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The strong longitudinal expansion of the reaction zone formed in relativistic heavy-ion collisions is found to significantly reduce the spatially averaged pion phase-space density, compared to naive estimates based on thermal distributions. This has important implications for data interpretation and leads to larger values for the extracted pion chemical potential at kinetic freeze-out.

The phase-space density of mesons produced in ultra-relativistic heavy ion collisions is an interesting quantity. Its spatial average $\langle f\rangle(\boldsymbol{p})$, taken over the effective emission region, can be measured by combining the single-particle momentum spectrum with the "homogeneity volume" extracted from Bose-Einstein correlation measurements [1]. Bertsch's original formula [1] was refined in [2] to exclude contributions from long-lived resonances decaying far outside the collision fireball, and a relativistically covariant derivation was given in [3–5]. With these improvements Bertsch's formula reads

$$\langle f\rangle(\boldsymbol{p}) = \frac{\sqrt{\lambda(\boldsymbol{p})}\; dn/(dy\, p_\perp dp_\perp\, d\phi)}{(E_p/\pi^{3/2}) R_s(\boldsymbol{p})\sqrt{R_o^2(\boldsymbol{p})R_l^2(\boldsymbol{p}) - R_{ol}^4(\boldsymbol{p})}}\,. \quad (1)$$

Here λ is the intercept at vanishing relative momentum of the Bose-Einstein correlation function, and the factor $\sqrt{\lambda(\boldsymbol{p})}$ corrects for contributions from decays of long-lived resonances. The numerator is the (thus corrected) invariant momentum spectrum, and the denominator contains the homogeneity volume calculated from the Hanbury Brown-Twiss (HBT) radii extracted from the Bose-Einstein correlation function [5].

The strongest motivation for measuring the pion phase-space density at *freeze-out* (i.e. on the last-scattering surface) comes from the search for multi-boson symmetrization effects [6]. Analyses of the hadron yields, spectra, and two-particle correlations measured in Pb+Pb collisions at the CERN SPS [7] and in Au+Au collisions at RHIC [8] show that freeze-out occurs in two stages: chemical freeze-out, at which the particle yields decouple, happens first, reflecting a temperature of around 170 MeV, whereas kinetic or thermal freeze-out takes place much later, at temperatures around 100–120 MeV. Between these two points the system cools at constant particle numbers which requires the build-up of a positive chemical potential. It has been speculated that for pions, with only 139 MeV rest mass, this chemical potential could approach the limit $\mu_\pi = m_\pi$ at which Bose condensation sets in. If this were true, measurements of the above type should exhibit large values of $\langle f\rangle(\boldsymbol{p})$ at small $\boldsymbol{p}$. Large phase-space densities also cause significant multi-boson symmetrization effects on the pion spectra [9] and Bose-Einstein correlation functions (see [10] and references therein), which lead to a reduction of the homogeneity volume extracted from the standard HBT correlation analysis [5]. This should cause an additional enhancement of $\langle f\rangle(\boldsymbol{p})$ as determined by Bertsch's formula (1), rendering it at the same time unreliable as an estimator of the real average phase-space density.

Previous analyses of heavy-ion data at the AGS [2] and SPS [3,4] concluded that there was no evidence for unusually large pion freeze-out phase-space densities at these collision energies. This conclusion was based on a comparison of $\langle f\rangle(\boldsymbol{p})$ from Eq. (1) with a Bose-Einstein distribution, yielding rough agreement when inserting a kinetic freeze-out temperature of around 100–120 MeV (as extracted by other methods [7]) and a small or vanishing pion chemical potential. In some cases [4,11] comparison was made with a transversely boosted Bose-Einstein distribution, in order to account for transverse flow effects. We here point out that all these comparisons neglected a strong reduction effect on the spatial averaging originating from the *longitudinal* expansion of the source. We show that for longitudinally expanding sources the spatially averaged phase-space density is smaller than its thermal value in the local rest frame, and that accounting for this effect is likely to considerably increase the extracted value of the pion chemical potential at kinetic freeze-out. In this Letter we concentrate on a theoretical exposition of the basic mechanism and leave a (re)analysis of existing data for later.

For a qualitative understanding of the effect under discussion, let us fix the momentum and focus on the averaging over position space. If the (thermalized) fireball expands, different parts moving with different velocities contribute to the production of particles with fixed momentum $\boldsymbol{p}$ with different rates, given by Bose-Einstein distributions *boosted* by the local flow velocity. The phase-space density at momentum $\boldsymbol{p}$ is then a function of position space, and its (weighted) spatial average can take any value between its spatial maximum and minimum; in general the $\boldsymbol{p}$-dependence of this spatial average will not be of Bose-Einstein form.

We illustrate this effect quantitatively within a specific model for the expanding source. We will assume a thermalized and longitudinally boost-invariant fireball. This simple model is expected to provide a good description of pions emitted near midrapidity at RHIC and LHC energies. We show that the calculated $p_\perp$-dependence of the spatially averaged phase-space density is flatter than a Bose-Einstein distribution with the assumed freeze-out temperature. The major contribution to this flattening arises from the *longitudinal* expansion which strongly *reduces* $\langle f\rangle(\boldsymbol{p})$ relative to the corresponding static Bose-Einstein distribution. Transverse flow introduces a weaker additional flattening by shifting weight from low to high transverse momenta $p_\perp$.

The phase-space density $f(t,\boldsymbol{x},\boldsymbol{p})$ is related to the source emission function $S(x,\boldsymbol{p})$ [5] by

$$f(t,\boldsymbol{x},\boldsymbol{p}) = \frac{(2\pi)^3}{E_p}\int_{-\infty}^{t} dt'\, S(t',\boldsymbol{x}+\boldsymbol{v}(t'-t),\boldsymbol{p})\,, \qquad (2)$$

where $\boldsymbol{v}=\boldsymbol{p}/E_p$ is the velocity corresponding to the momentum $\boldsymbol{p}$. The prefactor $(2\pi)^3/E_p$ ensures the standard normalization

$$\bar{N} = \int \frac{d^3x\, d^3p}{(2\pi)^3}\, f(t,\boldsymbol{x},\boldsymbol{p}) = \int \frac{d^3p}{E_p}\, d^4x\, S(x,\boldsymbol{p})\,. \qquad (3)$$

We take $\bar{N}$ as the average number of "directly emitted" pions, including those from the decays of shortlived resonances but excluding those from resonances with long lifetimes [2,3]. For simplicity, we follow Refs. [12,13] and assume that the change in shape and normalization of the thermal pion distribution at freeze-out caused by adding pions from shortlived resonance decays can be effectively absorbed by giving the pions a positive chemical potential. Although this cannot replace a full resonance decay calculation using the proper decay kinematics [14], it qualitatively reproduces the decay-induced low-$p_\perp$-enhancement of the measured pion spectra [12–14].

The space-averaged phase-space density, defined by [1]

$$\langle f\rangle(\boldsymbol{p}) = \frac{\int d^3x\, f^2(t,\boldsymbol{x},\boldsymbol{p})}{\int d^3x\, f(t,\boldsymbol{x},\boldsymbol{p})}\,, \qquad (4)$$

is brought into the form (1) by following the steps outlined in Sec. 3.4 of Ref. [5]. In order to include all "directly emitted" pions, the time t on the right hand side of Eq. (4) must be later than the time at which the last pion was emitted. Due to Liouville's theorem, the average phase-space density (4) remains constant after completion of the freeze-out process. This allows [4] to replace in (4) the spatial integrals at constant time t by integrals over the *freeze-out hypersurface* $\Sigma_\mathrm{f}(x)$ describing the last pion scattering or production points:

$$\langle f\rangle(\boldsymbol{p}) = \frac{\int_{\Sigma_\mathrm{f}} p\cdot d^3\sigma(x)\, f^2(x,\boldsymbol{p})}{\int_{\Sigma_\mathrm{f}} p\cdot d^3\sigma(x)\, f(x,\boldsymbol{p})}\,. \qquad (5)$$

If $\Sigma_\mathrm{f}(x)$ is parametrized by an $\boldsymbol{x}$-dependent freeze-out time $t_\mathrm{f}(\boldsymbol{x})$, then

$$\int_{\Sigma_\mathrm{f}} p\cdot d^3\sigma(x)\, f^n(t,\boldsymbol{x},\boldsymbol{p}) = E_p \int d^3x\, \left(1-\boldsymbol{v}\cdot\nabla_x\, t_\mathrm{f}(\boldsymbol{x})\right)\, f^n(t_\mathrm{f}(\boldsymbol{x}),\boldsymbol{x},\boldsymbol{p})\,. \qquad (6)$$

For the model emission function we take [15]

$$S(x,p) = \frac{m_\perp \cosh(y-\eta)}{(2\pi)^3}\, \frac{\delta(\tau-\tau_\mathrm{f})}{\exp\left[\frac{p\cdot u(x)-\mu(r)}{T}\right]-1} \qquad (7)$$

where

$$p\cdot u(x) = m_\perp \cosh(y-\eta)\cosh\zeta(r) - p_\perp\cos(\phi-\varphi)\sinh\zeta(r).$$

We use longitudinal proper time $\tau=\sqrt{t^2-z^2}$, space-time rapidity $\eta=\frac{1}{2}\ln[(t+z)/(t-z)]$, and transverse polar coordinates $(r,\,\varphi)$ to parametrize x. The pion momentum is $p=(m_\perp\cosh y,\, p_\perp\cos\phi,\, p_\perp\sin\phi,\, m_\perp\sinh y)$. Eq. (7) describes a longitudinally infinite source with boost-invariant longitudinal expansion and sharp freeze-out at proper time τ_f. Transverse expansion is parameterized by the transverse flow rapidity $\zeta(r)$, to be specified below. The r-dependent chemical potential $\mu(r)$ allows to discuss different transverse density profiles of the source at freeze-out. While probably too simple for a quantitative comparison with data, this model is technically convenient and allows to investigate the effects of longitudinal and transverse flow as well as the influence of the chemical potential on the average phase-space density $\langle f\rangle(\boldsymbol{p})$.

We now evaluate Eq. (5) for midrapidity pions ($y=0$). We can rotate the coordinate frame such that $\phi=0$. From (2) we obtain for the phase-space density immediately after freeze-out

$$f(t_\mathrm{f}(\boldsymbol{x}),\boldsymbol{x},\boldsymbol{p}) = \left[\exp\left(\frac{p\cdot u(x_\mathrm{f})-\mu(r)}{T}\right)-1\right]^{-1}\,, \qquad (8)$$

with $x_\mathrm{f}=(t_\mathrm{f}(\boldsymbol{x}),\boldsymbol{x})$. For the integration in Eq. (6) we use the freeze-out surface at $\tau=\tau_\mathrm{f}$ with integration measure

$$\left(1-\boldsymbol{v}\cdot\nabla_x\, t_\mathrm{f}(\boldsymbol{x})\right) d^3x = \tau_\mathrm{f}\cosh\eta\, d\eta\, r\, dr\, d\varphi\,. \qquad (9)$$

To obtain $\langle f\rangle(\boldsymbol{p})$ from (5) we need ($E_p=m_\perp$ for $y=0$)

$$A_n(p_\perp) = m_\perp\tau_\mathrm{f}\int \frac{\cosh\eta\, d\eta\, r\, dr\, d\varphi}{\left[\exp\left(\frac{p\cdot u(x_\mathrm{f})-\mu(r)}{T}\right)-1\right]^n} \qquad (10)$$

for $n=1,2$. Note that $A_1(p_\perp)$ is, up to a factor $(2\pi)^{-3}$, the invariant momentum spectrum $E_p(dN/d^3p)$.

We evaluate A_n by expanding the Bose distribution as $(e^x-1)^{-1}=\sum_{k=1}^{\infty} e^{-kx}$. This reduces (10) to a sum of exponential integrals. We now integrate over η and φ in the standard way [16], obtaining Bessel functions K_1 and I_0, respectively. Only the radial integration

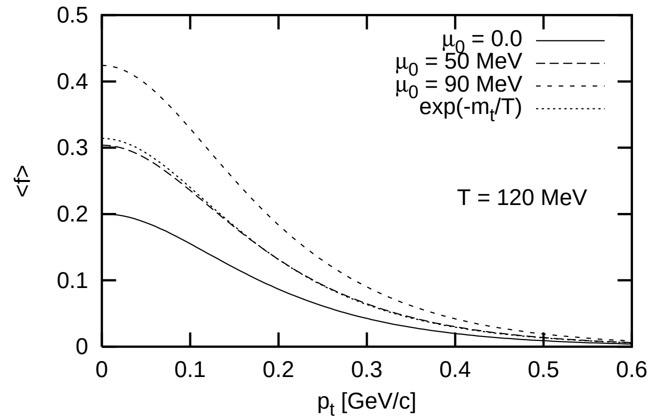

FIG. 1. Average phase-space density $\langle f \rangle (p_\perp)$ at rapidity $y=0$ for pions from a longitudinally expanding fireball with Boltzmann-distributed particle momenta, for a temperature of 120 MeV and three representative values for the pion chemical potential.

must be done numerically. With the shorthands $\alpha_\perp(r) = p_\perp \sinh\zeta(r)/T$ and $\beta_\perp(r) = m_\perp \cosh\zeta(r)/T$ we find

$$\langle f \rangle (p_\perp) = \frac{A_2(p_\perp)}{A_1(p_\perp)}\,, \tag{11a}$$

$$A_1(p_\perp) = \sum_{k=1}^{\infty} \tilde{A}_k(p_\perp)\,, \tag{11b}$$

$$A_2(p_\perp) = \sum_{k=2}^{\infty} (k-1)\tilde{A}_k(p_\perp)\,, \tag{11c}$$

$$\tilde{A}_k(p_\perp) = 4\pi m_\perp \tau_{\rm f} \int_0^\infty r\,dr\, e^{k[\mu(r)/T]} \times \mathrm{I}_0\big(k\,\alpha_\perp(r)\big)\,\mathrm{K}_1\big(k\,\beta_\perp(r)\big)\,. \tag{11d}$$

As a warm-up, let us consider the case without transverse flow, $\zeta(r)=0$, in the Boltzmann approximation (i.e. keeping only the first term in the sums over k). Then

$$\left\langle f^{\rm Boltz}_{\zeta=0} \right\rangle (p_\perp) = \bar{\lambda}_\pi \frac{\mathrm{K}_1\big(2\frac{m_\perp}{T}\big)}{\mathrm{K}_1\big(\frac{m_\perp}{T}\big)}\,, \tag{12}$$

where $\bar{\lambda}_\pi$ is the spatially averaged pion fugacity

$$\bar{\lambda}_\pi = \left\langle e^{\mu/T} \right\rangle \equiv \frac{\int r\,dr\, e^{2\mu(r)/T}}{\int r\,dr\, e^{\mu(r)/T}}\,, \tag{13}$$

the average being taken with the transverse matter density $\rho(r) \sim e^{\mu(r)/T}$. For a box-like density ($\mu(r)=\mu_0$ for $r \le R_{\rm box}$, $\mu(r)=-\infty$ for $r > R_{\rm box}$) this factor is $\bar{\lambda}_\pi = e^{\mu_0/T}$.

In Fig. 1 we show the result (12) for $T=120\,$MeV and three values for the average pion fugacity $\bar{\lambda}_\pi = e^{\mu_0/T}$. Also shown for reference as the dotted line is a Boltzmann distribution with vanishing pion chemical potential for the same temperature (i.e. the original thermal pion momentum distribution in the local rest frame). Comparing it with the solid curve one sees that, even without transverse flow, at low $p_\perp$ the spatially averaged phase-space density of the longitudinally expanding source is about 35% lower than its value in the local rest frame. In order to obtain an $\langle f \rangle (p_\perp)$ which looks like a Boltzmann distribution with vanishing pion chemical potential, we need in fact an average pion fugacity $\bar{\lambda}_\pi \approx 1.55$. Alternatively, thinking of the dashed line (corresponding to $\mu_0 = 50\,$MeV or an average pion fugacity $\bar{\lambda}_\pi \approx 1.52$) as experimental input and fitting it in the region $p_\perp \le 500\,$MeV by a Boltzmann distribution with $T=120\,$MeV, we would erroneously conclude $\mu_0 \approx 0$.

For large $m_\perp$, $\mathrm{K}_1(2m_\perp/T)/\mathrm{K}_1(m_\perp/T) \approx e^{-m_\perp/T}$, and this type of error no longer arises. Unfortunately, this requires for pions much larger values of $p_\perp$ than presently experimentally accessible. The reason for the strong reduction at low $p_\perp$, resulting from taking the spatial average, is that this average extends over a large longitudinal homogeneity length, and that for a longitudinally expanding source $f(x; y=0, p_\perp)$ agrees with the static Boltzmann distribution $\exp(-m_\perp/T)$ only at the point $\eta=0$, being smaller everywhere else by a factor $\exp[-m_\perp(\cosh\eta - 1)]$. For large $p_\perp$ the longitudinal homogeneity length decreases [5] and the reduction due to spatial averaging becomes less severe.

We now discuss the more general case of a source with additional transverse flow, including the full Bose-Einstein distribution by summing over all k in Eqs. (11). The transverse flow profile $\zeta(r)$ influences the radial averaging of the factors containing the pion chemical potential, and it is no longer possible as in (12) to factorize an effective average pion fugacity $\bar{\lambda}_\pi$ from the result for $\langle f \rangle (p_\perp)$. For simplicity we assume a constant transverse density distribution with radius $R_{\rm box}$, corresponding to a constant pion chemical potential μ_0. If the pion chemical potential is not too large, we expect our results to be qualitatively correct also for other transverse density profiles as long as we interpret in these cases $\exp(\mu_0/T)$ as the spatially averaged pion fugacity. For the transverse flow rapidity we take a linear profile with slope η_t,

$$\zeta(r) = \eta_t \frac{r}{R_{\rm box}}\,. \tag{14}$$

Fig. 2 shows that the longitudinal flow is again responsible for most of the suppression of $\langle f \rangle(\boldsymbol{p})$ compared to the local rest frame distribution. Transverse flow causes an additional suppression at low $p_\perp$ by up to 30% but, contrary to the case of longitudinal flow, this is compensated by large enhancement factors at high $p_\perp$ (where $\langle f \rangle(\boldsymbol{p})$ is, however, anyway very small). This effect is closely related to the well-known flattening of the transverse momentum spectrum ("blueshift") resulting from transverse flow which shifts particles from low to high values of $p_\perp$.

It is interesting to observe that in the relevant range of

T and μ_0, the curves corresponding to different transverse flow values η_t all cross at $p_\perp \approx 100\,\mathrm{MeV}/c$. If this "sweet spot" persists for models with different transverse density and flow profiles, it may be particularly well suited for a determination of μ_0/T [17].

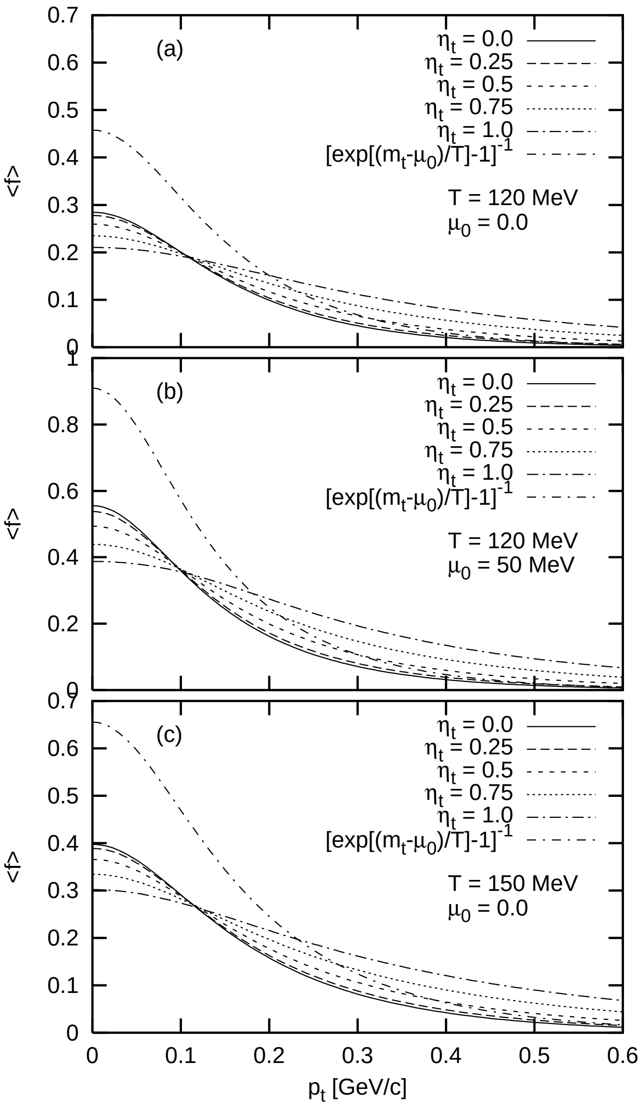

FIG. 2. Average phase-space density $\langle f \rangle$ for pions with $y = 0$ from a longitudinally and transversally expanding fireball with Bose-Einstein distributed momenta of temperature 120 MeV. The transverse flow is controlled by η_t. The transverse density profile has the shape of a box with radius $R_{\rm box}$.

We have demonstrated that the thermal model of ultra-relativistic heavy ion collisions does *not* predict an average phase-space density characterized by a Bose-Einstein distribution function. Comparing data for $\langle f \rangle(\boldsymbol{p})$ extracted from Eq. (1) with such a distribution is likely to severely underestimate the pion chemical potential or temperature. The previously used *ad hoc* introduction of a naive transverse blue-shift factor [4,11], without first properly accounting for the dominant effects from longitudinal expansion, is based on faulty intuition and misleading. Longitudinal expansion strongly reduces the average freeze-out phase-space density at low $p_\perp$, whereas transverse flow leads to an additional flattening of $\langle f \rangle\,(p_\perp)$. At low $p_\perp$, where $\langle f \rangle(\boldsymbol{p})$ and thus the danger of distortions due to strong multi-boson symmetrization effects [10] are largest, the strong longitudinal expansion of the heavy-ion collision fireball dominates the suppression of the average phase-space density. An extraction of the pion chemical potential should be based on a comparison of the measured values for $\langle f \rangle(\boldsymbol{p})$ with Eqs. (11), or a suitable generalization thereof which more accurately accounts for shortlived resonance decays (which we treated here rather superficially).

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