CPTh/S 017.0401
CERN-TH/2001-180
July 2001

Conformal expansions and renormalons*

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Abstract: The large-order behaviour of QCD is dominated by renormalons. On the other hand renormalons do not occur in conformal theories, such as the one describing the infrared fixed-point of QCD at small β_0 (the Banks–Zaks limit). Since the fixed-point has a perturbative realization, all-order perturbative relations exist between the conformal coefficients, which are renormalon-free, and the standard perturbative coefficients, which contain renormalons. Therefore, an explicit cancellation of renormalons should occur in these relations. The absence of renormalons in the conformal limit can thus be seen as a constraint on the structure of the QCD perturbative expansion. We show that the conformal constraint is non-trivial: a generic model for the large-order behaviour violates it. We also analyse a specific example, based on a renormalon-type integral over the two-loop running-coupling, where the required cancellation does occur.

*Research supported in part by the EC program "Training and Mobility of Researchers", Network "QCD and Particle Structure", contract ERBFMRXCT980194 and the U.S. Department of Energy, contract DE–AC03–76SF00515.

†CNRS UMR C7644

In QCD, as in other quantum field theories in four dimensions, the running-coupling depends logarithmically on the scale. As a consequence, the perturbative expansion is characterized by a renormalon factorial increase, which emerges from integrating over the running-coupling in loop diagrams [1]–[4].

Renormalons are believed to dominate the large-order behaviour of the series. In many physically interesting cases, their dominance sets in rather early [4]. Then renormalon resummation, i.e. the summation of the specific contributions associated with the running-coupling to all orders, becomes important for precision calculations (see e.g. [4]–[9]).

The idea that running-coupling effects can be resummed to all orders is based on an analogy with the Abelian theory [10, 11], where there is a systematic skeleton expansion. In the absence of the latter, the separation of running-coupling contributions from the conformal part of the expansion is not well defined. In practice, in most applications, the resummation is restricted to the level of a single dressed gluon, where the (Abelian) large-N_f calculation is sufficient to determine the renormalon contributions. The resummed result can then be written as

$$\begin{aligned} R_0(Q^2) &\equiv \int_0^\infty a(k^2)\,\phi\left(k^2/Q^2\right)\frac{dk^2}{k^2} \\ &= a(Q^2) + r_1^{(0)}\beta_0 a(Q^2)^2 + \left(r_2^{(0)}\beta_0^2 + r_1^{(0)}\beta_1\right)a(Q^2)^3 \\ &\quad + \left(r_3^{(0)}\beta_0^3 + \frac{5}{2}r_2^{(0)}\beta_1\beta_0 + r_1^{(0)}\beta_2\right)a(Q^2)^4 + \cdots, \end{aligned} \tag{1}$$

where k^2 is the virtuality of the exchanged gluon, $a(k^2) = \alpha_s(k^2)/\pi$ is the QCD running-coupling satisfying the renormalization group equation,

$$\frac{da(k^2)}{d\ln k^2} = \beta(a(k^2)) = -\beta_0 a(k^2)^2 - \beta_1 a(k^2)^3 - \beta_2 a(k^2)^4 - \cdots \tag{2}$$

and ϕ is an observable dependent Feynman integrand for a single gluon exchange diagram, which is interpreted as the gluon momentum distribution function [7].

Equation (1), should be viewed as the leading term in a hypothetical skeleton expansion [11]: $R(Q^2) = R_0(Q^2) + R_1(Q^2) + \cdots$. It has also proved useful [6, 5, 4] to view it as a model for the all-orders structure of the series. The Borel transform of $R_0(Q^2)$, satisfying

$$R_0(Q^2) = \int_0^\infty B(z)\,e^{-z/a(Q^2)}\,dz, \tag{3}$$

as that of the observable $R(Q^2)$, has singularities ("renormalons") on the real axis at $z = p/\beta_0$, where p are integers, which are responsible for the factorial increase of the standard perturbative coefficients.

On the other hand, a conformal expansion is totally free of renormalons. It may well contain other types of factorially increasing coefficients, e.g. due to the multiplicity of diagrams, but it cannot be influenced by running-coupling effects since the coupling in the conformal theory does not run.

A simple demonstration of how conformal relations become free of renormalons is the following: consider (1) in the case that the coupling does not run, $a(k^2) = a_{\mathrm{FP}}$:

the integral for R_0 in (1) simply reduces to $a_{\rm FP}$ (where the normalization of ϕ is 1 by definition). If instead the coupling $a(k^2)$ has an infrared fixed-point, changing variables in (1), $\epsilon \equiv k^2/Q^2$, one obtains

$$R_0(Q^2) = \int_0^\infty a\left(\epsilon Q^2\right)\,\phi(\epsilon)\frac{d\epsilon}{\epsilon}; \tag{4}$$

taking the limit $Q^2 \longrightarrow 0$ inside the integral one obtains again $R_0^{\rm FP} = a_{\rm FP}$, i.e. a trivial expansion. We shall return to this example below.

The fact that conformal relations are renormalon-free becomes relevant to QCD if an all-orders relation exists between the standard QCD perturbative expansion and a conformal expansion. Indeed, such a relation exists when a non-trivial infrared fixed-point is realized perturbatively, as occurs [12]–[15] for sufficiently small β_0. Using the Banks–Zaks expansion, an N_f-independent conformal relation between two generic effective charges can be written [11] (see also [14]–[19]). The coefficients in such a relation are renormalon-free. On the other hand, these coefficients are explicitly expressed as combinations of the standard (non-conformal) perturbative coefficients with those of the β function. It immediately follows that in these combinations renormalon factorials must conspire to cancel out. In this way, the absence of renormalons in conformal expansions translates into a constraint on the structure of the non-conformal perturbative expansion.

After briefly recalling the relation between conformal expansions and the standard perturbative expansion, we demonstrate that a generic model for the Borel function satisfying the expected large-order behaviour of the perturbative series is not necessarily consistent with the conformal limit. Next, since the conformal constraint holds by definition in (1) we analyse this example in detail, showing explicitly how the renormalon factorials eventually conspire to cancel out, leaving the conformal expansion renormalon-free.

Suppose that the perturbative expansion of $R(Q^2)$ is given by

$$R(Q^2) = a(Q^2) + r_1 a(Q^2)^2 + r_2 a(Q^2)^3 + \cdots, \tag{5}$$

where $a(Q^2)$ satisfies a renormalization group equation (2) which has a non-trivial infrared fixed-point $a_{\rm FP} \equiv a(Q^2=0) = -\beta_0/\beta_1 + \mathcal{O}(\beta_0)$, for sufficiently small β_0 (and $\beta_1 < 0$). One can then write a conformal relation between the fixed-point values of $R(Q^2=0) \equiv R_{\rm FP}$ and $a_{\rm FP}$,

$$R_{\rm FP} = a_{\rm FP} + c_1 {a_{\rm FP}}^2 + c_2 {a_{\rm FP}}^3 + c_3 {a_{\rm FP}}^4 + \cdots, \tag{6}$$

The conformal coefficients c_i can be related to r_i in (5) through the Banks–Zaks expansion

$$a_{\rm FP} = a_0 \,+\, v_1 a_0^2 \,+\, v_2 a_0^3 \,+\, v_3 a_0^4 + \cdots \tag{7}$$

where $a_0 \equiv -\frac{\beta_0}{\beta_1|_{\beta_0=0}}$, and v_i depend on the coefficients of $\beta(a)$ [15, 17, 18]. Since r_i are polynomials of order i in N_f, one can write

$$r_i \equiv \sum_{k=0}^{i} r_{i,k}\, {a_0}^k, \tag{8}$$

yielding the following N_f-independent relations [11],

$$\begin{aligned}
c_1 &= r_{1,0} \\
c_2 &= r_{1,1} + r_{2,0} \\
c_3 &= -r_{1,1}v_1 + r_{2,1} + r_{3,0} \\
c_4 &= 2r_{1,1}{v_1}^2 - r_{1,1}v_2 - r_{2,1}v_1 + r_{2,2} + r_{3,1} + r_{4,0}.
\end{aligned} \tag{9}$$

For simplicity we shall assume in the following that $a(Q^2)$ satisfies the two-loop renormalization group equation. In addition, we shall ignore the N_f dependence of β_1. Under these assumptions $a_{\rm FP} = a_0 = -\beta_0/\beta_1$, i.e. $v_i = 0$ for any $i \geq 1$. It obviously follows that

$$c_i = \sum_{k=0}^{[i/2]} r_{i-k,k} \tag{10}$$

for any i, where the square brackets indicate a (truncated) integer value. In this model then, c_i is simply the sum of all the possible $r_{j,k}$ coefficients such that $j + k = i$ and $j \geq k$. Relaxing these restrictions is possible, but the price will be more complicated expressions. The conclusions would not change, provided that $\beta(a)$ itself does not contain renormalons (this is probably true in $\overline{\rm MS}$).

Relation (10) is intriguing: the c_i on the l.h.s. are conformal coefficients, which must be totally free of renormalons, while $r_{i-k,k}$ on the r.h.s are factorially increasing because of renormalons. The only way in which the condition that c_i are renormalon-free can be realized is if explicit cancellations occur on the r.h.s. To show that this cancellation is non-trivial, we shall now consider simple models for the Borel transform $B(z)$, which are consistent with the expected large-order behaviour of the series and demonstrate that the corresponding c_i in (10) become factorially increasing, thus violating the conformal constraint.

We begin with the simplest example corresponding to a single simple pole in the Borel transform of the observable $R(Q^2)$:

$$B(z) = \frac{1}{1-(z/z_p)}, \tag{11}$$

where $z_p = p/\beta_0$ is the renormalon location. Note that in this example we choose the renormalon residue to be a constant, but in fact in QCD it depends on N_f.

The inverse Borel transform (3) yields

$$R(Q^2) = -z_p \mathrm{Ei}(1, -z_p/a) e^{-z_p/a}, \tag{12}$$

where $a = a(Q^2)$. The perturbative series of $R(Q^2)$ has the following factorially increasing coefficients

$$r_i = i! \left(\frac{\beta_0}{p}\right)^i. \tag{13}$$

Under the assumption that β_1 is N_f-independent, the decomposition of the coefficients of (13) in powers of a_0 according to (8) yields $r_{i,i} = (-\beta_1/p)^i\, i!$ and $r_{i,j} = 0$

for any $j \neq i$. The resulting coefficients (10) in this model are therefore

$$c_i = \begin{cases} 0 & i \text{ odd} \\ (i/2)! \, (-\beta_1/p)^{i/2} & i \text{ even} \end{cases} . \tag{14}$$

Thus, the would-be "conformal coefficients" do diverge factorially. In some sense the factorial divergence is slowed down: c_i contains just $(i/2)!$ rather than $i!$. Consequently we define $u = i/2$ and write the Banks–Zaks expansion as:

$$R_{\mathrm{FP}} = a_0 \sum_{u=0}^{\infty} u! \left(\frac{-\beta_1}{p}\right)^u {a_0}^{2u} = a_0 \sum_{u=0}^{\infty} u! (-\delta)^{-u}, \tag{15}$$

where

$$\delta \equiv \frac{p\beta_1}{\beta_0^2}. \tag{16}$$

We found that in the simple Borel pole example the factorial divergence of the perturbative series does enter the "conformal coefficients". The conformal constraint is therefore explicitly violated.

However, this example is not completely self-consistent: on one hand it was assumed that $a(Q^2)$ runs according to the two-loop β function (it has a fixed-point), but on the other hand we used the large-N_f (i.e. one-loop β function) form of the Borel singularity, namely a simple pole. It is known that a non-vanishing two-loop coefficient in the β function modifies the Borel singularity to be a branch point. For instance, for the leading infrared renormalon associated with the gluon condensate ($p = 2$) one has the following singularity structure in the Borel plane [2]

$$B(z) = \frac{1}{[1 - (z/z_p)]^{1+\delta}}, \tag{17}$$

where δ is defined in (16). The corresponding perturbative coefficients are

$$r_i = \frac{\Gamma(1+\delta+i)}{\Gamma(1+\delta)} \left(\frac{\beta_0}{p}\right)^i. \tag{18}$$

The large-order behaviour is

$$r_i^{as} = \frac{1}{\Gamma(1+\delta)} \, i! \left(\frac{\beta_0}{p}\right)^i i^\delta, \tag{19}$$

which is different from (13).

As opposed to the previous example, the r_i are not polynomials in β_0, so starting with (18) we cannot obtain the form (8). To see this, let us examine the expansion of the Γ function in r_i

$$f_i(\delta) \equiv \frac{\Gamma(1+\delta+i)}{\Gamma(1+\delta)} = (\delta+i)(\delta+i-1)(\delta+i-2)\dots(\delta+1). \tag{20}$$

It is clear that $f_i(\delta)$ can be written as a sum

$$f_i(\delta) = \sum_{k=0}^{i} f_k^{(i)} \delta^k, \tag{21}$$

where $f_k^{(i)}$ are numbers. Since δ is proportional to $1/{\beta_0}^2$, $f_i(\delta)$ contains all the even powers of $1/\beta_0$ from 0 up to $2i$. The additional positive power of β_0 in (18) finally leads to having half of the terms with positive power of β_0 and half with negative powers. The latter correspond to non-polynomial functions of N_f, which cannot be obtained in a perturbative calculation. This suggests that the current example is unrealistic.

The actual QCD situation, where the coefficients are polynomials in β_0 and behave asymptotically as (19), can be imitated by truncating the negative powers of β_0 in (18). We verified explicitly (see below) that this truncation does not alter the eventual large-order behaviour of r_i. Note that there is some ambiguity in the truncation: one can, in principle, truncate (21) at different k values and still obtain the same asymptotic behaviour. We shall choose the most natural possibility: truncate just the terms that lead to negative powers of β_0.

In order to proceed we should find the coefficients $f_k^{(i)}$. This can be done by writing a recursion relation using the property $f_i(\delta) = (\delta + i) f_{i-1}(\delta)$. This condition is equivalent to the following

$$f_k^{(i)} = \left\{ \begin{array}{ll} 1 & k = i \\ f_{k-1}^{(i-1)} + i f_k^{(i-1)} & 0 < k < i \\ i! & k = 0 \end{array} \right. \tag{22}$$

It is straightforward to use these recursion relations to obtain $f_k^{(i)}$ to arbitrarily high order.

After truncating the terms with negative powers of β_0, the coefficients (18) become

$$\tilde{r}_i = \left(\frac{\beta_0}{p}\right)^i \sum_{k=0}^{[\frac{i}{2}]} f_k^{(i)} \, \delta^k. \tag{23}$$

In order to verify that the truncation of the high powers of δ does not affect the large-order behaviour of the series, we calculated the ratio

$$\tilde{r}_i / r_i = \left[\sum_{k=0}^{[i/2]} f_k^{(i)} \delta^k\right] \Bigg/ \left[\sum_{k=0}^{i} f_k^{(i)} \delta^k\right] \tag{24}$$

for various values of δ, as a function of the order i. It turns out that this ratio approaches 1 fast, indicating a common asymptotic behaviour. For instance, for $\delta = 462/625$, corresponding to eq. (16) with $p = 1$ and the values of β_0 and β_1 in QCD with $N_f = 4$, we find $\tilde{r}_i / r_i \simeq 0.995$ for $i = 8$.

Next we write the decomposition of $\tilde{r}_i$ as a polynomial in a_0 according to (8),

$$\tilde{r}_{i,j} = f^{(i)}_{\frac{i-j}{2}} (-1)^j \left(\frac{\beta_1}{p}\right)^{\frac{i+j}{2}}, \tag{25}$$

where j is odd for odd i and even for even i (as always, $j \leq i$). Finally, the "conformal coefficients" (10) in this example are

$$\tilde{c}_{2u} = \sum_{j=0}^{u} \tilde{r}_{2u-j,j} = \left[\sum_{k=u}^{2u} f^{(k)}_{k-u} (-1)^k\right] \left(\frac{\beta_1}{p}\right)^u \tag{26}$$

and the expansion is

$$R_{\text{FP}} = a_0 \sum_{u=0}^{\infty} \left[\sum_{k=u}^{2u} f^{(k)}_{k-u} (-1)^{u+k}\right] (-\delta)^{-u}. \tag{27}$$

The square brackets should be compared with $u!$ in eq. (15), characterizing the simple Borel pole example. It turns out that the $\tilde{c}_{2u}$ increase *faster* than $u!$, but slower than $(2u)!$. Thus the factorial behaviour of the "conformal coefficients" persists also in this example. Again, the conformal constraint is violated.

Another possible approach to the analysis of the Borel cut example (17) is the following: consider the large-order behaviour of the coefficients in eq. (19). Let us now ignore the $1/\Gamma(1+\delta)$ factor, which does not depend on i and can be absorbed into the residue of the renormalon and expand $i^\delta \sim \exp(\delta \ln(i)) = 1 + \delta \ln(i) + \frac{1}{2}\delta^2 \ln^2(i) + \cdots$. Again we find that high powers of δ lead to non-polynomial dependence of the coefficients. As before we truncate these terms and write an approximation to r_i^{as}:

$$\bar{r}_i = i! \left(\frac{\beta_0}{p}\right)^i \sum_{k=0}^{[\frac{i}{2}]} \frac{1}{k!} \ln^k(i)\, \delta^k. \tag{28}$$

We checked numerically that the ratio $\bar{r}_i/r_i^{as}$ approaches 1 as i increases, so the asymptotic behaviour is not altered by this truncation. Note that the powers of $\ln(i)$ in (28) can be understood diagrammatically, as explained in [3].

We proceed and write

$$\bar{r}_{i,j} = \frac{i!}{\frac{i-j}{2}!} (-1)^j (\ln i)^{\frac{i-j}{2}} \left(\frac{\beta_1}{p}\right)^{\frac{i+j}{2}}, \tag{29}$$

and finally, using (10), the "conformal coefficients" are

$$\bar{c}_{2u} = \sum_{j=0}^{u} \bar{r}_{2u-j,j} = \left[\sum_{k=u}^{2u} \frac{k!}{(k-u)!} (-1)^k (\ln k)^{k-u}\right] \left(\frac{\beta_1}{p}\right)^u. \tag{30}$$

The large-order behaviour of $\bar{c}_{2u}$ turns out to be again between $u!$ and $2u!$. In fact, the two ways we used to construct the coefficients in this example lead to roughly the same asymptotic behaviour of the "conformal coefficients": the ratio between $\bar{c}_{2u}$ and $\tilde{c}_{2u}$ approaches a geometrical series at large orders. The reason for this is simply the fact that in both examples the dominant term in the sum is the one coming from the highest power of the coupling ($\tilde{r}_{2u,0}$ in (26) and $\bar{r}_{2u,0}$ in (30)). In fact, the contributions to the "conformal coefficients" from increasing orders in the coupling are monotonically increasing in both cases. We stress, however, that the decomposition of r_i into polynomials in β_0 ($r_{i,j}$) is not at all similar in the two cases.

We saw that, in general, consistency with the large-order behaviour of the perturbative series does not guarantee consistency with the conformal limit. In the above examples the renormalon factorials in $r_{i,j}$ do not cancel out in the sum (10) but rather induce a non-physical factorial increase in the "conformal coefficients". Apparently, the conformal constraint is non-trivial.

The examples above also teach us that the large-order behaviour of the perturbative coefficients r_i (or the nature of the Borel singularity) by itself does not uniquely determine the decomposition of r_i into powers of β_0: several choices of $r_{i,j}$ can fit. Indeed, we shall see below that the cancellation of renormalons in (10) crucially depends on having an "appropriate" decomposition of r_i. Since the model (1) is, by definition, consistent with the conformal constraint, the factorials should cancel out in (10) and the model should give an example of an "appropriate" decomposition.

In order to analyse the conformal coefficients in (1) we restrict ourselves to the contribution to $R_0(Q^2)$ from small k^2, which is the origin of infrared renormalons, and expand the momentum distribution function

$$\phi\left(k^2/Q^2\right) = \sum_n \left(\frac{k^2}{Q^2}\right)^n \Phi_n, \tag{31}$$

where Φ_n are numbers. It is sufficient to consider a specific infrared renormalon with $n = p$, so we choose our "observable" as

$$R_0(Q^2) \equiv \int_0^{Q^2} p \left(\frac{k^2}{Q^2}\right)^p a(k^2)\frac{dk^2}{k^2}, \tag{32}$$

where the upper integration limit is set for simplicity to Q^2.

It was shown in [20, 21] that if $a(k^2)$ satisfies the two-loop renormalization group equation, the Borel representation of R_0 is

$$R_0(Q^2) = \int_0^\infty e^{-\frac{\beta_1}{\beta_0}z}\frac{1}{\left[1-(z/z_p)\right]^{1+\delta}}\, e^{-z/a}\, dz, \tag{33}$$

where $a \equiv a(Q^2)$. This integral resums all those terms in eq. (1) that depend only on the first two coefficients of the β function. Note that eq. (32) is well defined thanks to the infrared fixed-point of the coupling $a(k^2)$. On the other hand, eq. (33) is not well defined because of the infrared renormalon, and it differs [20, 21] from eq. (32) by an ambiguous power correction. The equality between (32) and (33) should therefore be understood just as an equality of the (all-order) power series expansion of the two expressions.

To expand (33) we note that the Borel transform of R_0 with respect to the *modified coupling* $\tilde{a}$,

$$\frac{1}{\tilde{a}} \equiv \frac{1}{a} + \frac{\beta_1}{\beta_0}, \tag{34}$$

coincides with the Borel transform (17). Using the coefficients (18) we have

$$R_0 = \sum_{i=0}^{\infty} r_i \tilde{a}^{i+1} = \sum_{i=0}^{\infty} \frac{\Gamma(1+\delta+i)}{\Gamma(1+\delta)} \left(\frac{\beta_0}{p}\right)^i \tilde{a}^{i+1}. \tag{35}$$

Substituting $\tilde{a}^{i+1}$ for

$$\left(\frac{a}{1+a\beta_1/\beta_0}\right)^{i+1} = a^{i+1} \sum_{k=0}^{\infty} \frac{(i+k)!}{i!\,k!}\, a^k \left(-\frac{\beta_1}{\beta_0}\right)^k \tag{36}$$

we obtain

$$R_0 = a \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(1+\delta+i)}{\Gamma(1+\delta)} \frac{(i+k)!}{i!\,k!} \left(\frac{\beta_0}{p}\right)^i \left(-\frac{\beta_1}{\beta_0}\right)^k a^{i+k}. \tag{37}$$

Defining $n = k + i$ and performing first the summation over i we obtain

$$R_0 = \sum_{n=0}^{\infty} r_n a^{n+1} \tag{38}$$

with the perturbative coefficients r_n given by

$$r_n = \left(\frac{\beta_0}{p}\right)^n \sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!} \frac{\Gamma(1+\delta+i)}{\Gamma(1+\delta)} (-\delta)^{n-i}. \tag{39}$$

Note that the model eq. (18) corresponds to keeping only the $i = n$ term in eq. (39). We now use (21) to expand the Γ function and write explicitly the dependence on β_0 (or a_0). Defining $j = 2i - 2k - n$, we obtain

$$r_n \equiv \sum_{j=-n}^{n} r_{n,j} {a_0}^j, \tag{40}$$

with

$$r_{n,j} = \left(\frac{\beta_1}{p}\right)^u \sum_{i=u}^{n} \frac{n!}{(n-i)!\,i!} (-1)^i f_{i-u}^{(i)}, \tag{41}$$

where $u \equiv \frac{n+j}{2}$. Next, we note that for $0 \le 2u < n$,

$$\sum_{i=u}^{n} \frac{n!}{(n-i)!\,i!} (-1)^i f_{i-u}^{(i)} \equiv 0, \tag{42}$$

so the negative powers of a_0 are absent in eq. (40). We thus identify a major difference between this example and the examples considered above: here the decomposition of

r_n into powers of a_0 does not lead to any non-polynomial dependence, and truncation is not required. Note, however, that there are non-trivial cancellations.

Finally, using (10), the conformal coefficients‡ corresponding to R_0 are given by

$$c_{2u} = \sum_{j=0}^{u} r_{2u-j,j} = \sum_{k=u}^{2u} r_{k,2u-k} = \left[\sum_{k=u}^{2u}\sum_{i=u}^{k} \frac{k!}{(k-i)!\,i!}(-1)^i f_{i-u}^{(i)}\right]\left(\frac{\beta_1}{p}\right)^u = 0, \qquad (43)$$

where the last equality was checked explicitly. In other words, the final result is

$$R_0(Q^2 = 0) = a_{\rm FP}, \qquad (44)$$

in accordance with our expectations. As explained above, the vanishing of the conformal coefficients in this case can be understood directly from the defining integral R_0 (the polynomial N_f dependence of the r_n's is also transparent from this representation). We note that contrary to the previous examples (26) and (30), in (43) the term originating from the highest power of the coupling does not dominate. This is crucial for the eventual cancellation.

In conclusion, we saw that the absence of renormalons in conformal coefficients can be seen as a constraint on the form of the perturbative expansion: renormalon factorials must cancel out in certain combinations. We considered various examples for the Borel transform, which are consistent with the same large-order behaviour, finding that this cancellation is non-trivial. It would be interesting to find a concrete general form of this constraint, which still appears elusive. We note that in the two-loop example studied here the constraint is implemented through the *regular* factor $\exp\left(-\frac{\beta_1}{\beta_0}z\right)$ in eq. (33), and therefore it involves an infinite series of sub-leading terms in the large-order asymptotic behaviour of the coefficients.

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‡As in the previous examples c_i vanishes trivially for odd i, since $i+k$ is always even in $r_{i,k}$ (eq. (8)), making successive powers of a_0 decrease by a factor of 2. This reflects a property of the two-loop β function, namely the ultraviolet log structure is such that two powers of β_0 are replaced by one power of β_1 in the coefficients of successive powers of $\log(k^2/Q^2)$ when $a(k^2)$ is expanded in powers of $a(Q^2)$ (see eq. (1)).

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