^T\begin{pmatrix} 0 & 1\!\!1 \\ \mp 1\!\!1 & 0 \end{pmatrix} S^\dagger X_o\right] \; , \end{aligned} \tag{26}$$

where upper and lower signs are for real and pseudo-real representations, respectively, and $1\!\!1$ is the $N \times N$ unit matrix. The residual symmetry left unbroken is that which leaves invariant the anti-symmetric and symmetric $2N \times 2N$ form. Thus, for real representations the chiral symmetry breaking pattern of staggered fermions is

$$U(2N) \;\rightarrow\; Sp(2N) \; , \tag{27}$$

Table 1: Chiral symmetry breaking patterns for continuum and staggered fermions.

Rep r	Fermions	Coset	RMT ens
pseudo-real	Continuum	$\mathrm{SU}(2N_f)/\mathrm{Sp}(2N_f)$	chOE
complex	Continuum	$\mathrm{SU}(N_f)$	chUE
real	Continuum	$\mathrm{SU}(2N_f)/\mathrm{SO}(2N_f)$	chSE
pseudo-real	Staggered	$\mathrm{U}(2N)/\mathrm{SO}(2N)$	chSE
complex	Staggered	$\mathrm{U}(N)$	chUE
real	Staggered	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	chOE

while for pseudo-real representations it is

$$U(2N) \ \rightarrow \ O(2N) \ . \tag{28}$$

This is exactly the opposite of the continuum theory. In table 1 we summarize the results for both continuum and staggered fermions.

It is quite troubling that the patterns of spontaneous chiral symmetry breaking for staggered fermions in all pseudo-real and real representations are entirely different from those of continuum fermions. This is not a minor issue, but concerns the whole spectrum of (pseudo-)Goldstone bosons away from the continuum: at finite lattice spacing it is completely wrong. How can the correct Goldstone spectrum possibly be recovered in the continuum limit? Let us first do a simple counting of massless degrees of freedom. In the case of real representation staggered fermions the coset of chiral symmetry breaking is $\mathrm{U}(2N)/\mathrm{Sp}(2N)$. The dimension of this coset is $N(2N-1)$, and there are as many Goldstone bosons away from the continuum. For pseudo-real staggered fermions the coset is $\mathrm{U}(2N)/\mathrm{O}(2N)$, the dimension of which equals $N(2N+1)$. There are thus *more* Goldstone bosons in this case. As the continuum limit is reached, these two cosets should get interchanged. While it is fairly easy to imagine how more massless states can appear in the continuum limit (away from the continuum these states could have order-a artifacts that disappear in the continuum limit), it is difficult to imagine how strictly massless modes at any finite lattice spacing can become *massive* in the continuum limit. We find that the only resolution to this paradox is to assume that the interchange of symmetry breaking classes occurs simultaneously with the well-known enhancement of flavor symmetry: each staggered flavor becomes 4 flavors of continuum fermions. In this way the number of strictly massless states at finite lattice spacing will always be smaller than that expected in the continuum[1], and all that happens is that nearly massless states become massless in the continuum limit. Of course we have no proof of this, and it is clearly also intimately tied up also with the anomaly and the way the proper flavor singlet remains massive in the continuum limit. In any case it shows that we are extremely lucky that QCD is a theory of complex representation fermions, for which the issue of wrong Goldstone manifolds for staggered fermions is not relevant.

[1]The needed inequality based on a quadrupling of flavor degrees of freedom in the continuum limit is $N(2N+1) < 4N(8N-1)-1$, which is satisfied for all positive integers N.

Table 2: Details of the gauge field ensembles, including gauge group, β value, volume, number of configurations analyzed, representation considered (for gauge group SU(2) we label representations by isospin j), predicted Random Matrix Theory ensemble and value of the condensate Σ obtained as explained in the text.

Group	rep	RMT ens	β	V	N_{cfg}	Σ
SU(2)	3/2	chSE	2.2	4^4	4623	1.234(60)
SU(3)	6	chUE	5.1	4^4	8000	3.569(18)
SU(3)	6	chUE	5.1	6^4	4777	3.609(29)
SU(4)	6	chOE	10.3	4^4	9000	2.361(22)
SU(4)	6	chOE	10.3	6^4	5840	2.468(24)
SU(4)	4	chUE	10.3	4^4	9000	1.046(9)

4 Numerical Results

Let us now turn to the results of our numerical simulations. Some details about the ensembles are listed in Table 2. All ensembles are pure gauge, generated with a mixture of overrelaxation and heatbath sweeps, both working on the various SU(2) subgroups of the gauge group. The ensembles were generated in the fundamental representation of the gauge group with the standard Wilson action. For each of the three Random Matrix Theory ensemble, and hence chiral symmetry breaking patterns, we have chosen one example with a larger representation of the gauge group than the fundamental one.

We computed the low-lying part of the spectrum of the staggered Dirac operator in the desired representation as described in Ref. [28]. If chiral symmetry breaking occurs, the rescaled eigenvalues $\zeta = V\Sigma\lambda$ should have the microscopic spectral density eq. (3) corresponding to the appropriate chiral Random Matrix Theory ensemble. Similarly, the rescaled lowest eigenvalue should be distributed according to the distribution $P_{\min}(\zeta)$ of the same chiral Random Matrix Theory ensemble. Of course, to make the comparison, one needs to know the value Σ of the chiral condensate. If the distribution of the lowest eigenvalue agrees, in shape, with the predicted distribution, then Σ can be obtained from the rescaling that gives the best quantitative agreement between measured and theoretically predicted distribution. Alternatively, and somewhat simpler, one can obtain Σ from the average of the lowest eigenvalue, $\langle\lambda_{\min}\rangle$ and the theoretical mean value $\overline{\zeta}$

$$\overline{\zeta} = \int_0^\infty \zeta P_{\min}(\zeta) d\zeta \tag{29}$$

as $\Sigma = \overline{\zeta}/(V\langle\lambda_{\min}\rangle)$. The values obtained in this way are also given in Table 2. Some of the very earliest studies of chiral symmetry breaking and chiral symmetry restoration at finite temperature [29] also studied some of these exotic quark representations, but without aiming at answering the present questions.

One important issue regarding staggered fermions is the comparison with analytical results in sectors of fixed topological charge ν. For staggered fermions it is normally argued that at present-day gauge couplings these lattice fermions are insensitive to quantum gauge field configurations of non-trivial topological charge. Indeed, even if one restricts oneself to gauge field sectors that clearly carry non-zero topological charge, one finds that staggered fermions behave as if the charge ν were zero [28]. Consistent with this is the observation that the fermion zero modes that by the index theorem are present for continuum fermions in non-trivial sectors of topological charge appear completely mixed up with the other small Dirac operator eigenvalues [28]. Actually, there is an alternative way of understanding these results [30]. Note that we found that all cosets of spontaneous chiral symmetry breaking for staggered fermions differ by one additional U(1) factor from the cosets of the continuum. This additional U(1) factor plays a crucial role in determining the distributions of the smallest Dirac operator eigenvalues: to leading order in the finite-volume chiral Lagrangian expansion the integration over a coset enlarged by an additional U(1) factor is simply *equivalent* to a projection onto the $\nu = 0$ sector of the theory without this U(1) factor. This is easily illustrated. Consider, for example, the universality class of $\beta = 2$, for which the smallest Dirac operator eigenvalues of continuum fermions are determined by the zero-momentum modes of the following effective partition function [7]:

$$\mathcal{Z} \; = \; \int_{SU(N_f)} dU \;\exp\left[mV\Sigma \mathrm{Re}\; \mathrm{Tr}\left(e^{i\theta/N_f}\mathcal{M}U^\dagger\right)\right] \;, \tag{30}$$

where θ is the vacuum angle, and $\mathcal{M}$ is the quark mass matrix. Projection onto a sector of topological charge $\nu = 0$ yields

$$\mathcal{Z}_0 \; = \; \int_{U(N_f)} dU \;\exp\left[mV\Sigma \mathrm{Re}\; \mathrm{Tr}\left(\mathcal{M}U^\dagger\right)\right] \;, \tag{31}$$

which is completely equivalent to the partition function based on the coset $\mathrm{U}(N_f)\times\mathrm{U}(N_f)/\mathrm{U}(N_f)$ at vanishing vacuum angle. We learn that the additional U(1) factor of staggered fermions is entirely equivalent to a projection down on the $\nu = 0$ sector of the theory without this U(1) factor, at $\theta = 0$. The two other symmetry breaking patterns of staggered fermions, having also such additional U(1) factors in their cosets (see table 1 for a summary), are completely similar in this respect. Therefore, independently of the more empirical observations of ref. [28], we simply *should* compare with the predictions of the $\nu = 0$ sectors of the theories without these additional U(1) symmetries. It should also be remarked that this equivalence between the predictions for staggered fermions and those of theories without the U(1) symmetries for $\nu = 0$ is only valid to leading order in the finite-volume chiral expansion [30]. Already at the next-to-leading order differences show up, but we do not have sufficient statistics to probe these small deviations here.

The comparison of the measured and predicted distributions (recalling the remarks above) is shown in Figs. 1 to 5. For the smallest eigenvalue distributions we should thus compare with the $\nu = 0$ predictions [21, 22]

$$\begin{aligned}
P_{\mathrm{min}}(\zeta) \;&=\; \frac{\zeta+2}{4}e^{-\zeta/2-\zeta^2/8} \;, \qquad \beta = 1\\
P_{\mathrm{min}}(\zeta) \;&=\; \frac{\zeta}{2}e^{-\zeta^2/4} \;, \qquad \beta = 2\\
P_{\mathrm{min}}(\zeta) \;&=\; \sqrt{\frac{\pi}{2}}\zeta^{3/2}I_{3/2}(\zeta)e^{-\zeta^2/2} \;, \qquad \beta = 4 \;.
\end{aligned} \tag{32}$$

These eigenvalue distributions yield the following mean values (see eq. (29)) $\bar{\zeta}$: 1.311.. (for $\beta = 1$), 1.772.. (for $\beta = 2$), and 2.066.. (for $\beta = 4$).

Similarly, for the microscopic spectral densities we should compare with [9, 14, 31, 17, 18]

$$\rho_s(\zeta) \;=\; \frac{\zeta}{2}\left(J_0(\zeta)^2 + J_1(\zeta)^2\right) - \frac{1}{2}J_0(\zeta)\left(\int_0^\zeta dt\; J_0(t) - 1\right) \;, \quad \beta = 1$$

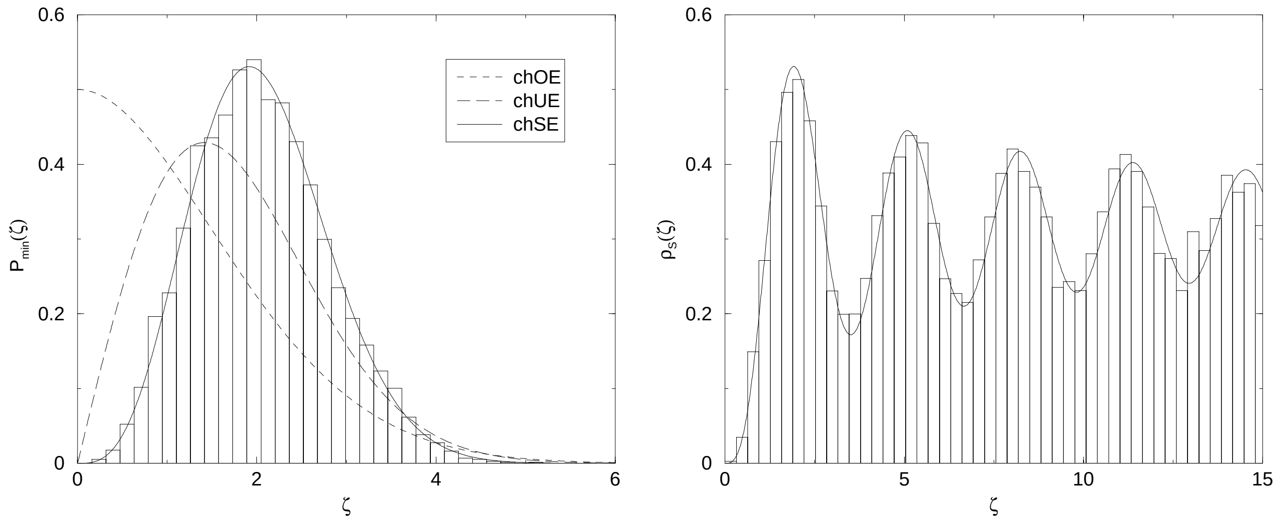

Figure 1: The left panel shows the microscopic distribution of the first eigenvalue of the SU(2) ensemble in the $j = 3/2$ representation, which is pseudo-real. The analytic predictions for the three ensembles chOE, chUE and chSE are included. The data clearly follow the curve for the (chiral) symplectic ensemble, as expected. In the right panel the beginning of the full microscopic spectral density of the same ensemble is displayed. The space-time volume used is $V = 4^4$.

$$\begin{aligned} \rho_s(\zeta) &= \frac{\zeta}{2}\left(J_0(\zeta)^2 + J_1(\zeta)^2\right) \quad , \quad \beta = 2 \\ \rho_s(\zeta) &= \zeta\left(J_0(2\zeta)^2 + J_1(2\zeta)^2\right) - \frac{1}{2}J_0(2\zeta)\int_0^{2\zeta} dt\ J_0(t) \quad , \quad \beta = 4 \end{aligned} \tag{33}$$

for the three different universality classes indicated. In all the figures we see beautiful agreement between the measured distributions and the theoretical predictions.

In two cases, for the 6-representations of SU(3) and SU(4), we considered two different volumes, in order to also illustrate the finite-size scaling that allows us to infer that indeed chiral symmetry is spontaneously broken. One quantitative measure of this is that the extracted value for the parameter Σ (which should be identified with the infinite-volume chiral condensate) should be independent of the volume. As can be seen in Table 2 the agreement between the values obtained on the different lattice sizes is very satisfactory. Altogether we have thus not only confirmed the theoretical prediction for the patterns of chiral symmetry breaking, but also shown that chiral symmetry breaking does indeed occur. The agreement is only possible if the average spacing between eigenvalues near the origin goes like $1/(V\rho(0))$ and therefore, in the infinite volume limit, a non-zero density $\rho(0)$ is built up.

Since this is the first study of the staggered eigenvalue distributions for gauge group SU(4) we decided to also look at the fundamental representation for which the corresponding Random Matrix Theory ensemble should be chiral unitary. As seen in Fig. 6 the agreement between measured distributions and the theoretical prediction is very nice here as well.

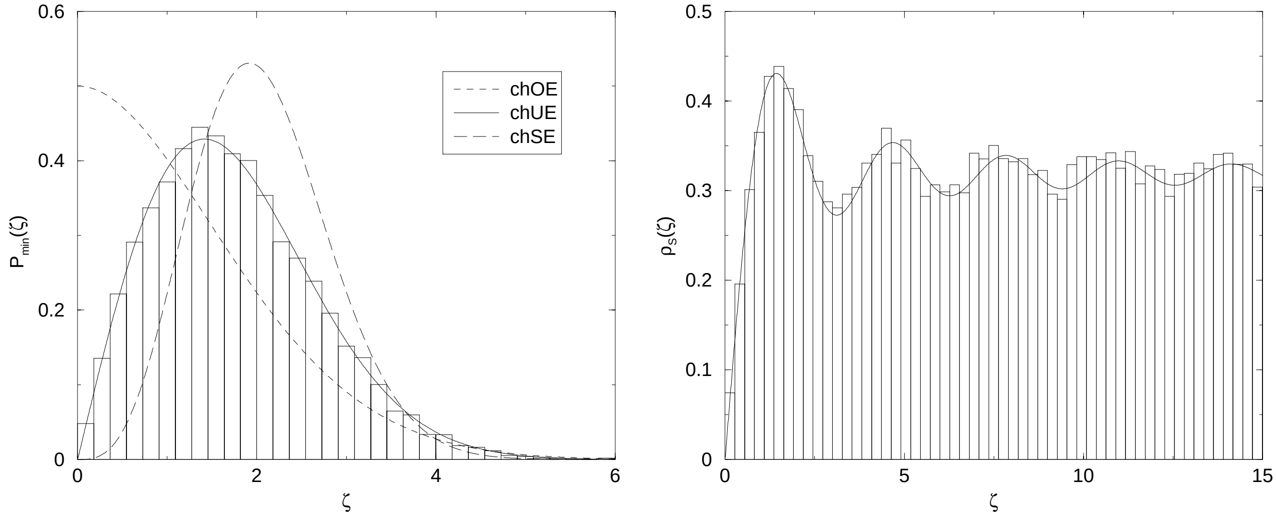

Figure 2: The microscopic distribution of the first eigenvalue (left) and the spectral density (right) for the SU(3) ensemble in the 6-representation (which is complex), from a 4^4 lattice. The data clearly follow the prediction for the (chiral) unitary ensemble.

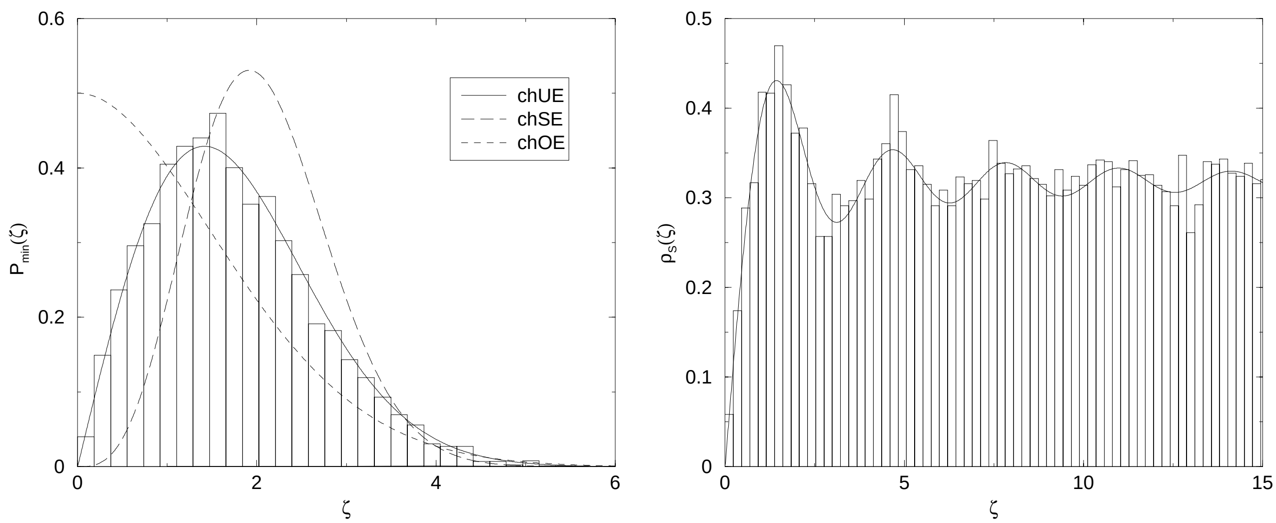

Figure 3: Same as Fig. 2 but for a 6^4 lattice. The fact that data are fit with the same parameter Σ shows that the eigenvalues indeed scale with the volume as required for spontaneous chiral symmetry breaking.

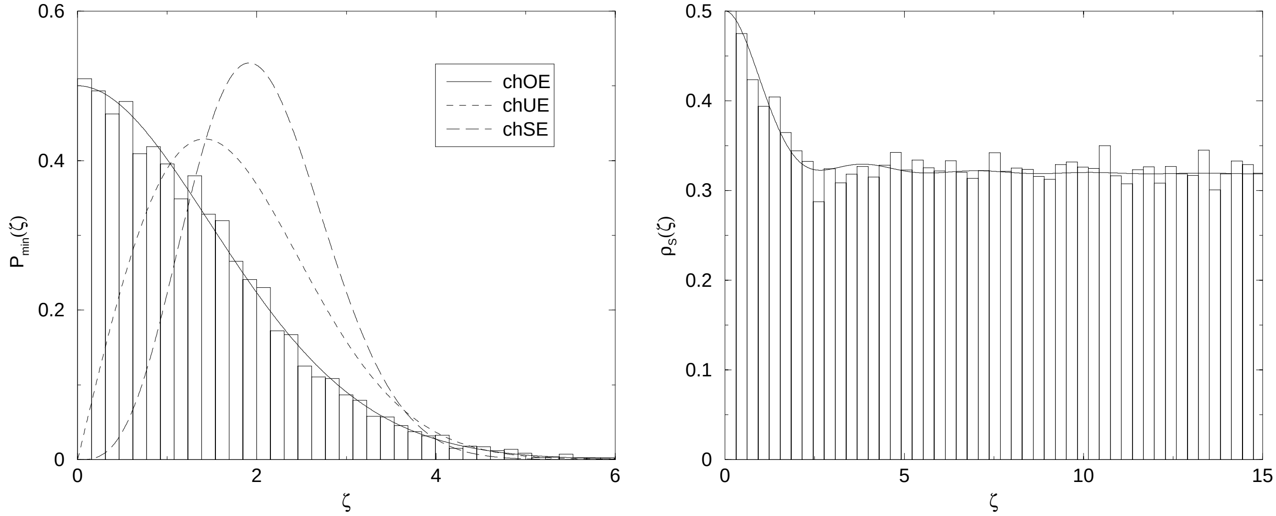

Figure 4: The microscopic distribution of the first eigenvalue (left) and the spectral density (right) for the SU(4) ensemble in the anti-symmetric 6-representation (which is real), from a 4^4 lattice. The data clearly follow the prediction for the (chiral) othogonal ensemble.

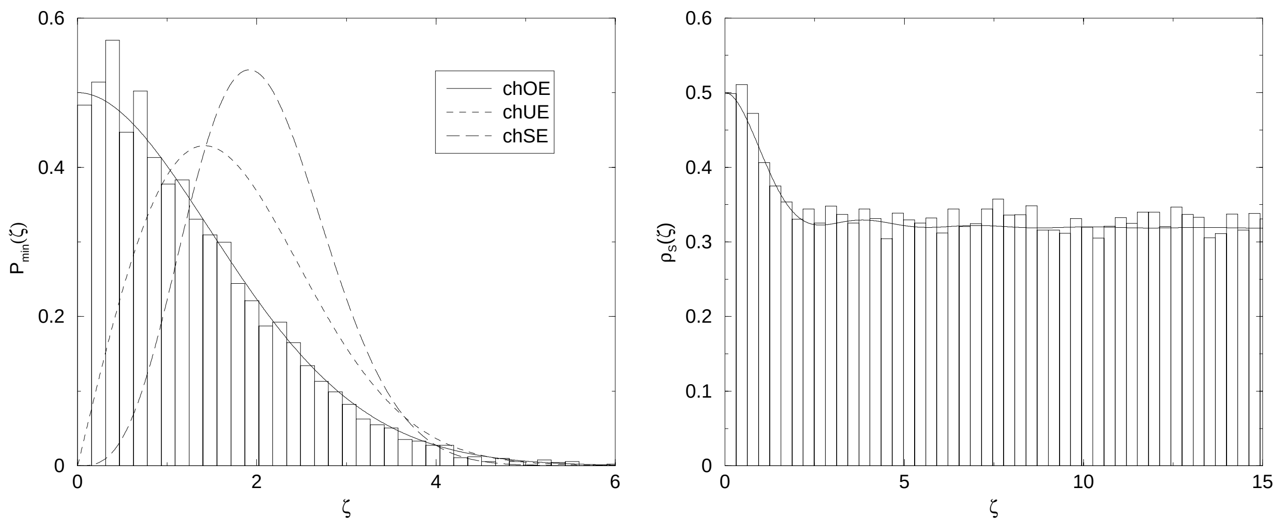

Figure 5: Same as Fig. 4 but for a 6^4 lattice.

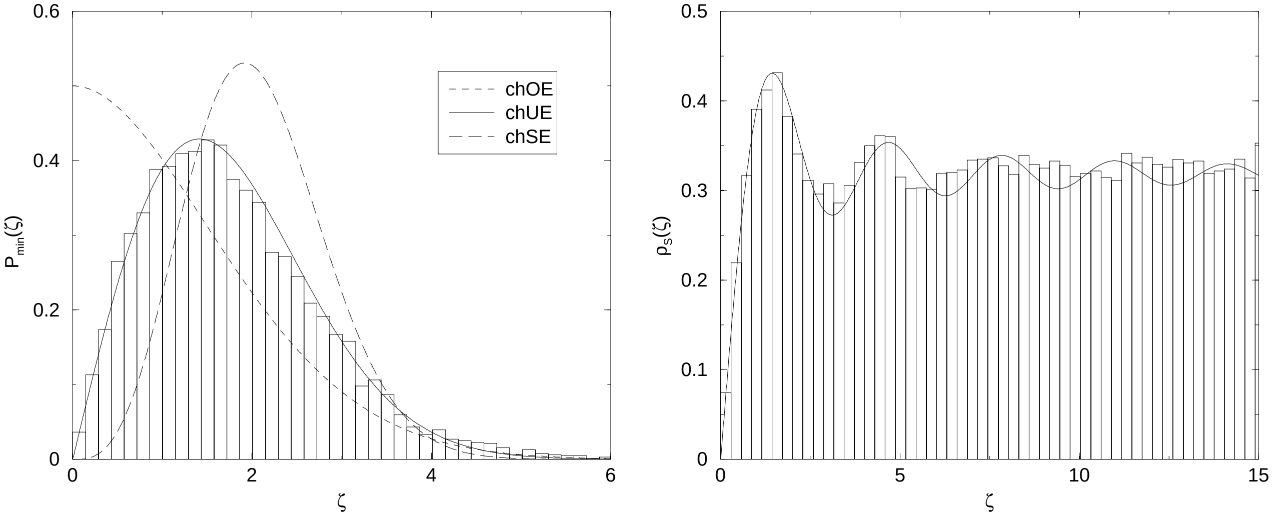

Figure 6: The microscopic distribution of the first eigenvalue (left) and the spectral density (right) for the SU(4) ensemble in the fundamental representation (which is complex), from a 4^4 lattice. The data clearly follow the prediction for the (chiral) unitary ensemble.

5 Conclusions

We have investigated the idea that there are only three distinct classes of spontaneous chiral symmetry breaking of vectorlike gauge theories in four dimensions. A particular issue concerns the lattice regularization of the fermions. It has been known for quite a while that staggered fermions in the fundamental representation of SU(2) have a spontaneous symmetry breaking pattern that falls in a wrong category as compared with continuum fermions, and it has likewise been known that staggered fermions in the adjoint representation of any SU(N) gauge group have a pattern of symmetry breaking that also falls in a wrong category compared with continuum fermions. Here we have generalized this argument to any irreducible representation r of any simple gauge group $\mathcal{G}$. It turns out that for all real and for all pseudo-real representations the symmetry breaking classes for staggered fermions are simply swapped as compared with continuum fermions. We have given some suggestions as to how the correct symmetry breaking classes may be recovered in the continuum limit.

We hope to have given convincing evidence that an extremely efficient tool for determining whether spontaneous chiral symmetry breaking occurs, and for simultaneously finding the universality class to which it belongs, is provided by the distributions of the smallest Dirac operator eigenvalues. As a by-product one obtains not only the answer to these two questions, but also an accurate determination of Σ, the infinite-volume chiral condensate. The method here is essentially a finite-size scaling analysis, where in the case of Dirac operator eigenvalues one actually knows the exact analytical form of the scaling function. By going one step further, and looking carefully at the corrections to finite-volume scaling, one can even extract one more parameter, the pion decay constant F, from just these eigenvalue distributions [30].

To illustrate the technique, we have performed Monte Carlo simulations of exotic representations of SU(2), SU(3) and SU(4) gauge groups with staggered fermions. In all cases we have found complete agreement with the expected patterns of chiral symmetry breaking. This, taken together with the four other known cases in the literature [5, 6], puts the conjecture of symmetry breaking classes on a quite solid footing.

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