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How to Quantize Phases and Moduli !

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Abstract

A typical classical interference pattern of two waves with intensities I_1, I_2 and relative phase $\varphi = \varphi_2 - \varphi_1$ may be characterized by the 3 observables $p = \sqrt{I_1 I_2}$, $p \cos \varphi$, and $-p \sin \varphi$. They are, e.g. the starting point for the semi-classical operational approach by Noh, Fougères and Mandel (NFM) to the old and notorious phase problem in quantum optics. Following a recent group theoretical quantization of the symplectic space $\mathcal{S} = \{(\varphi \in \mathbb{R} \bmod 2\pi, p > 0)\}$ in terms of irreducible unitary representations of the group $SO^\uparrow(1, 2)$ the present paper applies those results to that controversial problem of quantizing moduli and phases of complex numbers: The Poisson brackets of the classical observables $p \cos \varphi$, $-p \sin \varphi$ and $p > 0$ form the Lie algebra of the group $SO^\uparrow(1, 2)$. The corresponding self-adjoint generators $\widehat{p \cos \varphi} = K_1$, $-\widehat{p \sin \varphi} = K_2$ and $\widehat{p} = K_3$ of that group may be obtained from its irreducible unitary representations. For the positive discrete series the modulus operator K_3 has the spectrum $\{n + k, n = 0, 1, 2, \dots; k > 0\}$. Self-adjoint operators $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$ can be defined as $(K_3^{-1} K_1 + K_1 K_3^{-1})/2$ and $-(K_3^{-1} K_2 + K_2 K_3^{-1})/2$ which have the theoretically desired properties for $k \geq 0.5$. The approach advocated here solves, e.g. the modulus-phase quantization problem for the harmonic oscillator and appears to provide a full quantum theoretical basis for the NFM-formalism.

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1 Introduction

The problem how to quantize the modulus and the phase of a wave as some kind of canonically conjugate variables and relate them to genuine self-adjoint operators in Hilbert space is a very old one and is - according to the still ongoing controversial discussions in the field of quantum optics - not yet settled (see, e.g. the reviews [1]). A solution of that theoretical problem becomes more and more urgent, however, because the experiments in quantum optics are increasingly more refined and allow to differentiate between different theoretical schemes.

Perhaps the most successful one of the schemes proposed up to now is the semi-classical operational approach by Noh, Fougères and Mandel (NFM) [2, 3, 4, 5] which starts from well-known classical interference concepts and reinterprets them in terms of quantized observables. The approach works well as long as the properties of the quantum theoretical ground state do not become important, i.e. as long as one stays in the semi-classical regime. The crucial - idealized - elements of the NFM - scheme which are of interest here are the following:

Consider the sum

$$A = a_1 e^{i\varphi_1} + a_2 e^{i\varphi_2} \quad (1)$$

of two complex numbers A_j , where the phases φ_j are chosen such that $a_j > 0$, $j = 1, 2$. The quantities a_j and φ_j may be functions of other parameters, e.g. space or/and time variables etc. depending on the concrete experimental situation. The absolute square of A has the form

$$w_3(I_1, I_2, \varphi) = |A|^2(I_1, I_2, \varphi = \varphi_2 - \varphi_1) = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \varphi, \quad (2)$$

$$I_j = (a_j)^2, j = 1, 2.$$

Phase shifting one of the two amplitudes A_j by an appropriate device yields new intensities:

$$w_4(I_1, I_2, \varphi) = w_3(\varphi + \pi) = I_1 + I_2 - 2\sqrt{I_1 I_2} \cos \varphi, \quad (3)$$

$$w_5(I_1, I_2, \varphi) = w_3(\varphi + \pi/2) = I_1 + I_2 - 2\sqrt{I_1 I_2} \sin \varphi, \quad (4)$$

$$w_6(I_1, I_2, \varphi) = w_3(\varphi - \pi/2) = I_1 + I_2 + 2\sqrt{I_1 I_2} \sin \varphi. \quad (5)$$

The essential quantities for a classical description of the interference pattern are then

$$4P_1 = w_3 - w_4 = 4p \cos \varphi, \quad p = \sqrt{I_1 I_2}, \quad (6)$$

$$4P_2 = w_5 - w_6 = -4p \sin \varphi , \quad (7)$$

$$4P_3 = 4p = \sqrt{(w_4 - w_3)^2 + (w_6 - w_5)^2} > 0 . \quad (8)$$

The ratios P_1/P_3 and P_2/P_3 yield $\cos \varphi$ and $-\sin \varphi$.

In quantum optics the classical intensities $I_j, j = 1, 2, w_a, a = 3, 4, 5, 6$ become energies of a mean number of photons and the w_a are formally replaced by expectation values of number operators and so quantum theory comes into play in a semi-classical way.

This NFM-scheme has been quite successful, but justified theoretical doubts remain as to the applicability of the approach for small values of $|w_3 - w_4|$ and $|w_5 - w_6|$ and as to the commutativity of corresponding quantum operators, especially of those corresponding to $\cos \varphi$ and $\sin \varphi$.

In an attempt to find an appropriate quantized version of the above classical description of interferences let me start with the following observation: Suppose that $p > 0$. As any function $f(\varphi, p)$ periodic in φ with period 2π can – under certain mathematical conditions – be expanded in a Fourier series and as $\cos(n\varphi)$ and $\sin(n\varphi)$ can be expressed by polynomials of order n in $\cos \varphi$ and $\sin \varphi$, the observables $P_j, j = 1, 2, 3$, defined in Eqs. (6)-(8) are indeed the basic ones for such functions. Let

$$\{f_1, f_2\} = \partial_\varphi f_1 \partial_p f_2 - \partial_p f_1 \partial_\varphi f_2 \quad (9)$$

be the Poisson bracket for any two smooth functions $f_i(\varphi, p), i = 1, 2$. Then we have the closed algebra

$$\{P_3, P_1\} = -P_2, \quad \{P_3, P_2\} = P_1, \quad \{P_1, P_2\} = P_3 , \quad (10)$$

which is just the real Lie algebra of the group $SO^\dagger(1, 2)$ (identity component of the proper Lorentz group in 2+1 space-time dimensions) or of one of its infinitely many covering groups, e.g. the double covering $SU(1, 1)$ which is isomorphic to the group $SL(2, \mathbb{R})$ and the symplectic group $Sp(1, \mathbb{R})$. Quantizing the classical observables P_j then consists in replacing them by the self-adjoint generators $K_j \sim P_j$ of appropriate irreducible unitary representations of those groups.

The appropriate theoretical background for this approach is provided by the so-called “group theoretical quantization” which generalizes the usual quantization procedure to systems which cannot be dealt with in the naive manner where classical canonical variable pairs are replaced by multiplication and differential operators, respectively. This approach to quantizing a

classical system is a genuine extension of the conventional method which is included as a special case (see the reviews [6]).

As an application of that generalized quantization scheme in the present case of interest the symplectic manifold

$$\mathcal{S} = \{(\varphi \in \mathbb{R} \bmod 2\pi, p > 0)\} \quad (11)$$

(associated with the local symplectic form $d\varphi \wedge dp$) was quantized in terms of the group $SO^\uparrow(1, 2)$ for the purpose of quantizing Schwarzschild black holes [7]. We were not aware then that the same quantization had been performed previously by R. Loll in a different context [8].

In the meantime I realized that this quantization also sheds new light on the old unsolved problem how to represent phase and modulus as self-adjoint operators in a Hilbert space associated with a corresponding physical system (see the preliminary note [9]). Let me briefly point out some of the essential formal features of the approach:

The crucial point is that the manifold (11) has the nontrivial topology $S^1 \times \mathbb{R}^+$, \mathbb{R}^+ : real numbers > 0 . Such a manifold cannot be quantized in the usual naive way used for a phase space with the trivial topology \mathbb{R}^2 by converting a classical canonical pair (q, p) of phase space variables into operators and their Poisson bracket into a commutator. Here the group theoretical quantization scheme [6] as a generalization of the conventional one helps: The group $SO^\uparrow(1, 2)$ acts symplectically, transitively, effectively and (globally) Hamilton-like on the manifold (11) (which may also be characterized by the “forward light cone” $P_3^2 - P_1^2 - P_2^2 = 0, P_3 > 0$) and, therefore, its irreducible representations (or those of its covering groups) can provide the basic self-adjoint quantum observables and their Hilbert space of states (see Ref. [7] for more details): In the course of the group theoretical quantization one finds that the three basic classical observables $P_j, j = 1, 2, 3$, correspond to the three self-adjoint Lie algebra generators K_j of a positive discrete series irreducible unitary representation of the group $SO^\uparrow(1, 2)$ or one of its infinitely many covering groups. The generators K_j obey the commutation relations

$$[K_3, K_1] = iK_2, \quad [K_3, K_2] = -iK_1, \quad [K_1, K_2] = -iK_3 \quad . \quad (12)$$

Here K_3 is the generator of the compact sub-group $SO(2)$.

If the minus sign in front of iK_3 on the r.h.s. of the last commutator in Eqs. (12) is replaced by a plus sign we obtain the Lie algebra of the rotation group

or its covering group $SU(2)$. It is crucial for the following discussions that we are dealing with the non-compact group $SO^\uparrow(1, 2)$ instead.

It is essential to realize that a group theoretical quantization does *not* assume that the generators of the basic Lie algebra themselves may be expressed by some conventional canonical variables like in the case of angular momentum. This may be the case locally in special examples, but in general it will not be possible, especially not globally. For more details see the discussion below and the Refs. [6, 7, 10]. The paper is organized as follows:

In section 2 I collect the essential elements as to the self-adjoint Lie algebra generators of the irreducible unitary representations of the group $SO^\uparrow(1, 2)$ and of its covering groups and discuss important matrix elements of the “observables” $K_j, j = 1, 2, 3$, in the number state basis (eigenstates of K_3). Then the operators $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$ are introduced and some of their main matrix elements in the number state basis calculated, too.

Section 3 makes use of $SO^\uparrow(1, 2)$ Lie algebra related coherent states (with $K_-|k, z\rangle = z|k, z\rangle$), introduced by Barut and Girardello [11]. Properties of simple matrix elements are discussed and it is shown that the self-adjoint $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$ operators have the right support – the closed interval $[-1, +1]$ – provided the number $k > 0$ which characterizes an irreducible unitary representation has the lower bound $k \geq 0.5$.

Section 4 gives a physical interpretation of the previous results without referring to “field” variables, i.e. without the use of “underlying” creation and annihilation operators or corresponding “modes”. It is possible - at least theoretically - to show that a recorded interference pattern – like in the NFM-approach – can be described satisfactorily in terms of *observable quantities* like the intensities I_1, I_2, P_1, P_2, P_3 and their quantum mechanical counterparts, here especially the operators $K_j, j = 1, 2, 3$, and functions of them.

In section 5 I discuss briefly cases in which the Lie algebra operators K_j can be expressed in terms of creation and annihilation operators. The most interesting one is that in which the K_j are expressed non-linearly in terms of one creation and one annihilation operator a^+ and a acting in the Fock space of the harmonic oscillator. The general scheme immediately gives “decent” $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$ operators (i.e. self-adjoint and with the correct spectrum), solving an old and long discussed quantum mechanical problem [1]!

2 “Observable” operators and their matrix elements in the number state basis

In order to calculate expectation values and fluctuations we have to know the actions of the operators $K_i, i = 1, 2, 3$, on the Hilbert spaces associated with the positive discrete series of the irreducible unitary representations of $SO^\uparrow(1, 2)$ (or its covering groups). In the following I rely heavily on Ref. [7] where more (mathematical) details and Refs. to the corresponding literature can be found.

As the eigenfunctions of K_3 – the generator of the compact subgroup $SO(2)$ of $SO^\uparrow(1, 2)$ – form a complete basis of the associated Hilbert spaces, it is convenient to use them as a starting point. The operators

$$K_+ = K_1 + iK_2, \quad K_- = K_1 - iK_2 \quad (13)$$

$$[K_+, K_-] = -2K_3, \quad [K_3, K_\pm] = \pm K_\pm,$$

act as ladder operators. The positive discrete series is characterized by the property that there exists a state $|k, 0\rangle$ for which $K_-|k, 0\rangle = 0$. The number $k > 0$ characterizes the representation: For a general normalized eigenstate $|k, n\rangle$ of K_3 we have

$$K_3|k, n\rangle = (k + n)|k, n\rangle, \quad n = 0, 1, \dots, \quad (14)$$

$$K_+|k, n\rangle = \omega_n [(2k + n)(n + 1)]^{1/2} |k, n + 1\rangle, \quad |\omega_n| = 1, \quad (15)$$

$$K_-|k, n\rangle = \frac{1}{\omega_{n-1}} [(2k + n - 1)n]^{1/2} |k, n - 1\rangle. \quad (16)$$

In irreducible unitary representations the operator K_- is the adjoint operator of K_+ : $(f_1, K_+ f_2) = (K_- f_1, f_2)$. The phases ω_n serve to guarantee this property. Their choice depends on the concrete realization of the representations. In the examples discussed in Ref. [7] they have the values 1 or i . In the following I assume ω_n to be independent of n : $\omega_n = \omega$.

The Casimir operator

$$Q = K_1^2 + K_2^2 - K_3^2 = K_+ K_- + K_3(1 - K_3) = K_- K_+ - K_3(1 + K_3) \quad (17)$$

has the eigenvalues $q = k(1 - k)$. The allowed values of k depend on the group: For $SO^\uparrow(1, 2)$ itself one has $k = 1, 2, \dots$ and for the double covering $SU(1, 1)$ $k = 1/2, 1, 3/2, \dots$. For the universal covering group k may be any

real number > 0 (for details see Ref.[7]). The appropriate choice will depend on the physics to be described. In any case, for a unitary representation the number k has to be non-vanishing and positive!

The relation (15) implies

$$|k, n\rangle = \omega^{-n} \left[\frac{\Gamma(2k)}{n! \Gamma(2k+n)} \right]^{1/2} (K_+)^n |k, 0\rangle . \quad (18)$$

The expectation values of the self-adjoint operators $K_1 = (K_+ + K_-)/2$ and $K_2 = (K_+ - K_-)/2i$ (which correspond to the classical observables $p \cos \varphi$ and $-p \sin \varphi$) with respect to the eigenstates $|k, n\rangle$ and the associated fluctuations may be calculated with the help of the relations (13)-(16) :

$$\langle k, n | K_i | k, n \rangle = 0 , \quad i = 1, 2 . \quad (19)$$

The corresponding fluctuations are

$$(\Delta K_i)_{k,n}^2 = \langle k, n | K_i^2 | k, n \rangle = \frac{1}{2}(n^2 + 2nk + k) = \frac{1}{2}[(n+k)^2 + q] , \quad i = 1, 2 . \quad (20)$$

Because of $[K_1, K_2] = -iK_3$ the general uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle| \quad (21)$$

for self-adjoint operators A and B here takes the special form

$$(\Delta K_1)_{k,n} (\Delta K_2)_{k,n} = \frac{1}{2}(n^2 + 2kn + k) \geq \frac{1}{2} |\langle k, n | K_3 | k, n \rangle| = \frac{1}{2}(n+k) . \quad (22)$$

The equality sign holds for the ground state $|k, n=0\rangle$.

The eqs. (20) imply further (see also Eq. (17)) that

$$\langle k, n | K_1^2 | k, n \rangle + \langle k, n | K_2^2 | k, n \rangle = (n+k)^2 + q = \langle k, n | K_3^2 | k, n \rangle + q . \quad (23)$$

This means that for very large n the correspondence principle, $(p \cos \varphi)^2 + (p \sin \varphi)^2 = p^2$, is fulfilled! For $q=0$ (i.e. $k=1$) we even have $K_1^2 + K_2^2 = K_3^2$!

Next I define the self-adjoint operators [9] $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$ as follows:

$$\widehat{\cos \varphi} = \frac{1}{2}(K_3^{-1}K_1 + K_1K_3^{-1}) , \quad \widehat{\sin \varphi} = -\frac{1}{2}(K_3^{-1}K_2 + K_2K_3^{-1}) . \quad (24)$$

Notice that K_3^{-1} is well-defined because K_3 is a positive definite operator for the positive discrete series. One has $K_3^{-1}|k, n\rangle = |k, n\rangle/(n+k)$. We shall demonstrate below that the self-adjoint operators (24) have the right spectrum – the full interval $[-1, +1]$ – provided $k \geq 0.5$.

Using the commutation relations (12) we obtain for the operators (24):

$$[K_3, \widehat{\cos \varphi}] = -i \widehat{\sin \varphi} , \quad (25)$$

$$[K_3, \widehat{\sin \varphi}] = i \widehat{\cos \varphi} . \quad (26)$$

The validity of these relations has been considered important for the properties of number, \cos and \sin operators [1]. They were introduced by Louisell [12] who was the first to recognize that one should use the operator versions of $\cos \varphi$ and $\sin \varphi$ instead of φ itself in order to get a consistent quantization. The operators $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$ defined in Eq. (24) do not commute! Using the Jacobi identity $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$ we obtain from Eqs. (25) and (26)

$$[[\widehat{\cos \varphi}, \widehat{\sin \varphi}], K_3] = 0 . \quad (27)$$

This means that in an irreducible representation the commutator $[\widehat{\cos \varphi}, \widehat{\sin \varphi}]$ is a function of K_3 and the Casimir operator Q , that is to say the commutator is diagonal in the number basis $|k, n\rangle$. We shall use this property below in order to determine the commutator.

Similar arguments lead to the relation

$$[(\widehat{\cos \varphi}^2 + \widehat{\sin \varphi}^2), K_3] = 0 , \quad (28)$$

which we shall use in order to determine $\widehat{\cos \varphi}^2 + \widehat{\sin \varphi}^2$ which does not have the classical value 1!

From the Eqs. (13)-(16) we get the important relations

$$\widehat{\cos \varphi}|k, n\rangle = \frac{\omega}{4} f_{n+1}^{(k)}|k, n+1\rangle + \frac{1}{4\omega} f_n^{(k)}|k, n-1\rangle , \quad (29)$$

$$\widehat{\sin \varphi}|k, n\rangle = -\frac{\omega}{4i} f_{n+1}^{(k)}|k, n+1\rangle + \frac{1}{4i\omega} f_n^{(k)}|k, n-1\rangle , \quad (30)$$

$$f_n^{(k)} = [n(2k+n-1)]^{1/2} \left(\frac{1}{k+n} + \frac{1}{k+n-1} \right) , f_{n=0}^{(k)} = 0 . \quad (31)$$

They imply

$$\langle k, n|\widehat{\cos \varphi}|k, n\rangle = 0 , \quad \langle k, n|\widehat{\sin \varphi}|k, n\rangle = 0 \quad (32)$$

and

$$\langle k, n | (\widehat{\cos \varphi})^2 | k, n \rangle = \langle k, n | (\widehat{\sin \varphi})^2 | k, n \rangle = \frac{1}{16} [(f_{n+1}^{(k)})^2 + (f_n^{(k)})^2], \quad (33)$$

$$\langle k, n | [\widehat{\cos \varphi}, \widehat{\sin \varphi}] | k, n \rangle = \frac{i}{8} [(f_{n+1}^{(k)})^2 - (f_n^{(k)})^2]. \quad (34)$$

This gives for the number states $|k, n\rangle$ the uncertainty relation

$$(\Delta \widehat{\cos \varphi})_{k,n} (\Delta \widehat{\sin \varphi})_{k,n} = \frac{1}{16} [(f_{n+1}^{(k)})^2 + (f_n^{(k)})^2] \geq \frac{1}{16} [(f_{n+1}^{(k)})^2 - (f_n^{(k)})^2], \quad (35)$$

where again the equality sign holds for the ground state $|k, n=0\rangle$! for which

$$\langle k, 0 | (\widehat{\cos \varphi})^2 | k, 0 \rangle = \langle k, 0 | (\widehat{\sin \varphi})^2 | k, 0 \rangle = \frac{(2k+1)^2}{8k(k+1)^2}. \quad (36)$$

For $k=1/2$ the r.h.s. of Eq. (36) takes the value $4/9$ and for $k=1$ one has $9/32$. It also follows that an upper bound $\langle k, 0 | (\widehat{\cos \varphi})^2 | k, 0 \rangle \leq 1$ implies for k the lower bound $k \geq k_1 \equiv [(0.5 + 0.5\sqrt{23/27})^{1/3} + (0.5 - 0.5\sqrt{23/27})^{1/3} - 1]/2 = 0.162\dots$. A slightly higher lower bound for allowed values of k will be discussed in the next section.

For very large n we have the (correct) correspondence principle limits

$$\begin{aligned} \langle k, n | (\widehat{\cos \varphi})^2 | k, n \rangle &= \langle k, n | (\widehat{\sin \varphi})^2 | k, n \rangle \\ &\simeq \frac{1}{2} \left[1 + \frac{1}{4} \frac{1+4q}{n^2} + O(n^{-4}) \right] \text{ for } n \rightarrow \infty, \end{aligned} \quad (37)$$

The relations (33) and (34) may now be used in order to calculate the commutator and the sum of their squares for the operators $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$: We saw already above that (for an irreducible representation) these quantities have to be diagonal in the number basis $|k, n\rangle$! We just have to rewrite the r.h. sides of the Eqs. (33) and (34) in terms of the diagonal operators K_3 and Q . The results are

$$[\widehat{\cos \varphi}, \widehat{\sin \varphi}] = \frac{1}{4i} \frac{K_3^2 - 1 + 2Q(2K_3^2 - 1)}{K_3(K_3^2 - 1)^2} \quad (38)$$

and

$$\widehat{\cos \varphi}^2 + \widehat{\sin \varphi}^2 = \frac{1}{4} \left[\frac{4K_3^4 - 7K_3^2 + 3}{(K_3^2 - 1)^2} + Q \frac{4K_3^4 - 3K_3^2 + 1}{K_3^2(K_3^2 - 1)^2} \right]. \quad (39)$$

The r.h. sides of these two operator relations seem to have a problem if K_3 has the eigenvalue 1, i.e. for $k = 1, n = 0$. However, this is not so: if one sandwiches those r.h. sides between $\langle k, n = 0 |$ and $|k, n = 0 \rangle$ then all the factors $k - 1$ in the nominator and the numerator cancel nicely and one gets results according to Eqs. (33) and (34)!

For very large n we get from Eq. (38) the correspondence limit

$$\langle k, n | [\widehat{\cos \varphi}, \widehat{\sin \varphi}] | k, n \rangle = \frac{1}{4i} \frac{1 + 4q}{n^3} + O(n^{-5}) \text{ for } n \rightarrow \infty . \quad (40)$$

I would like to point out that phase operators associated with the Lie algebra of the group $SO^\uparrow(1, 2)$ different from those above have been discussed by other authors [13, 14, 15, 16, 10].

3 Matrix elements with respect to coherent states

Next I discuss some properties of coherent states. Contrary to the conventional coherent states (i.e. the eigenstates of the Bose annihilation operator associated with the harmonic oscillator, see e.g. the reviews [17, 18, 19] and the modern exposition [20]) there are several inequivalent ways [18] [21] to define coherent states related to the group $SO^\uparrow(1, 2)$ or $SU(1, 1)$ (see also the Refs. [22]). For our purposes the definition

$$K_- |k, z\rangle = z |k, z\rangle , \quad z = \rho e^{i\phi} \in \mathbb{C} , \quad \rho = |z| . \quad (41)$$

seems to be an interesting one, at least theoretically. About the possibility of their experimental realization I have nothing to say! The states (41), introduced by Barut and Girardello [11], have widely been discussed in the literature concerned with quantum optical applications of the group $SU(1, 1)$ [16, 23] and most of the general results I shall mention in the following are well-known.

Using the property (18) we get

$$\langle k, n | k, z \rangle = \frac{1}{\bar{\omega}^n} \left[\frac{\Gamma(2k)}{n! \Gamma(2k + n)} \right]^{1/2} z^n \langle k, n = 0 | k, z \rangle , \quad (42)$$

($\bar{\omega}$: compl. conj. of ω) so that

$$\langle k, z | k, z \rangle = \sum_{n=0}^{\infty} = \langle k, z | k, n \rangle \langle k, n | k, z \rangle \quad (43)$$

$$\begin{aligned}
&= \Gamma(2k) |\langle k, n=0 | k, z \rangle|^2 \sum_{n=0}^{\infty} \frac{\rho^{2n}}{n! \Gamma(2k+n)} \\
&= \Gamma(2k) |\langle k, n=0 | k, z \rangle|^2 \rho^{1-2k} I_{2k-1}(2\rho) ,
\end{aligned}$$

where

$$I_{\nu}(x) = \left(\frac{x}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{2n} \quad (44)$$

is the usual modified Bessel function of the first kind [24] which has the asymptotic expansion

$$I_{\nu}(x) \asymp \frac{e^x}{\sqrt{2\pi x}} \left[1 - \frac{4\nu^2 - 1}{8x} + 2 \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{16^2 x^2} + O(x^{-3}) \right] \quad (45)$$

for $x \rightarrow +\infty$.

If $\langle k, z | k, z \rangle = 1$ we have

$$|\langle k, n=0 | k, z \rangle|^2 \equiv |C_z|^2 = \frac{\rho^{2k-1}}{\Gamma(2k) I_{2k-1}(2\rho)} . \quad (46)$$

Choosing the phase of C_z appropriately and absorbing the phase ω into a redefinition of z we finally get the expansion

$$|k, z\rangle = \frac{\rho^{k-1/2}}{\sqrt{I_{2k-1}(2\rho)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n! \Gamma(2k+n)}} |k, n\rangle . \quad (47)$$

Notice that $|k, z=0\rangle = |k, n=0\rangle$.

Two different coherent states are not orthogonal:

$$\begin{aligned}
\langle k, z_2 | k, z_1 \rangle &= \sum_{n=0}^{\infty} \langle k, z_2 | k, n \rangle \langle k, n | k, z_1 \rangle \\
&= \frac{(\bar{z}_2 z_1)^{1/2-k} |z_2 z_1|^{k-1/2} I_{2k-1}(2\sqrt{\bar{z}_2 z_1})}{\sqrt{I_{2k-1}(2|z_2|) I_{2k-1}(2|z_1|)}} .
\end{aligned} \quad (48)$$

They are complete, however, in the sense that, with $z = \rho e^{i\phi}$, we have the relation

$$\begin{aligned}
\frac{2}{\pi} \int_0^{\infty} d\rho \rho K_{2k-1}(2\rho) I_{2k-1}(2\rho) \int_0^{2\pi} d\phi \langle k, n | k, \rho e^{i\phi} \rangle \langle k, \rho e^{i\phi} | k, n \rangle \\
= \langle k, n | k, n \rangle = 1
\end{aligned} \quad (49)$$

because [25]

$$\int_0^\infty d\rho \rho^{2(n+k)} K_{2k-1}(2\rho) = \frac{1}{4} n! \Gamma(2k+n) .$$

Here $K_\nu(x)$ is the modified Bessel function of the third kind [24].

The following expectation values are associated with the states $|k, z\rangle$:

$$\langle K_3 \rangle_{k,z} \equiv \langle k, z | K_3 | k, z \rangle = k + \rho b_k(\rho) , \quad (50)$$

$$b_k(\rho) = \frac{I_{2k}(2\rho)}{I_{2k-1}(2\rho)} , \quad (51)$$

$$\langle K_3^2 \rangle_{k,z} = k^2 + \rho^2 + \rho b_k(\rho) , \quad (52)$$

so that

$$(\Delta K_3)_{k,z}^2 = \rho^2(1 - b_k^2(\rho)) + (1 - 2k)\rho b_k(\rho) . \quad (53)$$

For $\rho \rightarrow 0$ one has

$$b_k(\rho) \rightarrow \rho \text{ for } \rho \rightarrow 0 \quad (54)$$

and for very large ρ we get from Eq. (45) that

$$b_k(\rho) \asymp 1 + \frac{1-4k}{4\rho} + \frac{3-16q}{32\rho^2} + O(\rho^{-3}) \quad (55)$$

and therefore for the r.h. sides of the relations (50) and (53) the leading terms

$$\langle K_3 \rangle_{k,z} \asymp \rho + \frac{1}{4} + O(\rho^{-1}) , \quad (\Delta K_3)_{k,z}^2 \asymp \frac{1}{2}\rho + O(\rho^{-1}) \text{ for } \rho \rightarrow +\infty . \quad (56)$$

This, together with the probability

$$|\langle k, n | k, z \rangle|^2 = \frac{\rho^{2(n+k)-1}}{n! \Gamma(2k+n) I_{2k-1}(2\rho)} \asymp 2\sqrt{\pi} \frac{\rho^{2(n+k)-1/2}}{n! \Gamma(2k+n)} e^{-2\rho} \quad (57)$$

shows that the corresponding distribution for large ρ is not Poisson-like!

In addition we have the following expectation values:

$$\langle K_1 \rangle_{k,z} = \frac{1}{2}(\bar{z} + z) = \rho \cos \phi , \quad \langle K_2 \rangle_{k,z} = \frac{1}{2i}(\bar{z} - z) = -\rho \sin \phi , \quad (58)$$

$$\langle K_1^2 \rangle_{k,z} = \rho^2 \cos^2 \phi + \frac{1}{2} \langle K_3 \rangle_{k,z} , \quad \langle K_2^2 \rangle_{k,z} = \rho^2 \sin^2 \phi + \frac{1}{2} \langle K_3 \rangle_{k,z} , \quad (59)$$

$$\langle K_1^2 + K_2^2 \rangle_{k,z} = \rho^2 + \langle K_3 \rangle_{k,z} ,$$

$$(\Delta K_1)_{k,z}^2 = (\Delta K_2)_{k,z}^2 = \frac{1}{2} \langle K_3 \rangle_{k,z} . \quad (60)$$

In deriving the relations (59) the equality $K_-K_+ = K_+K_- + 2K_3$ (Eqs. (13)) has been used.

Comparing Eqs. (6),(7) and (58) we see the close relationship between the expectation values $\langle K_i \rangle_{k,z}$, $i = 1, 2$, and their classical counterparts. This supports the above choice (41) as coherent states. Further support comes from their property to realize the minimal uncertainty relation: From the third commutator in Eqs. (12) we get the general inequality

$$(\Delta K_1)^2(\Delta K_2)^2 \geq \frac{1}{4}|\langle K_3 \rangle|^2. \quad (61)$$

The relations (60) show that the coherent states (41) realize the minimum of the uncertainty relation (61). One can, of course, extend the discussion to associated squeezed states [16, 23]. Their use in the present context will be of considerable interest.

For matrix elements of the operators $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$ we get (from Eqs. (29), (30) and (47))

$$\begin{aligned} \langle k, n | \widehat{\cos \varphi} | k, z \rangle &= \frac{\rho^{k-1/2}}{4\sqrt{I_{2k-1}(2\rho)}} \left[\frac{z^{n-1}}{\sqrt{(n-1)!\Gamma(2k+n-1)}} f_n^{(k)} \right. \\ &\quad \left. + \frac{z^{n+1}}{\sqrt{(n+1)!\Gamma(2k+n+1)}} f_{n+1}^{(k)} \right], \end{aligned} \quad (62)$$

$$\begin{aligned} \langle k, n | \widehat{\sin \varphi} | k, z \rangle &= \frac{\rho^{k-1/2}}{4i\sqrt{I_{2k-1}(2\rho)}} \left[-\frac{z^{n-1}}{\sqrt{(n-1)!\Gamma(2k+n-1)}} f_n^{(k)} \right. \\ &\quad \left. + \frac{z^{n+1}}{\sqrt{(n+1)!\Gamma(2k+n+1)}} f_{n+1}^{(k)} \right]. \end{aligned} \quad (63)$$

From them and (47) we obtain the expectation values

$$\langle \widehat{\cos \varphi} \rangle_{k,z} = \frac{\bar{z} + z}{2\rho} \frac{g^{(k)}(\rho)}{I_{2k-1}(2\rho)} = \cos \phi \frac{g^{(k)}(\rho)}{I_{2k-1}(2\rho)}, \quad (64)$$

$$\langle \widehat{\sin \varphi} \rangle_{k,z} = \frac{-\bar{z} + z}{2i\rho} \frac{g^{(k)}(\rho)}{I_{2k-1}(2\rho)} = \sin \phi \frac{g^{(k)}(\rho)}{I_{2k-1}(2\rho)}, \quad (65)$$

$$g^{(k)}(\rho) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\rho^{2(n+k)}}{n! \Gamma(2k+n)} \left(\frac{1}{n+k} + \frac{1}{n+k+1} \right). \quad (66)$$

One has

$$g^{(k)}(\rho) = \frac{1}{2} \int_0^{2\rho} du I_{2k-1}(u) + \frac{1}{8\rho^2} \int_0^{2\rho} du u^2 I_{2k-1}(u) . \quad (67)$$

The right hand side may be expressed by a combination of modified Bessel and Lommel functions [26]. For large ρ one obtains [27]

$$\frac{g^{(k)}(\rho)}{I_{2k-1}(2\rho)} \simeq 1 - \frac{1}{4\rho} + O(\rho^{-2}) \text{ for large } \rho \quad (68)$$

which again gives the expected correspondence principle limits for $\langle \widehat{\cos \varphi} \rangle_{k,z}$ and $\langle \widehat{\sin \varphi} \rangle_{k,z}$. Notice that the ratio $\langle \widehat{\sin \varphi} \rangle_{k,z} / \langle \widehat{\cos \varphi} \rangle_{k,z} = \tan \phi$ is independent of k and ρ !

Other expectation values like $\langle \widehat{\cos \varphi}^2 \rangle_{k,z}$ etc. may be calculated by observing that

$$\langle \widehat{\cos \varphi}^2 \rangle_{k,z} = \sum_n \langle k, z | \widehat{\cos \varphi} | k, n \rangle \langle k, n | \widehat{\cos \varphi} | k, z \rangle = \sum_n |\langle k, n | \widehat{\cos \varphi} | k, z \rangle|^2$$

etc. The resulting expressions are not simple.

The operators $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$ are bounded self-adjoint operators (see their definition (24) and Eqs. (33)). They have a continuous spectrum covering the interval $[-1, +1]$ for $k \geq 0.5$. The last assertion follows from Eqs. (64) and (65) together with a numerical analysis of the ratio $g^{(k)}(\rho)/I_{2k-1}(2\rho)$ which shows that ratio to be < 1 for all finite ρ if $k \geq 0.5$. That is not so, e.g. for $k = 0.25$ for which $g^{(k)}(\rho)/I_{2k-1}(2\rho)$ becomes larger than 1 for certain ρ -values. The bound $k \geq 0.5$ is definitely a sufficient one because $g^{(k)}(\rho)/I_{2k-1}(2\rho) \leq 1$ numerically also for $k = 0.4$. These - not yet very detailed - numerical results and the relation (68) imply that at least for $k \geq 0.5$ we have

$$\sup_z |\langle \widehat{\cos \varphi} \rangle_{k,z}| = \sup_z |\langle \widehat{\sin \varphi} \rangle_{k,z}| = 1 \text{ for } k \geq 0.5 \quad (69)$$

from which the support of the spectrum follows [28]. Thus, for the groups $SO^\uparrow(1, 2)$ and $SU(1, 1)$ which have $k = 1$ and $k = 1/2$ respectively as their lowest k -values we are on the safe side.

The reason for a lower bound for k in order to ensure that the operators $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$ have the right spectrum can be seen qualitatively from the

first ($n = 0$) term in the series (66). That term (divided by I_{2k-1}) diverges for $k \rightarrow 0$.

The ansatz $|\mu\rangle = \sum_{n=0}^{\infty} a_n |k, n\rangle$ for the improper “eigenfunctions” of $\widehat{\cos\varphi}$ with “eigenvalues” μ , $\widehat{\cos\varphi}|\mu\rangle = \mu|\mu\rangle$, leads to the recursion formula $a_{n+1} = (4\mu a_n - f_n^{(k)} a_{n-1})/f_{n+1}^{(k)}$, $f_0^{(k)} = 0$, which allows to express the a_n by μ, a_0 and the $f_n^{(k)}$.

4 Interpretation in terms of “observables”

Let me start with a very important remark which is crucial for the following physical interpretation of the results in the last two sections:

In the case of a *group theoretical quantization* of a classical system it is *not* required that the generators of the corresponding Lie algebra are expressible in terms of pairs (\hat{q}_j, \hat{p}_j) , $j = 1, 2, \dots$ of canonical operators or in terms of the associated annihilation and creation operators $a_j = (\hat{q}_j + i\hat{p}_j)/2$, $a_j^\dagger = (\hat{q}_j - i\hat{p}_j)/2$. A well-known example is the angular momentum: Its components \hat{l}_j , $j = 1, 2, 3$, are expressible in terms of the 3 pairs (\hat{q}_j, \hat{p}_j) but this is not essential at all for the quantum theory of the angular momentum. That can be constructed from the single property that the 3 operators \hat{l}_j generate the Lie algebra of the group $SO(3)$ or that of its covering group $SU(2)$ the representations of which allow for half-integer spins not expected from semi-classical arguments! Classically the Poisson brackets of the 3 components $l_1 = q_2 p_3 - q_3 p_2, l_2 = \dots$ fulfill the $SO(3)$ Lie algebra, too. However, this applies to the orbital angular momentum only.

Let us apply a similar analysis to the quantities (1)-(8) from the introduction: On the classical level the actual “observables” for a complete description of the interference pattern are the two intensities I_1, I_2 and the 3 quantities P_j , $j = 1, 2, 3$. The latter are not independent because

$$P_1^2 + P_2^2 = P_3^2, \quad \cos\varphi = P_1/P_3, \quad \sin\varphi = -P_2/P_3. \quad (70)$$

The individual phases φ_j , $j = 1, 2$, do not have to be known, only their difference $\varphi = \varphi_2 - \varphi_1$. All those observables may be determined by measuring the 6 intensities I_1, I_2 and w_a , $a = 3, 4, 5, 6$. I_1 and I_2 may be measured by shielding one of the two interfering waves completely. Notice that $P_3^2 = p^2$ may also be obtained from $(I_1 + I_2)^2$:

$$2p^2 = (I_1 + I_2)^2 - I_1^2 - I_2^2, \quad I_1 + I_2 = w_3 + w_4 = w_5 + w_6. \quad (71)$$

We are, of course, assuming a very idealized situation, namely that all the quantities are stationary in time, that there is no absorption in the $\lambda/2$ and $\lambda/4$ phase shifters etc. etc.

As a mere pedagogical example take Young's interference experiment with a completely coherent source the light of which has a definite frequency and falls onto two (equal) pinholes as origins for the two waves which interfere on a screen on the other side of the pinholes where the interference pattern is recorded [29]. At a certain position on the screen one has the intensity w_3 . When shielding pinhole 1 one observes the intensity $w_3 = I_2$ and when shielding pinhole 2 one observes I_1 . Placing an ideal $\lambda/2$ phase shifter behind pinhole 1 or 2 yields w_4 . Using instead a $\lambda/4$ shifter gives w_5 and placing in addition a $\lambda/2$ shifter in behind the other hole yields w_6 .

Let us turn now to the quantum theory of the system [30]: Crucial new features are that the 3 self-adjoint operators $K_j = \hat{P}_j$ do no longer commute but obey the Lie algebra (12) and that K_3 which corresponds to the classical quantity $\sqrt{I_1 I_2}$ is not a pure number operator! K_3 has the eigenvalues $n + k$, $n = 0, 1, \dots$ and the value of $k > 0$ depends on the type of global group one associates with the Lie algebra (12). As already mentioned before: for the group $SO^\uparrow(1, 2)$ one has $k = 1$, for $SU(1, 1)$ $k = 1/2$ etc. (for more details see Ref. [7]). If we define the number operator

$$N = K_3 - kI, \quad I: \text{unity operator}, \quad (72)$$

then we have the commutators

$$[N, K_1] = iK_2, \quad [N, K_2] = -iK_1, \quad [K_1, K_2] = -i(N + k \cdot I), \quad k > 0, \quad (73)$$

which is a Lie algebra formed by K_1, K_2 and N with a central extension [31] characterized by k . In the following the unity operator I is no longer exhibited explicitly.

Instead of the classical relation (70) we now have from Eq. (17) for an irreducible representation

$$K_1^2 + K_2^2 = K_3^2 + q = N^2 + (2N + 1)k \quad (74)$$

which requires a corresponding modification of the NFM-type analysis for small numbers n of the quanta where the non-vanishing k makes itself felt! In a quantum optical experiment where mean photon *numbers* $\bar{n}_a = \langle |N_a| \rangle$,

$a = 3, 4, 5, 6$, are recorded instead of the intensities w_a one expects the relations (6) and (7) to have the quantum expectation value correspondences

$$\bar{n}_3 - \bar{n}_4 = 4 \langle |K_1| \rangle, \quad (75)$$

and

$$\bar{n}_5 - \bar{n}_6 = 4 \langle |K_2| \rangle, \quad (76)$$

where $|\rangle$ is a general state and the N_a some appropriate number operators corresponding to the quantum version of the w_a .

However, because of Eq. (74) one can no longer expect a simple correspondence relation for the classical equality (8). For a state $|\rangle$ in an irreducible representation one has from Eq. (74)

$$\langle |K_1^2 + K_2^2| \rangle = \langle |N^2| \rangle + k(2\langle |N| \rangle + 1). \quad (77)$$

On the other hand one expects

$$\langle |K_1^2| \rangle = \frac{1}{16} \langle |(N_3 - N_4)^2| \rangle, \quad (78)$$

$$\langle |K_2^2| \rangle = \frac{1}{16} \langle |(N_5 - N_6)^2| \rangle, \quad (79)$$

In addition one might get information about $\langle |K_3^2| \rangle$ from the quantum version of Eq. (71):

$$\begin{aligned} 2 \langle |K_3^2| \rangle &= \langle |(N_3 + N_4)^2| \rangle - \langle |N_1^2| \rangle - \langle |N_2^2| \rangle, \\ 2 \langle |K_3^2| \rangle &= \langle |(N_5 + N_6)^2| \rangle - \langle |N_1^2| \rangle - \langle |N_2^2| \rangle. \end{aligned} \quad (80)$$

As an additional general information one has the inequality

$$\Delta K_1 \cdot \Delta K_2 \geq \frac{1}{2} \langle |K_3| \rangle. \quad (81)$$

In this elementary approach there appears to be no obvious way to measure $\langle |K_3 = N + k| \rangle$ itself directly for an arbitrary state $|\rangle$.

Let us see next how these more general considerations look for the special states $|k, n\rangle$ and $|k, z\rangle$ from sections 2 and 3. I have nothing to say about their possible experimental realizations!

In case of the number eigenstates $|k, n\rangle$ we have the relations (19) for the expectation values of K_1 and K_2 and (32) for those of $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$, i.e.

there is no interference pattern at all! According to Eqs. (75) and (76) this should correspond to the relations $\bar{n}_4 = \bar{n}_3, \bar{n}_6 = \bar{n}_5$. The associated fluctuations for $K_j, j = 1, 2$, are given by Eq. (20) and their sum by Eq. (23) which relates those fluctuations to the exact eigenvalues $n + k$ of K_3 and $(n + k)^2$ of K_3^2 . Supplemented with the relations (80) this should allow for a determination of n and, very important, of k . The value of the non-vanishing positive parameter k which characterizes the properties of the ground state – see e.g. Eq. (36) – does not appear to be determined by general considerations alone, at least not to me. So it should be determined or confirmed experimentally, like in the case of the ground state for the harmonic oscillator.

If one knows n and k then one can infer the fluctuations of $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$ from Eq. (33).

In the case of the coherent states $|k, z\rangle$ the situation is quite pleasant – theoretically: With the help of Eqs. (75), (76) and (58) one determines $\rho \cos \varphi$ and $\rho \sin \varphi$. ρ itself can in addition be obtained by combining the relations (78), (79) and (59). These determine $\langle K_3 \rangle_{k,z}$ and, because of Eq. (74), $\langle K_3^2 \rangle_{k,z}$ as well. The results may be cross-checked by means of the relations (80)!

It appears that the emerging picture of describing the quantum theory of moduli and phases in interference experiments in terms of the Lie algebra of the group $SO^\uparrow(1, 2)$ or one of its (infinitely many) covering groups is quite promising and may lead to progress in that field!

Up to now I have not specified the concrete form of the Hilbert space, the operators $K_i, i = 1, 2, 3$, and the eigenfunctions $|k, n\rangle$ and $|k, z\rangle$. Several interesting examples may be found in Ref. [7] and the coherent states $|k, z\rangle$ can be constructed explicitly from the concrete form of the operators K_- given there. Many examples are also contained in numerous of the Refs. [22].

5 Interpretation in terms of creation and annihilation operators: phase operators for the harmonic oscillator

In the following I discuss three – well-known – examples in which the 3 Lie algebra operators K_j are expressed in terms of the “beloved” bosonic creation

and annihilation operators a^+ and a with

$$[a, a^+] = 1 \quad (82)$$

which act on a n-quanta state $|n\rangle$ as

$$a^+|n\rangle = \sqrt{n+1}|n+1\rangle \quad a|n\rangle = \sqrt{n}|n-1\rangle, \quad N|n\rangle = n|n\rangle, \quad N = a^+a. \quad (83)$$

Comparing these relation with the Eqs. (14)-(16) suggests the ansatz

$$K_+ = a^+\sqrt{N+2k}, \quad K_- = \sqrt{N+2k}a, \quad K_3 = N+k. \quad (84)$$

Using the commutation relation (82) it is easy to verify that the operators (84) have the properties (14)-(16) and form the Lie algebra (13). We know from our general discussion that the operators (84) are self-adjoint in the Fock space of the harmonic oscillator provided $k \geq 0.5$. The operators $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$ here take the explicit forms

$$\begin{aligned} \widehat{\cos \varphi} &= \frac{1}{4} \frac{1}{N+k} [8a^+\sqrt{N+2k} + \sqrt{N+2k}a] + \\ &+ \frac{1}{4} [a^+\sqrt{N+2k} + \sqrt{N+2k}a] \frac{1}{N+k}, \end{aligned} \quad (85)$$

$$\begin{aligned} \widehat{\sin \varphi} &= \frac{i}{4} \frac{1}{N+k} [a^+\sqrt{N+2k} - \sqrt{N+2k}a] + \\ &+ \frac{i}{4} [a^+\sqrt{N+2k} - \sqrt{N+2k}a] \frac{1}{N+k}. \end{aligned} \quad (86)$$

Again, according to our general results these operators are self-adjoint with a spectrum in the interval $[-1, +1]$ provided $k \geq 0.5$ (as mentioned above: this is a sufficient lower bound which was found by numerical methods. For $k = 0.25$ the same analysis shows that the spectrum exceeds that interval!) For $k = 1/2$ or $k = 1$ we are on the safe side and therefore the operators are decent self-adjoint $\cos \varphi$ and $\sin \varphi$ operators for the harmonic oscillator! Actually there is no obvious reason up to now to identify the modulus operator K_3 with the Hamiltonian $H = N + 1/2$ of the harmonic oscillator though one might be inclined to do so. The relations (25) and (26) here follow immediately from $[N, a^+] = a^+$ and $[N, a] = -a$.

If we compare the expressions (85) and (86), e.g. for $k = 1/2$ or $k = 1$, with the questionable operators previously suggested by Dirac (and Heitler),

$$\widehat{\cos \varphi}_D = \frac{1}{2}(aN^{-1/2} + N^{-1/2}a^+), \quad \widehat{\sin \varphi}_D = \frac{1}{2i}(aN^{-1/2} - N^{-1/2}a^+), \quad (87)$$

or by Susskind and Glogower,

$$\widehat{\cos \varphi}_{SG} = \frac{1}{2}[(N+1)^{-1/2} a + a^+ (N+1)^{-1/2}], \quad (88)$$

$$\widehat{\sin \varphi}_{SG} = \frac{1}{2i}[(N+1)^{-1/2} a - a^+ (N+1)^{-1/2}], \quad (89)$$

one sees immediately by which kind of approximations of the expressions (85) and (86) one arrives at the disputable operators (87)-(89)! In order to do so it is helpful to use the relations

$$f(N) a^+ = a^+ f(N+1), \quad a f(N) = f(N+1) a \quad (90)$$

for appropriate functions $f(N)$ of the operator N , here applied to $f(N) = (N+k)^{-1}$ (provided $k > 0$). They are a consequence of the basic relations (83). We then get for the expressions (85) and (86)

$$\widehat{\cos \varphi} = \frac{1}{2}[a^+ F_k(N) + F_k(N) a], \quad (91)$$

$$\widehat{\sin \varphi} = \frac{i}{2}[a^+ F_k(N) - F_k(N) a], \quad (92)$$

$$F_k(N) = \frac{1}{2} \sqrt{N+2k} \left(\frac{1}{N+k} + \frac{1}{N+k+1} \right).$$

As to the history of the operators (87)-(89) see the reviews [1].

From the expressions (91), (92) and the relation (90) one gets, for instance,

$$[\widehat{\cos \varphi}, \widehat{\sin \varphi}] = \frac{i}{2} \{ [N+1] F_k^2(N) - N F_k^2(N-1) \} \quad (93)$$

which is the operator version of the relation (34).

It is interesting to have a look at the expectation values of the operators K_j and $\widehat{\cos \varphi}$ and $\widehat{\sin \varphi}$ with respect to the conventional coherent states $|\alpha\rangle$ defined by [23]

$$a |\alpha\rangle = \alpha |\alpha\rangle, \quad \alpha = r e^{i\beta} \in \mathbb{C}, \quad |\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2} |n\rangle. \quad (94)$$

We get

$$\langle \alpha | K_1 | \alpha \rangle = r \cos \beta \langle \alpha | \sqrt{N+2k} | \alpha \rangle, \quad (95)$$

$$\langle \alpha | K_2 | \alpha \rangle = -r \sin \beta \langle \alpha | \sqrt{N+2k} | \alpha \rangle, \quad (96)$$

$$\langle \alpha | K_3 | \alpha \rangle = r^2 + k, \quad (97)$$

where

$$\langle \alpha | \sqrt{N+2k} | \alpha \rangle = h_1^{(k)}(r^2) = e^{-r^2} \sum_{n=0}^{\infty} \sqrt{n+2k} \frac{r^{2n}}{n!}. \quad (98)$$

From the expressions (91)-(92) one gets

$$\langle \alpha | \widehat{\cos \varphi} | \alpha \rangle = \cos \beta h_2^{(k)}(r), \quad \langle \alpha | \widehat{\sin \varphi} | \alpha \rangle = \sin \beta h_2^{(k)}(r), \quad (99)$$

where now

$$h_2^{(k)}(r) = \frac{r}{2} e^{-r^2} \sum_{n=0}^{\infty} \sqrt{n+2k} \left(\frac{1}{n+k} + \frac{1}{n+k+1} \right) \frac{r^{2n}}{n!}. \quad (100)$$

Due to the factor $\sqrt{n+2k}$ inside the sums (98) and (100) the functions $h_j^{(k)}$, $j = 1, 2$, do not seem to be summable in an elementary way. Numerical inspections show that $h_2^{(k)} \leq 1$ for $k \geq 0.5$ with $h_2^{(k)}(r) \rightarrow 1$ for $r \rightarrow \infty$ as it should be.

Notice the differences between the relations (58) and (50) on the one hand and the relations (95)-(97) on the other, but also notice that - like in the case of the coherent states $|k, z\rangle$ with the expectation values (64) and (65) - the expectation values (99) are proportional to $\cos \beta$ and $\sin \beta$ and the common function $h_2^{(k)}(r)$ which drops out of the ratio yielding $\cot \beta$ or $\tan \beta$, independent of k and r !

It is obvious that the realization (84) of the $SO^\dagger(1, 2)$ Lie algebra does not represent an interference situation we originally started from. It rather represents the self-adjoint Lie algebra generators in the state space of the harmonic oscillator and provides self-adjoint operators for sin and cos. For $k = 1/2$ the operator K_3 coincides with the Hamiltonian of the harmonic oscillator.

Thus, the expressions (85) and (86) appear to present a whole class of solutions for that old problem of quantizing the pair “modulus” and “phase” in a satisfactory manner for the harmonic oscillator. The elements of the class are specified by the parameter k which characterizes an irreducible unitary representation from the positive discrete series of the group $SO^\dagger(1, 2)$ or of one of its covering groups!

The expressions (84) form a highly non-linear realization of the Lie algebra in terms of one pair of creation and annihilation operators. In the literature it has been called a “Holstein-Primakoff realization” [32]. These

two were among the first to express Lie algebra generators of $SU(2)$ in terms of creation and annihilation operators [33].

The following combination of a and a^+ also fulfills the Lie algebra (13) [18, 22]:

$$K_+ = \frac{1}{2}(a^+)^2, \quad K_- = \frac{1}{2}a^2, \quad K_3 = \frac{1}{2}(a^+a + 1/2). \quad (101)$$

As K_- annihilates $|n=0\rangle$ as well as $|n=1\rangle$ the representation decomposes into one with states of even quanta and one with odd quanta. For even one has $k = 1/4$ and for odd $k = 3/4$. Although the representation (101) is discussed quite frequently in the quantum optics literature [22] it, too, does not appear to be related to an interfering system as discussed above.

A realization of the Lie algebra (12) or (13) in terms of a pair $a_j^+, a_j, j = 1, 2$, of creation and annihilation operators has been known for a long time [34, 18, 7] and has been discussed frequently in quantum optics in the context of 2-mode problems [22], especially in connection with squeezed states.

The operators

$$K_3 = \frac{1}{2}(a_1^+a_1 + a_2^+a_2 + 1), \quad K_+ = a_1^+a_2^+, \quad K_- = a_1a_2 \quad (102)$$

obey the commutation relations (13) and the tensor product $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc}$ of the two harmonic oscillator Hilbert spaces contains all the irreducible unitary representations of the group $SU(1,1)$ (for which $k = 1/2, 1, 3/2, \dots$) in the following way: Let $|n_i\rangle_j, n_j \in \mathbb{N}_0, j = 1, 2$, be the eigenstates of the number operators N_j generated by a_j^+ from the oscillator ground states. Then each of those two subspaces of $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc} = \{|n_1\rangle_1 \otimes |n_2\rangle_2\}$ with fixed $|n_1 - n_2| \neq 0$ contains an irreducible representation with $k = 1/2 + |n_1 - n_2|/2 = 1, 3/2, 2, \dots$ and for which the number n in the eigenvalue $n + k$ is given by $n = \min\{n_1, n_2\}$.

For the “diagonal” case $n_2 = n_1$ one gets the unitary representation with $k = 1/2$.

Again, at first sight the operators (102) do not correspond to the interference (2) I started from, if one adheres to the correspondence $a_j \sim A_j, a^+ \sim \bar{A}_j$. This can be seen immediately: The operator $K_1 = (a_1a_2 + a_1^+a_2^+)/2$ corresponds to the classical quantity $(A_1A_2 + \bar{A}_1\bar{A}_2)/2$, not to $(A_1\bar{A}_2 + \bar{A}_1A_2)/2$, i.e. it corresponds to $\sqrt{I_1I_2} \cos(\varphi_1 + \varphi_2)$ with the sum of the angle φ_j , not their difference! However, one should be careful here: if one of the angles, e.g. φ_1 , vanishes the two situations are not so different. This situation corresponds to a certain fixed “gauge” of the angles, whereas the difference

$\varphi = \varphi_2 - \varphi_1$ is “gauge invariant”: In addition, K_3 in Eq. (102) is given by the sum of the energies of the 2 modes, whereas p in Eq. (6) is given by the square root of the product! Nevertheless the realization (102) has been used for other interfering devices [35] and our general analysis may be useful for those systems, too. It also might be of interest if $n_1 = n_2$, i.e. if the energies of the 2 modes are the same as is the case classically if the two pinholes in Young’s experiment have the same size, so that $I_2 = I_1$.

All these considerations do not impede our original analysis of the observables in an interference pattern, e.g. in the case of the NFM set-up. I repeat again that a group theoretical quantization like the one above does *not* suppose that there is a “deeper” conventional canonical structure in terms of the usual q and p of a and a^+ . It claims to provide an appropriate quantum framework for topologically nontrivial symplectic manifolds like (11) by itself. Quantum optics (or other quantum interference phenomena) may very well be able to test such claims experimentally. In addition it may test the identification (24) as an operator version of $\cos \varphi$ and $\sin \varphi$. That definition is a new ansatz *within* - not a basic ingredient *of* - the group theoretical quantization scheme.

It even may turn out that the operators K_1 and K_2 which correspond to the classical observables $\sqrt{I_1 I_2} \cos \varphi$ and $-\sqrt{I_1 I_2} \sin \varphi$ are actually more convenient to use than the operators (24). The future will tell.

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