

N=1 Mirror Symmetry and Open/Closed String Duality

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Abstract

We show that the exact $\mathcal{N} = 1$ superpotential of a class of 4d string compactifications is computed by the closed topological string compactified to two dimensions. A relation to the open topological string is used to define a special geometry for $\mathcal{N} = 1$ mirror symmetry. Flat coordinates, an $\mathcal{N} = 1$ mirror map for chiral multiplets and the exact instanton corrected superpotential are obtained from the periods of a system of differential equations. The result points to a new class of open/closed string dualities which map individual string world-sheets with boundary to ones without. It predicts an mathematically unexpected coincidence of the closed string Gromov–Witten invariants of one Calabi–Yau geometry with the open string invariants of the dual Calabi–Yau .

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1. Introduction

It has been known that the exact non-perturbative holomorphic prepotential that determines the effective action of an $\mathcal{N} = 2$ effective field theory, can be determined from the periods of a geometric object, both in the pure field theory [1] and in the low energy string theory [2] context. The purpose of this note is to show that a similar statement is true for the holomorphic superpotential

$$\int d^4x d^2\theta W(\Phi), \tag{1.1}$$

for a class of $\mathcal{N} = 1$ effective string theories. The argument will involve a new relation between certain open and closed string backgrounds which relates the superpotential (1.1) to an amplitude of the closed string compactified on a $\hat{c} = 4$ background to two dimensions.

The $\mathcal{N} = 1$ supersymmetric situation that we consider arises in a type II Calabi–Yau compactification with D-branes wrapped on cycles in the interior manifold and filling space-time. There is a closely related open topological string theory [3] which has been used in [4]¹ to propose an extension of mirror symmetry to include D-branes. A vital break-through in this direction has been made in the paper by Aganagic and Vafa [6] for a class of non-compact D-brane geometries. The exact instanton correct superpotential has been obtained by a combination of closed string mirror symmetry and open string methods [6]; see also [7] for a further discussion and applications and [8] - [13] for other related work in this direction.

We will follow a different route by observing that the connection to the $\hat{c} = 4$ topological closed string predicts a certain geometry of the holomorphic F-terms inherited from the 2d amplitudes. This $\mathcal{N} = 1$ *special geometry* leads to a definition of mirror symmetry for the chiral multiplets which is completely analogous to $\mathcal{N} = 2$ closed string mirror symmetry. In particular we derive a system of differential equations for the special geometry that treats the chiral multiplets from the open and closed string sector on a complete equal footing. An $\mathcal{N} = 1$ mirror map for the chiral multiplets and the exact holomorphic $\mathcal{N} = 1$ function $W(\Phi)$ are then defined by the solutions, or “periods”, of this system, very similar to the case of closed string mirror symmetry.

We will first recover the results of [6] on the superpotential from this definition of $\mathcal{N} = 1$ mirror symmetry. To appreciate that this is not merely a technicality, note firstly that by treating the closed and open string moduli on the same footing,

¹ See also [5].

we obtain a complete, global description of the holomorphic $\mathcal{N} = 1$ moduli space², including phase transitions, singularities and the associated analytic continuations and monodromies. Also this approach to $\mathcal{N} = 1$ mirror symmetry appears to be a framework for generalizations to many D-branes and higher genera amplitudes.

A central argument will be the new relation between open and closed string backgrounds which identifies certain topological amplitudes of the open string compactification with central charge $\hat{c} = 3$ and a “dual” closed string with $\hat{c} = 4$. These topological amplitudes are related to physical space-time amplitudes in the type II string on a Calabi–Yau 3-fold with D-branes and a type II string on a Calabi–Yau 4-fold without branes but with certain fluxes, respectively. We will separate the case of disc and sphere world-sheet topologies, where a relation of the above kind will be verified, from the case of general world-sheet topologies which will remain at the level of a speculation. It will be interesting study the conjectural open/closed string duality in other world-sheet topologies, both from the mathematics and physics point of view. If true, an infinite number of holomorphic four-dimensional $\mathcal{N} = 1$ amplitudes would be computed by the 2d string theory. Some evidence in favor of a true open/closed string duality will be given by studying M-theory on a nine-dimensional manifold.

A mathematically quite surprising connection implied by the genus zero³ duality is one between the moduli spaces of discs in the D-brane geometry on a Calabi–Yau 3-fold Y , and the moduli spaces of spheres in the dual Calabi–Yau 4-fold X . Specifically the closed string Gromov–Witten invariants of X , which count the appropriately defined number of holomorphic spheres [14], and the integral open string invariants [15] of the D-brane geometry (L, Y) , which count the number of discs in Y with boundary on the D-brane, coincide for a dual pair! In other words the open/closed string relation maps *individual* open string world-sheets with boundary to those without⁴.

In this paper, we will discuss the basic ideas and arguments outlined above, mostly for the type IIB D-branes. Other aspects, such as the global structure of the $\mathcal{N} = 1$ moduli space, phase transitions, the type IIA (D-brane) geometry, more explicit calculations and various generalizations will be discussed in [16].

The organization of this paper is as follows. In sect. 2 we formulate the precise proposal for the relation between the topological amplitudes at genus zero of two string backgrounds with and without branes. We explain why the $\hat{c} = 4$ closed topological string computes the 4d $\mathcal{N} = 1$ superpotential (1.1). In sect. 3 we identify pairs of closed

² That is moduli modulo D-terms.

³ More precisely open $g = 0, h = 1$ vs. closed $g = 0$.

⁴ This is similar to a duality described in [13].

and open string backgrounds which are supposedly related by a duality. We verify their equivalence at the level of the genus zero amplitudes related to the superpotential (a calculation of period integrals is relegated to app. A for readability.). In sect. 4 we use the established ($g = 0$) connection between the open and closed topological string theories to define $\mathcal{N} = 1$ mirror symmetry for scalars of chiral multiplets in a way very similar to closed string, $\mathcal{N} = 2$ mirror symmetry. The central element is a system of differential equations, whose solutions, or periods, define the flat coordinates, the $\mathcal{N} = 1$ mirror map and the 4d superpotential. The approach is illustrated with a simple example, including an explicit demonstration of the agreement of the closed string (sphere) and open string (disc) Gromov–Witten invariants in the dual backgrounds. In sect. 5 we provide some evidence for an extension of the open/closed string relation to a true string duality, by studying the M-theory limit of the type IIA compactifications.

2. Topological string amplitudes in type II Calabi–Yau compactifications

In this section we review the space-time interpretation of various topological string theories on a Calabi–Yau background and formulate the precise proposal for a coincidence of certain genus zero topological amplitudes of an open string theory on a Calabi–Yau 3-fold and a conjecturally dual closed string on a Calabi–Yau of one dimension higher.

Topological closed $\mathcal{N} = 2$ amplitudes and 3-cycle integrals

The $\mathcal{N} = 2$ closed topological string [17] on a Calabi–Yau 3-fold has non-zero partition function \mathcal{F}_g for all world-sheet genera g . At $g = 0$ the topological string computes the holomorphic prepotential of the four-dimensional effective $\mathcal{N} = 2$ string theory. For B-twist, the result may be expressed by the geometric period integrals [18] of the holomorphic 3-form over special Lagrangian 3-cycles in the Calabi–Yau Y^*

$$\Pi(z_a^{(c)}) = \int_{C_3} \Omega^{(3,0)}(Y^*), \quad C_3 \in H_3(Y^*) \quad (2.1)$$

from which \mathcal{F}_0 can be recovered by integration. As indicated the period Π depends on the complex structure moduli $z_a^{(c)}$ of Y^* only. The superscript will be used to indicate that these moduli arise in the closed string sector. Although the expression (2.1) looks completely classical, it comprises a highly intricate sum of instanton corrections. This is best seen in the equivalent A-twisted model on the mirror manifold Y , where these instantons arise from Euclidean world-sheets wrapped on holomorphic 2-spheres. Most of the complexity of the geometric expression (2.1) is hidden in the relation between

the vev's of physical scalar components $t_a^{(c)}$ and the geometric moduli $z_a^{(c)}$, which is given by a ratio of periods

$$t_a^{(c)}(z_b^{(c)}) = \Pi_a(z_b^{(c)})/\Pi_0(z_b^{(c)}), \quad (2.2)$$

and is called the mirror map. Geometrically, the real scalars $\text{Im} t_a^{(c)}$ measure the size of holomorphic spheres in Y on which the type IIA world-sheet instantons wrap; as the BPS action of these instantons is proportional to the area of the sphere, the instanton weight is $\sim q_a = e^{2\pi i t_a^{(c)}}$. Knowing the mirror map (2.2), one obtains from the remaining periods $\tilde{\Pi}_a$ the first derivatives of the prepotential \mathcal{F}_0 , the latter having an integral instanton expansion in the large t limit of the form [14]:

$$\tilde{\Pi}_a^{inst} = \partial_a \mathcal{F}_0 = \sum_{\vec{n}} n_a D_{\vec{n}} \sum_{m=1}^{\infty} \frac{q^{m\vec{n}}}{m^2}. \quad (2.3)$$

Here $q^{\vec{n}} = \prod_a q_a^{n_a}$ and the integral coefficients $D_{\vec{n}}$ are the Gromov–Witten invariants which count the appropriately defined number of holomorphic spheres in X .

Topological open $\mathcal{N} = 1$ amplitudes and 3-chain integrals

One may add D-branes to the previous compactification on the Calabi–Yau 3-fold such that supersymmetry is broken to $\mathcal{N} = 1$. This situation is described by the $\mathcal{N} = 1$ open topological string theory [3]. The partition function has an expansion $\mathcal{F} = \mathcal{F}_{g,h}$ in world-sheets of genus g and with h holes. The holomorphic functions $\mathcal{F}_{g,h}$ of the A-twisted model are related to $d = 4$ $\mathcal{N} = 1$ superpotential terms on the world-volume of a D6-brane wrapped on a special Lagrangian 3-cycle L in Y [19][15][20]:

$$h \int d^4x d^2\theta \mathcal{F}_{g,h}(\mathcal{W}^2)^g (\mathcal{F}^2)^{h-1}, \quad (2.4)$$

where \mathcal{W} is the gravitational chiral superfield and \mathcal{F} is the chiral superfield for the gauge supermultiplet on the D-brane. In particular the partition function $\mathcal{F}_{0,1}$ computes the superpotential (1.1). There is a formula similar to (2.1) that describes the same superpotential in terms of the mirror D-brane geometry which is, for appropriate choice of L , a D5-brane wrapped on a 2-cycle C in the mirror manifold Y^* [6]⁵

$$W = \int_{D_3} \Omega = \int_{\partial D} \omega = W(z_a^{(c)}, z_i^{(o)}). \quad (2.5)$$

Here D_3 is a 3-chain in Y^* with C as a boundary component and Ω is the holomorphic (3,0) form on Y^* , which may locally be written as $\Omega = d\omega$. Due to the B-twist, $W(C)$

⁵ See also [21][10].

depends only on the complex structure moduli $z_a^{(c)}$ of Y and the moduli $z_i^{(o)}$ of the 2-cycle C . Note that the former are from the closed string sector while the latter are moduli of the open string sector.

Again, the geometric expression W describes highly delicate instanton physics on the brane world-volume, which is best seen in terms of *disc* world-sheet instantons in the type IIA geometry (Y, L) . Knowing the relation between the scalar vev's $(t_a^{(c)}; t_i^{(o)})$ of the chiral multiplets and the geometric moduli $(z_a^{(c)}; z_i^{(o)})$, the instanton expansion may be extracted from the geometric integral (2.5), which is predicted [15] to have an integral large t expansion of the form:

$$W = \sum_{\vec{k}, \vec{m}} d_{\vec{k}, \vec{m}} \sum_n \frac{\vec{q}_{(c)}^{n\vec{k}} \vec{q}_{(o)}^{n\vec{m}}}{n^2}, \quad (2.6)$$

where $q_{(c)} = e^{2\pi i t_a^{(c)}}$ ($q_{(o)} = e^{2\pi i t_i^{(o)}}$) are the exponentials of the (complexified) volumes of holomorphic spheres (holomorphic discs) in a CY 3-fold Y . Moreover, the expansion coefficients $d_{\vec{k}, \vec{m}}$ are the integral open string invariants that count the appropriately defined number of discs in the class (\vec{k}, \vec{m}) . The integrality predictions of [15] have been verified by now in a large number of highly non-trivial examples [6][7][12][13][22].

Topological closed “ $\mathcal{N} = 1$ ” amplitudes

A topological closed string compactification with $\hat{c} = 4$ is defined by a Calabi–Yau 4-fold X . This is related to a 2d space-time theory with the same number of supercharges as $\mathcal{N} = 1$ in $d = 4$. The two different topological field theories can be used to study mirror symmetry for 4-folds [23][24], with the details being slightly different from the “critical” case of Calabi–Yau 3-folds due to different ghost number of the vacuum. The basic genus zero amplitude is the topological triple coupling $C_{\alpha\beta\gamma} = \langle \mathcal{O}_\alpha^{(1)} \mathcal{O}_\beta^{(1)} \mathcal{O}_\gamma^{(2)} \rangle$, where $\mathcal{O}^{(i)}$ is an operator localizing on a codimension i hypersurface in X . They are related to the periods of the holomorphic 4-form of the mirror X^*

$$\Pi_\alpha(z_\beta^{(c)}) = \int_{\gamma_\alpha} \Omega^{(4,0)}, \quad \gamma_\alpha \in H_4^\Omega(X^*), \quad (2.7)$$

where the superscript Ω denotes the sub-sector of homology 4-cycles which has a representative calibrated by the holomorphic $(4, 0)$ form. Specifically the topological triple couplings $C_{\alpha\beta\gamma}$ are the double derivatives [24]

$$C_{\alpha\beta\gamma} = \frac{\partial}{\partial t_\alpha} \frac{\partial}{\partial t_\beta} \Pi_\gamma^{\ln^2} \quad (2.8)$$

of the middle periods with leading double logarithmic behavior in the large t limit. Moreover these periods have the integral large t expansion [23][24]

$$\Pi_\gamma^{\ln^2} = P_\gamma^2(t_\alpha) + \sum_{\vec{k}} D_{\vec{k}}^\gamma \sum_n \frac{\vec{q}^{n\vec{k}}}{n^2}, \quad (2.9)$$

where P_γ^2 is a certain degree two polynomial. Note that this expansion is precisely of the same form as the $\mathcal{N} = 1$ open string expansion (2.6).

The proposal

Our proposal is that for appropriate choice of the open string D-brane geometry (Y, L) and a dual closed string geometry X , the open and closed string expansions (2.6) and (2.9), are identical. This is the same as saying that the topological amplitudes of the open string at $g = 0, h = 1$ defined on the D-brane geometry (Y, L) and the closed string at $g = 0$ on X are identical.

We will identify appropriate duals (Y, L) and X in the next section and verify the agreement of the genus zero topological amplitudes. Clearly it will be extremely interesting to see whether a similar relation exists for other world-sheet topologies and the relation at genus zero extends to a true string theory duality [16]. An M-theory argument in favor of this conjectural string duality will be given in sect. 5.

The coincidence of the $g = 0$ ($h = 1$) open/closed string amplitudes has the following *two-dimensional* space-time interpretation. Consider the D-brane geometry (Y, L) , however with a D4-brane wrapped on the 3-cycle L instead of a D6-brane. The open topological amplitudes $\mathcal{F}_{g,h}$ compute the following terms in the 2d effective world-volume [15]:

$$h \int d^4x d^2\theta \delta^2(x) F_{g,h}(t_a^{(c)}; t_i^{(o)}) (\mathcal{W}^2)^g (\mathcal{W} \cdot v)^{h-1}, \quad (2.10)$$

where the delta function localizes on the D4 world-volume and the tensor v points into the orthogonal directions. For $g = 0, h = 1$ this is the two-dimensional superpotential in a theory with 4 supercharges. Precisely the same superpotential arises in the type II Calabi–Yau 4-fold compactification with background fluxes [25]⁶:

$$W = \sum N_a \int_{\gamma_a} \Omega^{(4,0)}, \quad \gamma_a \in H_4(X^*). \quad (2.11)$$

Here N_a are integers that specify the background fluxes. Moreover we have used the type IIB language on the mirror X^* for simplicity; see [25][20] for the corresponding

⁶ See also [26][27].

type IIA superpotential on X . The superpotentials (2.10) and (2.11) are defined in the same class of 2d theories with four supercharges. The conjecture says that for the appropriate choice of Y , L and X , they are identical. Note that the flux on the 4-fold is important for the duality to work. This is very similar as for the AdS/CFT correspondence [28] and Vafa’s large N duality [20]⁷.

To be more precise, as we consider non-compact manifolds, some of the periods (2.7) are defined on non-compact cycles. The moduli dependent contribution of these periods to the superpotential (2.11) is described by the *variation* of the volume of the non-compact cycle with the complex structure of X^* . Rather than in terms of flux on the non-compact cycle, this contribution is better thought of as the response to variations of the complex structure relative to a fixed behavior at infinity, similarly as in [6][21].

Note that, once the closed topological amplitude (2.9) is identified with $\mathcal{F}_{0,1}$ by the D4-brane interpretation (2.10), it translates immediately to the holomorphic superpotential W in the *four-dimensional* $\mathcal{N} = 1$ theory, by the alternative space-time interpretation (2.4) of $F_{0,1}$ on the D6-brane world-volume. It is by this chain of arguments that the 2d closed string may be seen to compute 4d holomorphic $\mathcal{N} = 1$ space-time couplings.

It is worth stressing that the integral coefficients of the identified expansions, $d_{\vec{k}, \vec{m}}$ in (2.6) and $D_{\vec{k}}$ in (2.9), have vastly different meanings. The first counts the appropriately defined number of discs in the D-brane geometry (Y, L) , while the other the appropriately defined number spheres in X . This is the aforementioned new feature of this class of open/closed string “dualities”: world-sheets with boundaries are mapped individually to world-sheets without.

Similarly, the duality identifies the closed and open string moduli $(t_a^{(c)}; t_i^{(o)})$ of the D-brane geometry (Y, L) with only the closed string moduli $t_\alpha^{(c)}$ of X . The first are defined as the areas of spheres and discs in Y , respectively, while the latter measure only volumes of spheres in X .

⁷ In fact background fluxes in type II compactifications in Calabi–Yau *3-folds* give also rise to superpotentials in four-dimensional compactifications with a perturbative $\mathcal{N} = 1$ supersymmetry [29][30]. This is the only known other case where an infinite sum of instanton corrections to a $\mathcal{N} = 1$ superpotential can be computed systematically, by conventional $\mathcal{N} = 2$ mirror symmetry [29][30][20][31].

3. Open/closed string duals in two dimensions

While it has been argued in the previous section that the open topological amplitude $\mathcal{F}_{0,1}$ in the D-brane geometry (Y, L) can in principle coincide with the topological genus zero amplitude on X , it is the purpose of this section to identify the appropriate X for a given type IIA D6-brane configuration (Y, L) . In fact it is more convenient to use the equivalent mirror geometries and to identify a Calabi–Yau 4-fold X^* for a given type IIB D5-brane geometry (Y^*, C) ⁸.

Recall that the D-brane geometry consists of a D5-brane wrapped on the 2-cycle C of a Calabi–Yau 3-fold Y^* , with the closed string moduli for the Calabi–Yau geometry related to the 3-cycle integrals (2.1) and the open string moduli for the geometry of the D-brane related to the 3-chain integrals (2.5). An appropriate formulation of the problem is to find for a given geometry (Y^*, C) a 4-fold X^* such that there is an injective and surjective map

$$H_3(C, Y^*) \rightarrow H_4^\Omega(X^*), \quad (3.1)$$

where $H_3(C, Y^*)$ denotes the relative homology modulo boundaries on C . This asserts that all 3-cycle (2.1) and 3-chain (2.5) integrals in Y^* may be associated with an appropriate 4-cycle integral (2.7) in X^* .⁹

The triple (Y^, L, X^*) in the 2d linear sigma model*

In the following we will use the 2d linear sigma model [32] to study the relevant Calabi–Yau geometries with and without D-branes. The appropriate 4-fold X^* for the closed string compactification may be constructed from the linear sigma model for the 3-fold Y^* by a simple procedure that adds a *linear 2d superpotential in a new variable v* . Concretely we consider non-compact Calabi–Yau d -folds defined by a 2d superpotential¹⁰

$$\mathcal{W}_{D=2} = \sum_{i=0}^{d-1+h_{1,d-1}} a_i y_i, \quad \prod_i y_i^{l_i^{(a)}} = 1, \quad a = 1, \dots, h_{1,d-1}, \quad (3.2)$$

where the variables y_i take value in \mathbf{C}^* and the a_i are constants that parametrize the complex structure. Moreover the $l^{(a)}$ are $h_{1,d-1}$ linearly independent vectors with

⁸ The type IIA mirror geometries will be discussed in [16].

⁹ As alluded to earlier, the manifold X^* and some of the 4-cycles will be non-compact.

¹⁰ We use $\mathcal{W}_{D=2}$ to denote the 2d *world-sheet* superpotential of the linear sigma model. This should not be confused with the space-time superpotential W in two dimensions.

integral entries that define the specific Calabi–Yau manifold¹¹. For further background on the definition of Calabi–Yau manifolds and mirror symmetry we refer to [33][34].

By rescalings of the y_i , the superpotential $\mathcal{W}_{D=2}$ depends only on $h_{1,d-1}$ combinations of the a_i , which is the dimension of the complex structure moduli space \mathcal{M}_{CS} . A canonical choice of coordinates on this space is provided by the so-called algebraic coordinates $z_a^{(c)}$ defined as

$$z_a^{(c)} = \prod_i a_i^{l_i^{(a)}}. \quad (3.3)$$

There will in general be several proper choices of linear combinations of the $l^{(a)}$ which define good coordinates on various patches of \mathcal{M}_{CS} .

After solving for the relations in (3.2), the superpotential depends on d variables for a Calabi–Yau d -fold, $\mathcal{W}_{D=2} = \mathcal{W}_{D=2}(y_0, \dots, y_{d-1})$. An equivalent definition [34] is in terms of the hypersurface¹²

$$\mathcal{W}_{D=2}(\tilde{y}_0, \dots, \tilde{y}_{d-2}) + xz = 0, \quad \tilde{y}_i = \frac{y_i}{y_{d-1}}. \quad (3.4)$$

We are interested in a D5 brane on a 2-cycle C in the 3-fold Y^* , such that there is an $\mathcal{N} = 1$ world-volume theory on it with classically zero, but non-perturbatively (in the world-volume sense) non-zero superpotential W . For holomorphic C , the superpotential (2.5) is identically zero. A non-zero brane superpotential arises for a non-compact 2-cycle with fixed behavior at infinity [21][6]. An appropriate C in the hypersurface (3.4) is defined by

$$C : x = 0, \quad \tilde{y}_i = \tilde{y}_i(r), \quad (3.5)$$

where z is the holomorphic coordinate on C , $r = |z|$ and near infinity, the coordinates \tilde{y}_i , $i = 0, 1$ approach a fixed value \tilde{y}_i^∞ . The 4d superpotential W depends only on the values of the \tilde{y}_i at $r = 0$ which may be specified by the ratio

$$z_0^{(o)} = \tilde{y}_0/\tilde{y}_1|_{r=0}. \quad (3.6)$$

Eventually the superpotential W for the D5 brane on C evaluates to [6]

$$W(z_a^{(c)}; z_0^{(o)}) = \int_{\tilde{y}_1^\infty}^{\tilde{y}_1|_{r=0}} \ln(\tilde{y}_0) d \ln(\tilde{y}_1), \quad (3.7)$$

¹¹ The vectors $l^{(a)}$ are also the charge vectors of the type IIA gauged linear sigma model on the mirror manifold.

¹² See app. A for the derivation.

where the value of \tilde{y}_0 is fixed in terms of \tilde{y}_1 by the fact that $\mathcal{W}_{D=2}$ vanishes on C . Note that $z_0^{(o)}$ is the relevant open string modulus for the D-brane geometry. It is related to the scalar component of a chiral multiplet by a field redefinition described below.

We may finally formulate the Calabi–Yau 4-fold X^* on which the conjecturally dual closed string is compactified. Let $\mathcal{W}_{D=2}(Y^*)$ denote the 2d superpotential for the Calabi–Yau 3-fold Y^* on which the previously described D-brane geometry is defined. The 2d linear sigma model for X^* is defined by the superpotential and relation

$$\mathcal{W}_{D=2}(X^*) = \mathcal{W}_{D=2}(Y^*) + a_1 v_1 + a_2 v_2, \quad y_0 v_1 = y_1 v_2. \quad (3.8)$$

Here v_i are two new variables in \mathbf{C}^* . The $h_{13}(X^*) = h_{12}(Y^*) + 1$ charge vectors $l^{(a)}$ for X^* are

$$\begin{aligned} l^{(0)} &= (1, -1, 0, \dots; 1, -1), \\ l^{(a)} &= (\quad l_Y^{(a)} \quad ; 0, 0), \quad a = 1, \dots, h_{13} - 1, \end{aligned} \quad (3.9)$$

where the last two entries refer to the new fields v_i . The $l^{(a)}$ define h_{13} coordinates (3.3) on the complex structure moduli space $\mathcal{M}_{CS}(X^*)$. In the duality to the D-brane geometry (C, Y^*) , the complex structure moduli of X^* will be identified with the moduli of the D-brane geometry (C, Y^*) as described in Tab. 1:

Y^* (4d)		X^* (2d)	
complex structure $z_a^{(c)}$		$z_a, a>0$ complex structure	
D5-brane geometry $z_0^{(o)}$		z_0 “	(3.10)

Tab. 1: The identifications of the parameters in the effective 4d $\mathcal{N} = 1$ compactification with the parameters of the related 2d closed string compactification.

It is instructive to have an intuitive picture of the geometry of X^* . If we add a further term $\sim v^{-1}$ to the superpotential (3.8), then X^* is a fibration of Y^* over a cylinder \mathbf{C}_v^* parameterized by the variable v . There are also branch points on \mathbf{C}_v^* where the fiber degenerates. This is sketched in Fig. 1. In the limit where the coefficient of the term v^{-1} goes to zero, and we recover X^* , the cylinder becomes infinitely long.

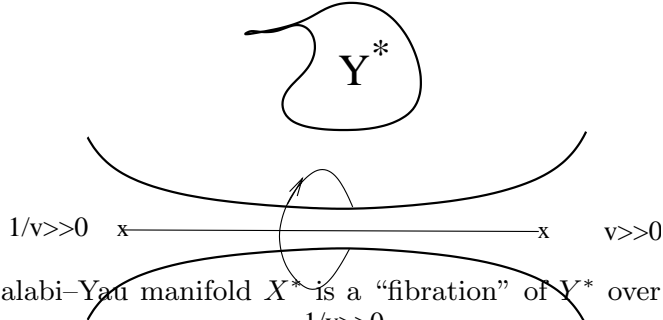


Fig. 1: The Calabi-Yau manifold X^* is a “fibration” of Y^* over a cylinder with branch points, in the limit where the cylinder becomes infinitely long.

There are two types of 4-cycles¹³ in X^* , made from 3-cycles in Y^* and a closed cycle on \mathbf{C}_v^* . The ones of the first type are of topology $S^1 \times C_3^{Y^*}$ and arise from pulling a 3-cycle in Y^* around the periodic direction of the cylinder. The period integrals of the holomorphic 4-form evaluated on such a cycle reduce to the periods of the holomorphic 3-form in Y^* , times an irrelevant constant. The other type of cycles is obtained by integrating between branch points on \mathbf{C}_v^* where a 3-cycle in the fiber vanishes. The topology of such a cycle is in general S^4 . We show in app. A that the period integral over this type of 4-cycle that is well-defined in the limit of the very long cylinder, agrees with the *3-chain* integral (3.7) in Y^* . This establishes the relation (3.1) and justifies the identifications made in Tab. 1.

4. Special geometry for $\mathcal{N} = 1$ mirror symmetry

The 2d topological field theory amplitudes define a certain special geometry with a flat connection for the vacuum bundle in special coordinates [19][23]. This is a generalization of the well-known case of $\mathcal{N} = 2$ special geometry which follows from the so-called tt^* equations [19].

Although there is no known physical duality in four dimensions, it follows from the arguments in sect. 2 that the $\mathcal{N} = 1$ 4d F-term inherits this structure from the

¹³ See also [2] for a very similar discussion and [35] for closely related Calabi-Yau geometries and a connection to the Jacobi map.

2d amplitudes. This “ $\mathcal{N} = 1$ special geometry” leads to a definition of 4d $\mathcal{N} = 1$ open string mirror symmetry which is completely analogous to that of $\mathcal{N} = 2$ closed string mirror symmetry. In the following derive a system of differential equations for the $\mathcal{N} = 1$ special geometry and use it to describe 4d $\mathcal{N} = 1$ mirror symmetry of the chiral multiplets in terms of the solutions, or “periods” of this system.

Note that the same differential equations describe also the true periods of the two-dimensional type IIA compactified on the 4-fold X , although this theory is not physically equivalent (as it is defined in 2d). The system of differential equations satisfied by the periods $\Pi(X^*)$ of the Calabi–Yau 4-fold geometry for the 2d compactification is of a generalized hyper-geometric, so-called GKZ type [36]¹⁴. This has been discussed in the study of mirror symmetry for 4-folds [23][24].

It is important to note that the only data on which the special geometry depends, are the charge vectors $l^{(a)}$ in (3.9). Once the relation between the D-brane geometry (C, Y^*) and X^* described in the last section is understood, one may immediately define the GKZ system for $\mathcal{N} = 1$ mirror symmetry from the classical D-brane geometry (C, Y^*) in the following way. Let C be the D5-brane on Y with the asymptotic behavior

$$C : x = 0, \quad y_i = z_0 y_j \text{ at } r = 0, \quad (4.1)$$

such that z_0 is small and $z_0 \rightarrow 0$ describes the classical limit on the D-brane world-volume. This is simply the appropriate definition of the D-brane in the coordinates adapted to the relevant patch in Y^* , where the D-brane is located. The D5-brane with the above asymptotic behavior defines the following basis of charge vectors for $\mathcal{N} = 1$ mirror symmetry:

$$l^{(a)} = (\quad \quad \quad l_{Y^*}^{(a)} \quad \quad \quad ; 0 \quad 0), \quad a = 1, \dots, h_{12}$$

$$l^{(0)} = (0, \quad \dots, 0, 1, 0, \dots \quad \dots, 0, -1, 0, \dots \quad \dots, 0; 1 \quad -1). \quad (4.2)$$

i-th position *j-th position*

The rank of the set of charge vectors of the above form equals the rank of the lattice of all possible supersymmetric D5-brane charges on Y^* . In other words any D5-brane which is mirror to a D6-brane on a special Lagrangian 3-cycle of topology $\mathbf{C} \times S^1$ may be described by the charge vectors (4.2) for appropriate i and j . The good local coordinates (3.3) are then defined by a linear combination of the $l^{(\alpha)}$ within a

¹⁴ See also [37] for a nice discussion of the similar GKZ systems of Calabi–Yau 3-folds.

given basis (4.2). The precise linear combination is obtained by finding the minimal generators of the Kähler cone as will be explained in more detail in [16]; see also the example below for a brief explanation.

For ease of notation we will then use the form (3.9) corresponding to $i = 0, j = 1$ in the following; the general case is recovered by a trivial redefinition of coordinates. The searched for differential equations may be straightforwardly obtained from the LG definition of the periods on X [38]

$$\Pi(X^*) \sim \int \prod_0^{d-1} \frac{dy_i}{y_i} e^{-\mathcal{W}_{D=2}(X^*)}. \quad (4.3)$$

From (3.2) it is easy to see that they satisfy the differential equations $\mathcal{D}_\alpha \Pi = 0$ with

$$\mathcal{D}_\alpha = \prod_{l_i^{(\alpha)} > 0} \left(\frac{\partial}{\partial a_i} \right)^{l_i^{(\alpha)}} - \prod_{l_i^{(\alpha)} < 0} \left(\frac{\partial}{\partial a_i} \right)^{-l_i^{(\alpha)}}, \quad (4.4)$$

where $l^{(\alpha)}$, $\alpha = 0, \dots, h_{12}$, are the charge vectors. The same operators rewritten in terms of logarithmic derivatives $\theta_\alpha = z_\alpha \frac{\partial}{\partial z_\alpha}$ of the proper coordinates (3.3) are

$$\mathcal{D}_\alpha = \prod_{l_i^{(\alpha)} > 0} \prod_{j=0}^{l_i^{(\alpha)}-1} \left(\sum_\beta l_i^{(\beta)} \theta_\beta - j \right) - z_\alpha \prod_{l_i^{(\alpha)} < 0} \prod_{j=0}^{-l_i^{(\alpha)}-1} \left(\sum_\beta l_i^{(\beta)} \theta_\beta - j \right). \quad (4.5)$$

From the charge vector $l^{(0)}$ in (3.9) we obtain the following operator \mathcal{D}_0 for the open string modulus $z_0^{(0)}$:

$$\mathcal{D}_0 = (1 - z_0) \theta_0^2 + \sum_{b>0} (l_b^{(0)} + z_0 l_b^{(1)}) \theta_b \theta_0. \quad (4.6)$$

In particular \mathcal{D}_0 consists of only terms of second degree.

The classical limit, which defines the instanton expansion, corresponds to small z_α ; in particular the leading behavior of the relation between the physical scalar fields t_α in the chiral multiplets and the geometric parameters z_α of the 2d world-sheet superpotential is $t_\alpha \sim \frac{1}{2\pi i} \ln(z_\alpha)$. The classical limit corresponds to a so-called point of maximally unipotent monodromy¹⁵, which also implies that the leading behavior of the solutions of the GKZ system expanded around the classical point $z_\alpha = 0$ are in agreement with the physical expectations. Concretely there is one holomorphic solution, which is just a constant

$$\Pi^{(0)} = 1,$$

¹⁵ See e.g. [39].

as follows from the fact that the \mathcal{D}_α do not have a constant piece. This is a consequence of the non-compactness of X^* .

The next set of solutions have a single logarithmic leading behavior

$$\Pi_\alpha^1 = \frac{1}{2\pi i} \ln(z_\alpha) + S_\alpha(z_\beta), \quad \alpha, \beta = 0, \dots, h_{12}, \quad (4.7)$$

where S_α are a power series in the geometric parameters z_β . These periods specify the instanton corrections to the classical relations $t_\alpha \sim \frac{1}{2\pi i} \ln(z_\alpha)$; indeed $z_\alpha = e^{2\pi i t_\alpha} + \dots$ is the weight of the string world-sheet instanton wrapped on a holomorphic 2-cycle with area $\text{Im } t_\alpha$. The above periods define the flat coordinates and the

$$\underline{N = 1 \text{ mirror map}}: \quad t_\alpha(z_0^{(o)}; z_a^{(c)}) = \frac{\Pi_\alpha^1(z_0^{(o)}; z_a^{(c)})}{\Pi^0(z_0^{(o)}; z_a^{(c)})}, \quad (4.8)$$

where α runs over the indices of both the open and closed string moduli. The mirror map (4.8) describes the exact functional dependence of the scalars vev's t_α of the chiral $\mathcal{N} = 1$ multiplets on the geometric open and closed string moduli $z_\alpha = (z_0^{(o)}; z_a^{(c)})$.

The $\mathcal{N} = 1$ mirror map (4.8) has a peculiar property inherited from the special set of charge vectors (4.2). It has already been mentioned that there is no constant term in all of the operators \mathcal{D}_α . The logarithms in the ansatz (4.7) produce simple source terms in the differential equation for the corrections S_α :

$$\mathcal{D}_\alpha S_\beta + z_\alpha A_\beta^\alpha = 0,$$

where the A_β^α are integers that characterize the linear part $\mathcal{D}_\alpha^{lin} = z_\alpha \sum A_\beta^\alpha \theta_\beta$ of the operators \mathcal{D}_α .

From the special form of the operator \mathcal{D}_0 in (4.6) it follows immediately that the source terms A_β^0 are zero and the power series $S_\alpha(z_\beta)$ are independent of the open string modulus z_0 . This is in full agreement with the result of [7] derived from open string methods. Moreover, it follows from the above that the correction $S_0(z_b)$ is given by the following linear combination of the $S_a(z_b)$, $a, b > 0$:

$$S_0 = \sum_{b=1}^{h_{13}-1} r_b S_b, \quad r_b = (A_b^a)^{-1} A_0^a.$$

Here we have assumed that the matrix A_a^b , $a, b > 0$ is invertible. In the degenerate case there are further relations between the corrections S_a with $a > 0$ and a similar reasoning applies after choosing a linearly independent basis of power series S_a .

The next class of solutions has a double logarithmic leading behavior

$$\Pi_\alpha^2 = c_\alpha^{\beta\gamma} t_\beta t_\gamma + b_\alpha^\beta t_\beta + a_\alpha + \mathcal{O}(e^{2\pi i t_\beta}) \quad (4.9)$$

where the coefficients $a_\alpha, b_\alpha, c_\alpha$ are constants and we have inverted the mirror map (4.8) to eliminate the z_α in favor of the physical fields t_α . These periods encode the topological triple couplings (2.8) and the Gromov–Witten invariants [24]. In order that the ansatz (4.9) solves the operator \mathcal{D}_0 , the leading coefficients c_α must obey

$$c_\alpha^{00} + \sum l_0^{(a)} c_\alpha^{0a} = 0.$$

The solution of this equation with non-zero c_α^{00} determines the period Π_W^2 that describes the

$\mathcal{N} = 1$ superpotential :

$$W_{d=4}^{\mathcal{N}=1}(t_0^{(o)}; t_a^{(c)}) = W_{d=2}^{\hat{c}=4}(t_0^{(o)}; t_a^{(c)}) = \frac{\Pi_W^2(t_0^{(o)}; t_a^{(c)})}{\Pi^0(t_0^{(o)}; t_a^{(c)})}. \quad (4.10)$$

An example: $\mathcal{N} = 1$ mirror symmetry and open/closed string duality at work.

It is instructive to compare the above framework for $\mathcal{N} = 1$ mirror with the different approach of ref.[6]. In virtue of the genus zero duality (4.10) this example will also illustrate the surprising one-to-one mapping between closed string sphere (2.9) and open string disc (2.6) instantons predicted by the genus zero part of the open/closed string duality conjecture. We will only sketch the result for a specific 3-fold Y here and refer to [16] more details on the computation, which involves a study of type IIA geometry and the phase structure of the open string moduli space.

We consider a non-compact D-brane on the non-compact Calabi–Yau Y defined as the canonical bundle $\mathcal{O}(-3)_{\mathbf{P}^2}$. We start from the linear sigma model for the Calabi–Yau 4-fold X^* defined by eq. (3.8), where the charge vectors (3.9) take the form

$$I : \quad l^{(0)} = (1, -1, 0, 0, 1, -1), \quad l^{(1)} = (-3, 1, 1, 1, 0, 0)$$

This describes the D5-brane with classical limit $\tilde{y}_0 = z_0 \tilde{y}_1 \ll \tilde{y}_1$. From the differential equations (4.5), we obtain the $\mathcal{N} = 1$ mirror map (4.8) and the superpotential (4.10).

By the relation (4.10), the superpotential W describes at the same time the disc instanton corrections to the 4d $\mathcal{N} = 1$ superpotential in the D-brane geometry (Y, L) as well as the sphere instanton corrections to the 2d superpotential on the dual 4-fold

X . The open/closed string invariants are obtained from the integral large t expansions (2.6) or (2.9) respectively. The result is displayed in Tab. 2 for small degree.

n_0	$n_1=1$	2	3	4	5	6	7	8	9
1	2	-5	32	-286	3038	-35870	454880	-6073311	84302270
2	1	-4	21	-180	1885	-21952	275481	-3650196	50370000
3	1	-3	18	-153	1560	-17910	222588	-2926959	40148496
4	1	-4	20	-160	1595	-17976	220371	-2869120	39055518

(4.11)

Table 2: Closed string invariants $D_{\vec{k}}$ of the toric variety X in phase I. These agree with the open string invariants $d_{k,m}$ on Y , with the vertical direction corresponding to the class of the basic disc in Y ending on L and the horizontal directions to the class of the basic 2-sphere in Y .

The above result is in full agreement with the calculation of the open string invariants $d_{k,m}$ in the approach of ref.[6], see in particular Tab. 6 in ref.[7].

As z_0 grows, the perturbative expansion around the classical point $z_0 = 0$ breaks down near $z_0 \sim 1$. The behavior of the superpotential near $z_0 = 1$ can be studied from the solutions of the GKZ system and it corresponds to a birational transformation, a “flop”, on the Calabi–Yau 4-fold X [16]. Continuing to large $z_0 \gg 1$ there is a new classical regime defined by $y_1 = z_0^{-1} y_0 \ll y_0$. The appropriate charge vectors $l^{(\alpha)}$ that describe the perturbative expansion in the new classical phase for $z_0 \gg 1$ are defined by the Kähler cone of X :

$$II: \quad l^{(0)} = (-1, 1, 0, 0, -1, 1), \quad l^{(1)} = (-2, 0, 1, 1, 1, -1) \quad (4.12)$$

and lead to the new invariants reported in Tab. 3.

n_0	$n_1=1$	2	3	4	5	6	7	8	9
0	1	-1	1	-2	5	-13	35	-100	300
1	0	-2	4	-10	28	-84	264	-858	2860
2	-1	0	12	-32	102	-344	1200	-4304	15730
3	-1	5	0	-104	326	-1160	4360	-16854	66222
4	-1	7	-40	0	1085	-3708	14274	-57760	239404
5	-1	9	-61	399	0	-12660	45722	-185988	793502
6	-1	12	-93	648	-4524	0	159208	-598088	2530946
7	-1	15	-140	1070	-7661	55771	0	-2112456	8171400

(4.13)

Table 3: Closed string Gromov–Witten invariants of the toric variety X in phase II.

The closed string invariants $D_{\vec{k}}$ in Tab. 3 should be compared with the open string invariants $d_{k,m}$ of the D-brane geometry specified by the above classical behavior.

Again the latter have been evaluated in the open string approach in [7], see Tab. 5 therein. Again we find complete agreement up to a trivial relabelling $n_0 \rightarrow n_0 - n_1$. The relabelling corresponds to taking a linear combination of the charge vectors $l^{(\alpha)}$, which is fixed for us by the requirement that the $l^{(\alpha)}$ span the dual of the Kähler cone of X . In particular it is the choice (4.12) which leads to positive intersection in the phase II of X - note that there is no such simple concept in the open string language where the same data would correspond to the intersection calculus on bundles over Y .

In [7] it was observed, that the non-perturbative superpotential depends on the choice of an integral parameter, or “frame”, that specifies the IR behavior of the D-brane¹⁶. The above solution from $\mathcal{N} = 1$ mirror symmetry does not have such a parameter, and defines a preferred frame. It would be interesting to understand how the preferred frame is selected. It appears that the fixing of the frame is linked to the expansion around the point with maximal unipotent monodromy [16].

5. A local M-theory lift

In the previous sections it has been proposed that the open D-brane geometries (Y, L) are in some sense dual to certain closed string Calabi–Yau 4-fold compactifications without branes. It has also been verified that the topological string amplitudes agree at $g = 0, h = 1$ and $g = 0$, respectively. Here we want to add some evidence that this coincidence might be the first term of an expansion of a true string theory duality between the two backgrounds, using M-theory.

To start with, one may always take the large volume limit of Y to reduce the type IIA geometry locally to a D6-brane on a SL cycle L in the trivial \mathbf{C}^3 . It has been argued in [13] that this open string background is dual to a closed string on the small resolution of the conifold, by a combination of a lift to M-theory [7], and the “large N ” duality¹⁷ between the open string on T^*S^3 and the conifold [20][40]. A necessary condition for a duality between the D-brane geometries (Y, L) and the Calabi–Yau 4-fold X is therefore that the latter must reduce to the conifold in this limit. The 4-fold X is the mirror of the manifold X^* (3.8), which has been proposed on the basis of its period structure in sect. 3. It is defined as a gauged linear sigma model with gauge group $U(1)^{h_{13}}$ and matter fields with $U(1)$ charges defined by the vectors $l^{(\alpha)}$

¹⁶ See also the last reference in [22] for a further discussion.

¹⁷ In fact $N = 1$ in the present context.

in (3.9). One can verify that taking the image of the large volume limit of Y in the moduli of X leads to the local geometry

$$X \xrightarrow{\text{Vol}(Y) \rightarrow \infty} \mathcal{O}(-1)_{\mathbf{P}^1}^{\oplus 2} \times \mathbf{C}.$$

This is precisely the small resolution of the conifold times two flat directions. Thus the proposed string duality for the geometries (Y, L) and X has passed the first non-trivial test: in the local limit it reproduces the known string duality of [13].

One suspects that there is a generalization of the M-theory argument which extends to finite volume of Y - and thus to non-trivial 3-folds. However the proof of [13] relies also on the large N duality for which there is no known equivalent in the present case. In the following we alter the argument for the duality of \mathbf{C}^3 in a way that avoids the large N duality and can be applied to the case of non-trivial Y .

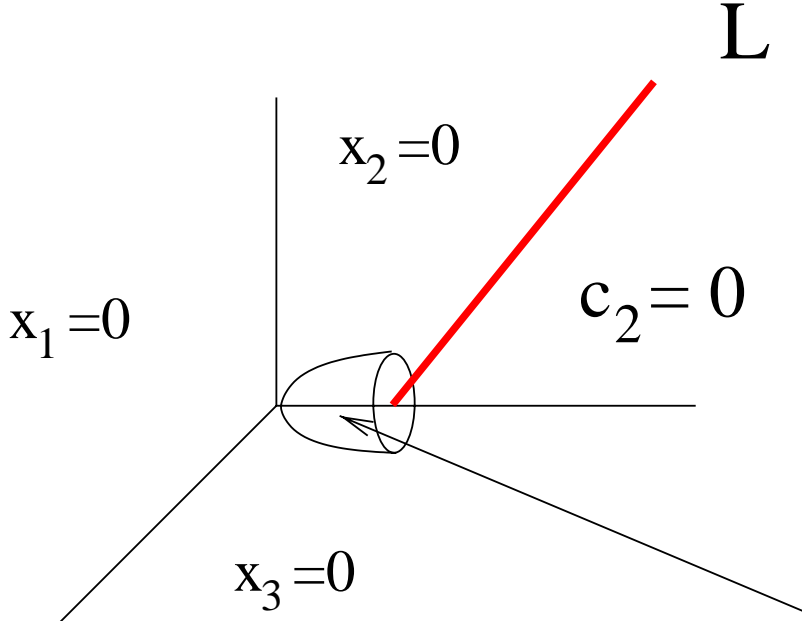
Consider \mathbf{C}^3 , parametrized by x_1, x_2, x_3 and a D-brane on a special Lagrangian 3-cycle L defined by [6]

$$L : \quad |x_1|^2 - |x_3|^2 = c_1, \quad |x_2|^2 - |x_3|^2 = c_2$$

and $\sum \theta_i = \text{const}$, where $x_i = |x_i|e^{i\theta_i}$. The 3-cycle L has a boundary unless it ends on the locus $x_a = x_b = 0$ for arbitrary a, b . Consider the phase with $c_2 = 0$, where the brane ends on $x_2 = x_3 = 0$. A holomorphic disc ending on L is defined by $D : x_2 = x_3 = 0, |x_1|^2 \leq c_1$. One may parametrize the radial direction of the disc by a real number $|x_1|^2 = c_1 - r_m$ and introduce the M-theory circle as the phase of a coordinate $x_m = r_m^{1/2} e^{i\theta_m}$. The S^1 fibration over D defines the S^3 [7][13]

$$S^3 : \quad x_2 = x_3 = 0, \quad |x_1|^2 + |x_m|^2 = c_1. \quad (5.1)$$

Moreover r_m vanishes on L and defines the D6-brane on L in the M-theory reduction on the circle $x_m \rightarrow x_m e^{i\alpha}$. On the other hand, as the only non-trivial cycle in this geometry is the S^3 , the local G_2 manifold for the M-theory must be the spin bundle Σ_{S^3} over S^3 , which in turn allows another circle reduction to a type IIA string on the cotangent bundle T^*S^3 . The transition from the phase $c_1 > 0, c_2 = 0$ to a second phase with $c_1 = 0, c_2 > 0$ describe the large N duality [20][40] and establishes the equivalence of the two open string backgrounds to the closed string on the small resolution $\mathcal{O}(-1)_{\mathbf{P}^1}^{\oplus 2}$ of the conifold [13]. The image of this transition under the moment map $x_i \rightarrow |x_i|^2$ is summarized in the figure below. The subscript IIA₁ refer to the geometry of the original type IIA theory with the D6-brane on L and the subscript IIA₂ to the alternative S^1 reduction leading to a type IIA with D6-brane on T^*S^3 .



$\tilde{S}_M^3 \sim$

Fig. 2: The moment map for the D6-brane on \mathbf{C}^3 with the fundamental disc instanton D for the two phases $c_2 = 0$ and $c_1 = 0$. The subscripts $\text{IIA}_1, \text{IIA}_2, \text{IIA}_3$ refer to the two alternative S^1 reductions of the M-theory completion described in the text.

Since the three edges $x_a = x_b = 0$ of \mathbf{C}^3 are equivalent, it should be possible to avoid the large N transition in the above argument and to see the circle fibration that leads to type IIA on $\mathcal{O}(-1)_{\mathbf{P}^1}^{\oplus 2}$ already in the original phase. The local G_2 manifold Σ_{S^3} , which is topologically $\sim \mathbf{R}^4 \times S^3$, may be described by the equation

$$\Sigma_{S^3} : |x_1|^2 + |x_m|^2 - |x_2|^2 - |x_3|^2 = c_1, \quad (5.2)$$

where $x_2 = x_3 = 0$ defines the S^3 and the x_2, x_3 parameterize \mathbf{R}^4 .

We are interested in a $U(1)$ acting on (5.2) without fixed point which describes a third type IIA₃ compactification in the phase $c_2 = 0$ (the result of a transition that starts from a type IIA with D6-brane on T^*S^3 in the phase with $c_1 = 0$). Specifically the $U(1)$ should act as the Hopf fibration $S^3 \rightarrow S^2$ on the non-trivial S^3 , defined by

$(x_1, x_m) \rightarrow (e^{i\alpha}x_1, e^{i\alpha}x_m)$ in the above coordinates. This is interpreted as the type IIA theory on the base S^2 with flux through the latter.

We make now the ansatz that the searched for M-theory $U(1)$ can be represented as a gauge symmetry of a 2d linear sigma model. The motivation is that without the real constraint (5.2), the coordinates x_1, x_2, x_3, x_m define a Calabi–Yau 4-fold \mathbf{C}^4 . Once we add the real constraint (5.2) to reduce to the local G_2 manifold, there is essentially one way to divide further by a $U(1)$ to obtain a smooth Calabi–Yau 3-fold with a 2-cycle. Namely we interpret (5.2) as the D-term $q_i |x_i|^2 = c_1$ of the gauged $U(1)$. This determines in turn the $U(1)$ charges q_i of the fields x_i . Note that the gauge transformation $(e^{i\alpha}x_1, e^{i\alpha}x_m, e^{-i\alpha}x_2, e^{-i\alpha}x_3)$ with $\alpha \in \mathbf{R}$ acts precisely as the Hopf fibration on the S^3 . It also acts non-trivially in the x_2, x_3 directions, which may be interpreted as flux in the directions transverse to the S^2 base.

The geometry of this S^1 reduction is precisely the same as that of the linear sigma model for the small resolution $\mathcal{O}(-1)_{\mathbf{P}^1}^{\oplus 2}$. The momentum map for the LSM with D-term (5.2) is shown in Fig. 3. It is in fact identical to the “enlarged” momentum map for the M-theory manifold described in [13].

A similar reasoning may now be applied to the case of D-branes on a non-trivial 3-fold Y with compact directions. For simplicity we consider the example $Y = \mathcal{O}(-3)_{P^2}$ described in sect. 4. The argument applies more generally to other choices for Y with some obvious modifications. Let (x_0, x_1, x_2, x_3) be the coordinates on Y with the projective action defined by the multiplication $(\lambda^{-3}x_0, \lambda x_1, \lambda x_2, \lambda x_3)$ where $\lambda \in \mathbf{C}^*$. Let L be the special Lagrangian 3-cycle in Y defined by

$$|x_1|^2 - |x_0|^2 = c_1, \quad |x_2|^2 - |x_0|^2 = c_2. \quad (5.3)$$

The D-term for the gauged $U(1)$ linear sigma model for Y is

$$|x_1|^2 + |x_2|^2 + |x_3|^2 - 3|x_0|^2 = t. \quad (5.4)$$

Consider the phase where the brane ends on an interior edge, say $x_0 = x_2 = 0$, which requires $c_2 = 0$. There are two holomorphic discs ending on L :

$$\begin{aligned} D_1 : x_0 = x_2 = 0, \quad 0 \leq |x_1|^2 \leq c_1, \\ D_2 : x_0 = x_2 = 0, \quad c_1 \leq |x_1|^2 \leq t. \end{aligned}$$

To describe an M-theory limit we are looking for an S^1 fibration over D_i that promotes an IIA instanton on the disc to a M2-brane wrapped on S^3 . For a single disc this can be achieved as in (5.1). However a similar ansatz for the two discs is

inconsistent with the global structure constrained by the D-term $|x_1|^2 + |x_3|^2 = t$ on the edge $x_2 = x_0 = 0$.

The best we can do is to introduce *two* new variables x_4 and x_5 that parametrize the radial direction of the two discs D_i and define two 3-spheres by $x_0 = x_2 = 0$ and

$$\begin{aligned} S_1^3 & : x_4 = 0, |x_1|^2 + |x_5|^2 = c_1 + |x_4|^2, \\ S_2^3 & : x_5 = 0, |x_3|^2 + |x_4|^2 = t - c_1 + |x_5|^2, \end{aligned} \tag{5.5}$$

These equations are now consistent with the D-term on the edge $x_0 = x_2 = 0$. Moreover the M-theory S^1 vanishing over L is defined by the $U(1)$ action $(x_4, x_5) \rightarrow (e^{i\alpha}x_4, e^{i\alpha}x_5)$.

Adding a term $|x_0|^2$ on the r.h.s. of the second equations in (5.5), the local geometry is that of two spin bundles over the two homology 3-spheres, with the latter intersecting at the point $x_4 = x_5 = 0$, and with the transverse directions identified in a non-trivial way. To make these equations globally consistent with the D-term (5.4) we add another term $|x_0|^2 - |x_2|^2$ that is locally zero on the edge and obtain¹⁸

$$\begin{aligned} S_1^3 & : |x_1|^2 + |x_5|^2 = c_1 + |x_4|^2 + |x_0|^2, & x_4 = 0, \\ S_2^3 & : |x_3|^2 + |x_4|^2 = t - c_1 + |x_5|^2 + |x_0|^2 - (|x_2|^2 - |x_0|^2), & x_5 = 0. \end{aligned} \tag{5.6}$$

A new M-theory $U(1)$ without fixed points that defines a type IIA compactification without branes can now be obtained in a similar way as before. Namely, we interpret the equations (5.6) as the D-terms of a gauged linear sigma model with gauge symmetry $U(1)^2$. This leads to the following $U(1)$ charges $l_i^{(a)}$ for the matter fields x_i :

$$l^{(1)} = (-1, 1, 0, 0, -1, 1), \quad l^{(2)} = (-2, 0, 1, 1, 1, -1). \tag{5.7}$$

Note that the the $U(1)$ action defined by $l^{(1)}$ describes the Hopf-fibration of the first S^3 and the $U(1)$ $l^{(2)}$ the Hopf fibration of the second S^3 in (5.6). Moreover the diagonal $U(1)$ coincides with the gauge symmetry of the GLSM for Y .

The geometry for the GLSM defined above is precisely the mirror of the Calabi–Yau 4-fold X^* defined in (3.8)! Moreover the two Hopf-fibrations of the M-theory circle over the S^2 bases indicates the presence of flux on the 4-fold. The flux generates (part of) the superpotential in the two-dimensional type IIA theory on X .

Note that to realize the M-theory lift of the discs to 3-spheres in (5.5), we needed to add two complex variables x_4 and x_5 . Subtracting the one real constraint this adds

¹⁸ Adding the vanishing term instead to the other equation is equivalent by a change of moduli.

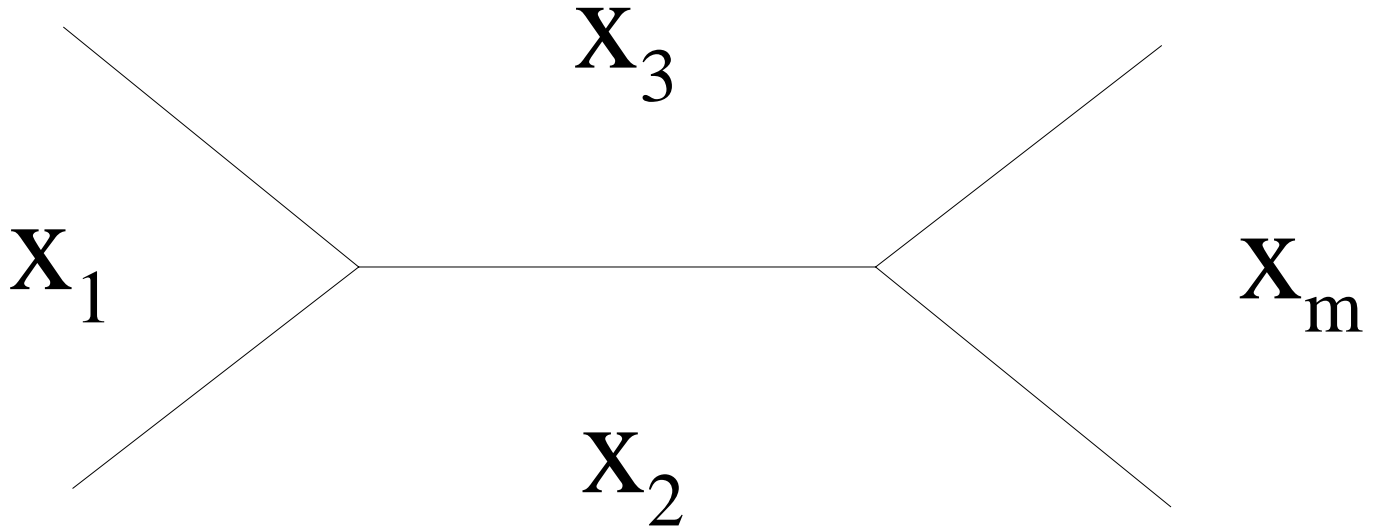


Fig. 3: The moment map for the LSM defined by (5.2) and the projection to $x_2 = 0$ of the momentum map for the LSM on the 4-fold (5.6). We have also indicated how the momentum map for the spin bundle $\Sigma_{S^3_1}$ sits in the latter (blue lines).

three real dimensions to the six of the Calabi–Yau 3-fold Y . This is in agreement with the 2d interpretation of the duality in sect. 2.

In the above argument we have clearly changed the global geometry transverse to the edge $x_2 = x_0 = 0$ on which the discs and their M-theory lifts to S^3 are defined. This is not unexpected, as the closed string includes the back-reaction of the geometry to the D-branes; yet it makes this reasoning somewhat heuristic. Note however that the same is already true for the M-theory lift for \mathbf{C}^3 (5.1), where the string duality has been tested at all genera. One may therefore hope that this *local* M-theory lift captures the essential geometry and the transverse geometry is uniquely fixed by the consistency of the background. Clearly it would be very satisfying to have an independent check of this type of arguments by comparing amplitudes other than the superpotential.

6. Outlook

There are several interesting questions arising from the previously described connection between the exact 4d $\mathcal{N} = 1$ superpotential and the 2d topological closed string amplitudes. Firstly one would like to generalize these ideas to more general D-brane geometries, involving many D-branes and compact Calabi–Yau 3-folds Y . Also the generalization to higher genera will be interesting, both from the math/physics point of view as well as to test the conjectured duality. On more conceptual grounds, a quite interesting structure is the $\mathcal{N} = 1$ 4d special geometry inherited from the 2d amplitudes. It would be interesting to derive it from the open string equivalent of the tt^* equations. These questions will be addressed in [16].

Another interesting direction is F-theory on the 4-fold as a natural candidate to lift up the conjectural duality to four dimensions¹⁹. In fact the same 2d topological amplitudes that we have considered here have been proposed to be relevant for F-theory in the earlier study of mirror symmetry for 4-folds [24]. Our result indicates that this is indeed the case. It would be interesting to have an alternative 2d world-sheet formulation - describing F-theory - that leads to the same amplitudes as that of the closed type IIA string considered in this paper.

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Appendix A. Geometric integrals for the dual open/closed string pairs

In the following we show that the period integrals on the closed string background X^* without branes reproduce the 3-cycle and 3-chain integrals of the proposed dual open string background (C, Y^*) . The Landau–Ginzburg expression of the period integrals over the holomorphic d -form $\Omega^{(d,0)}$ is [38]

$$\begin{aligned}
 \Pi &\sim \int \prod_0^{d-1} \frac{dy_i}{y_i} e^{-\mathcal{W}_{D=2}(y_i)} \\
 &= \int \prod_0^{d-2} \frac{d\tilde{y}_i}{\tilde{y}_i} dy_{d-1} dx dz e^{-y_{d-1}(\mathcal{W}_{D=2}(\tilde{y}_i) + xz)}, \\
 &= \int_S \prod_0^{d-2} \frac{d\tilde{y}_i}{\tilde{y}_i} \frac{dz}{z}, \quad \tilde{y}_i = \frac{y_i}{y_{d-1}}, \quad i = 1, \dots, d-2,
 \end{aligned} \tag{A.1}$$

¹⁹ An F-theory argument in favor of the proposed open/closed string duality has been pointed out to me by Cumrun Vafa.

where in the last, alternative expression the integral is on the hypersurface

$$S : \mathcal{W}_{D=2}(\tilde{y}_i) + xz = 0, \quad (\text{A.2})$$

with $x, z \in \mathbf{C}$ [34]. The equivalence of the above representations follows readily from the integration of linear variables, $\int dx e^{xf} = \delta(f)$. In particular integrating out x, z in the second line gives back the first one, while integrating out y_{d-1} leads to the expression in the third line. We have also used the homogeneity of $\mathcal{W}_{D=2}$ in the variables y_i which assures that y_{d-1} factors out of $\mathcal{W}_{D=2}$ in the variables \tilde{y}_i .

To show that periods of the holomorphic form $\Omega^{(4,0)}(X^*)$ on the closed string geometry X^* (3.8) describes the 3-cycle and 3-chain integrals over $\Omega^{(3,0)}$ in the D-brane geometry (Y^*, C) , consider the integrals

$$\Pi \sim \int \prod_0^2 \frac{dy_i}{y_i} \frac{dv}{v} e^{-\mathcal{W}_{D=2}(Y^*) - v(1+z_0^{-1} \frac{y_0}{y_1})}, \quad (\text{A.3})$$

where we have solved for v_2 and write v for v_1 for ease of notation. Note that the variable v enters only *linearly* into $\mathcal{W}_{D=2}(X^*)$.

As explained in sect. 3, there are two types of 4-cycles in the manifold X^* , considered as a fibration of Y^* over an infinitely long cylinder parametrized by v . The period integrals for the first type of cycles is obtained by integrating on a small cycle around $v = 0$ and one finds

$$\Pi \sim \int \prod_0^2 \frac{dy_i}{y_i} e^{-\mathcal{W}_{D=2}(Y^*)}. \quad (\text{A.4})$$

These are simply the periods of Y^* , times an irrelevant constant. This is also obvious from the fibration.

The second type of 4-cycles projects onto a path between branch points on \mathbf{C}_v^* , above which a 3-cycle shrinks in the fiber Y^* . To evaluate this type of integral, we take a derivative with respect to $\tau_0 = \frac{1}{2\pi i} \ln z_0$:

$$\begin{aligned} \frac{\partial}{\partial \tau_0} \Pi &\sim z_0^{-1} \int \prod_{i=0}^2 \frac{dy_i}{y_i} dv \frac{y_0}{y_1} e^{-\mathcal{W}_{D=2}(Y^*) - v(1+z_0^{-1} \frac{y_0}{y_1})} \\ &= z_0^{-1} \int \prod_{i=0}^2 \frac{dy_i}{y_i} \frac{y_0}{y_1} \delta(1 + z_0^{-1} \frac{y_0}{y_1}) e^{-\mathcal{W}_{D=2}(Y^*)}. \end{aligned}$$

Note that left-over integral in the last expression is again defined on the Calabi–Yau 3-fold Y^* , however this time with an extra delta function produced by the integration over the linear variable v .

To proceed we assume small z_0 and integrate out y_0 . Moreover we use the simple manipulation described below (A.1) to obtain the equivalent expression

$$\frac{\partial}{\partial \tau_0} \Pi \sim \int_S \frac{d\tilde{y}_1}{\tilde{y}_1} \frac{dz}{z},$$

where S is the surface

$$S : \mathcal{W}_{D=2}(Y^*)|_{\tilde{y}_0=z_0\tilde{y}_1} + xz = 0,$$

The surface S has the structure of a \mathbf{C}^* fibration over the \tilde{y}_1 plane and there are again two types of 2-cycles made from a 1-cycle in the \mathbf{C}^* fiber and a path in the base. Integration on small circles around $z = \tilde{y}_1 = 0$ gives a constant. This is the expected result for the period that describes the flat coordinate

$$t_0 = \tau_0 + S_0(z_\alpha), \quad \alpha = 0, \dots, h_{13} - 1.$$

where S_0 is the the sub-leading correction to the period integral discussed in sect.4. However note that the derivative $\partial/\partial \tau_0 = \sum_{a>0} (\partial t_a / \partial \tau_0) \partial / \partial t_a + (\partial t_0 / \partial \tau_0) \partial / \partial t_0$ is in general different from $\partial/\partial t_0$. The above outcome suggests that $\partial t_a / \partial \tau_0 = 0$ for $a > 0$ and $\partial t_0 / \partial \tau = 1$, which implies that the sub-leading corrections $S(z_\alpha)$ to the flat coordinates are functions of *only* the $h_{13} - 1$ coordinates z_a , $a > 0$. It is not too difficult to verify this relation by showing that the periods that satisfy $\partial \Pi / \partial t_a = 1$, $a > 1$ are independent of z_0 . A more powerful approach is to consider the system (4.5) of differential equations satisfied by the periods (A.3), and it has been already shown in sect. 4 that this leads to the desired result $\partial/\partial \tau_0 = \partial/\partial t_0$.

There is another 2-cycle in the surface S made from integrating \tilde{y}_1 instead from infinity to the special points \tilde{y}_1^0 , where the cycle in the \mathbf{C}^* fiber shrinks. The result for this type of integral is

$$\frac{\partial}{\partial t_0} \Pi \sim \int_\infty^{\tilde{y}_1^0} d(\ln(y_1)) = \ln(y_1^0) + \dots,$$

where the dots denote the contribution from infinity which is independent of the moduli. This is precisely the first derivative of the superpotential (3.7) for the D5-brane on Y^* .

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