DEFORMED BOOST TRANSFORMATIONS THAT SATURATE AT THE PLANCK SCALE

N.R. Bruno and **G. Amelino-Camelia**
Dipart. Fisica, Univ. Roma "La Sapienza", and INFN Sez. Roma1
P.le Moro 2, 00185 Roma, Italy

J. Kowalski-Glikman
Institute for Theoretical Physics, University of Wrocław,
pl. Maxa Borna 9, 50–204 Wrocław, Poland

We derive finite boost transformations based on the Lorentz sector of the bicross-product-basis κ-Poincaré Hopf albegra. We emphasize the role of these boost transformations in a recently-proposed new relativistic theory. We find that when the (dimensionful) deformation parameter is identified with the Planck length, which together with the speed-of-light constant has the status of observer-independent scale in the new relativistic theory, the deformed boosts saturate at the value of momentum that corresponds to the inverse of the Planck length.

I. INTRODUCTION

Several approaches to the unification of General Relativity and Quantum Mechanics have led to arguments in favour of the emergence of a minimum length and/or a maximum momentum, possibly connected with the Planck length ($L_p \sim 10^{-33} cm$) and this has also motivated some authors (see, *e.g.*, Refs. [1,2]) to argue that somehow boosts should saturate at the Planck scale. A framework that can be used to implement this type of concepts in consistent relativistic theories was recently proposed in Ref. [3], where, following a close analogy with our present description of fundamental physics (in which Galileo's Relativity Principle coexists with the observer-independent scale c, the speed-of-light fundamental constant), it was shown that logically consistent relativistic theories can host both c and a second scale, possibly connected with the Planck length, as observer-independent scales. One example of this new type of Relativity postulates, an example based on the introduction of a deformed dispersion relation, was analyzed in detail [3] obtaining various results in leading order in the second (Planck-scale related) observer-independent scale, including the nature of the transformation rules between different inertial observers and kinematic rules[1] for particle-production processes, emphasizing those aspects that lead to the emergence of a minimum length, a minimum length uncertainty and a maximum momentum (minimum wavelength). In particular, it was found (again within a leading-order analysis [3]) that consistency with the new postulates requires that (infinitesimal and finite) transformations between different inertial observers be described in terms of the generators of one of the κ-Poincaré Hopf albegras [5–9], and that the action of boosts saturates at momenta corresponding to the second observer-independent scale. This connection with κ-Poincaré was then exploited in Ref. [10] as a guiding principle for obtaining exact (all-order) results on the maximum, observer-independent, momentum scale, and on some asymptotic (infinite-energy) features of the new relativistic theory.

In this paper we intend to further exploit the connection with κ-Poincaré to obtain the exact (all-order) form of the finite boost transformations, generalizing one of the leading-order results of Ref. [3]. The relevant properties of operators obtained by exponentiation of the generators of the κ-Poincaré algebra here of interest were not previously analyzed, but for another example of κ-Poincaré algebra a similar analysis was attempted [11], encountering several difficulties (in particular, the exponentiatied action of the generators could only be described through implicit formulas). We shall show that no such difficulties is encountered in the analysis of the κ-Poincaré algebra here of interest.

As additional motivation for the analysis reported in the following Sections we observe that, in addition to their significance for the logical consistency of the new type of relativistic theories, the possibility of deformed boosts has recently attracted interest also as a tool to address certain problems in phenomenology. In particular, puzzling

[1] Additional work on these kinematic rules was then reported in Ref. [4].

observations of ultra-high-energy cosmic rays and multi-TeV photons from Markarian 501 have been analyzed as an indication of boost deformation [12–15]. Also some cosmological scenarios based on κ-Poincaré were analyzed, leading to encouraging preliminary results [16,17].

II. FINITE BOOST TRANSFORMATIONS

The κ-Poincaré Hopf algebra that is relevant for the example of new relativistic theory studied in Refs. [3,10] is the one proposed by Majid and Ruegg in Ref. [6] (the so-called "bicrossproduct basis"). For our analysis of finite boosts only the algebra sector of this Hopf algebra is relevant, and we note the commutation relations here for completeness:[2]

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\tau}] &= i\left(\eta_{\mu\tau} M_{\nu\rho} - \eta_{\nu\rho} M_{\nu\tau} + \eta_{\nu\rho} M_{\mu\tau} - \ \eta_{\nu\tau} M_{\mu\rho}\right), \\
[M_i, k_j] &= i\epsilon_{ijk} k_k\,, \quad [M_i, \omega] = 0\,, \\
[N_i, k_j] &= i\delta_{ij}\left(\frac{1}{2\lambda}\left(1 - e^{-2\omega\lambda}\right) + \frac{\lambda}{2}\vec{k}^{\,2}\right) - \ i\lambda k_i k_j, \\
[N_i, \omega] &= i k_i\,, \\
[P_\mu, P_\nu] &= 0\,,
\end{aligned} \tag{1}$$

where $P_\mu = (\omega, k_i)$ are the time and space components of the four-momentum generators and $M_{\mu\nu}$ are modified Lorentz generators with rotations $M_k = \frac{1}{2}\epsilon_{ijk} M_{ij}$ and boosts $N_i = M_{0i}$. Also important for our analysis is the dispersion relation that follows from these κ-Poincaré algebraic relations

$$\frac{e^{\lambda\omega} + e^{-\lambda\omega} - 2}{\lambda^2} - \vec{k}^2 e^{\lambda\omega} = m^2\ , \tag{2}$$

which of course corresponds to a Casimir of (1) (just as the special-relativistic dispersion relation corresponds to a Casimir of the standard Lorentz algebra). The eigenvalues m^2 of the κ-deformed Casimir are related to the physical mass M (rest energy) by

$$\frac{e^{\lambda M} + e^{-\lambda M} - 2}{\lambda^2} = m^2\ . \tag{3}$$

Clearly according to (1) the action of rotations is completely conventional (undeformed). The action of the boosts N_i needs instead a deformation, to reflect the properties of the λ-dependent commutator $[N_i, k_j]$. As announced, we intend to construct finite deformed boosts transformations. This can be done in close analogy with the corresponding analysis of ordinary Lorentz boosts. A particle which, for a given first observer, has four-momentum $(\omega^0, \vec{k}^0)$ will have four-momentum $(\omega, \vec{k})$ for a second observer in relative motion, with boost/rapidity parameter ξ, with respect to the first observer. The starting point for obtaining the relation between $(\omega, \vec{k})$ and $(\omega^0, \vec{k}^0)$ is the differential representation of the deformed boost generators. From the algebra (1) one can easily derive this differential representation (of course, without loss of generality we can choose to focus on the boost that acts along the axis 1)

$$N_1 = i k_1 \frac{\partial}{\partial\omega} + i\left(\frac{\lambda}{2}\vec{k}^2 + \frac{1 - e^{-2\lambda\omega}}{2\lambda}\right)\frac{\partial}{\partial k_1} - i\lambda k_1\left(k_j \frac{\partial}{\partial k_j}\right), \tag{4}$$

and accordingly[3] the differential equations to be satisfied by $(\omega, \vec{k})$ are

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[2] Since the second observer-independent scale appears several times in several of our formulas we find convenient to denote it with λ rather than the more cumbersome notation $\tilde{L}_p$ adopted in Ref. [3]. We of course maintain the intuition that this dimensionful parameter should be naturally identified with the Planck length, up to a numerical factor of order 1 and a possible sign difference. Also notice that in the κ-Poincaré literature the mass parameter κ is used instead instead of the length parameter λ ($\kappa \equiv 1/\lambda$).

[3] Here we are making the implicit assumption that the action of the adjoint representation be described by exponentiation of the generators. This assumption is in fact justified [8] in the specific κ-Poincaré Hopf algebra here of interest, differently from the case of the Lorentz sector of other κ-Poincaré Hopf algebra [8,11].

$$\frac{d}{d\xi}k_1(\xi) + \frac{\lambda}{2}(k_1^2(\xi) - k_2^2(\xi) - k_3^2(\xi)) + \frac{e^{-2\lambda\omega(\xi)} - 1}{2\lambda} = 0 \tag{5}$$

$$\frac{d}{d\xi}\omega(\xi) - k_1(\xi) = 0 \tag{6}$$

$$\frac{d}{d\xi}k_2(\xi) + \lambda k_1(\xi)k_2(\xi) = 0 \tag{7}$$

$$\frac{d}{d\xi}k_3(\xi) + \lambda k_1(\xi)k_3(\xi) = 0 \tag{8}$$

Differentiating (5) and making use of the other equations one obtains a non-linear second-order equation for $k_1(\xi)$:

$$\frac{d^2}{d\xi^2}k_1(\xi) + 3\,\lambda\,k_1(\xi)\frac{d}{d\xi}k_1(\xi) + \lambda^2 k_1^3(\xi) - k_1(\xi) = 0\ . \tag{9}$$

We find that the solutions of this equation are of the form

$$k_1(\xi) = -\frac{B}{\lambda}\frac{\cosh(\xi+\beta)}{(1 - B\sinh(\xi+\beta))} \tag{10}$$

where B and β are integration constants.

Corresponding solutions for the other components of the four-momentum can be obtained by substituting Eq. (10) in Eqs. (6–8). Our problem is therefore already reduced to the identification of a few integration constants.

We determine the integration constants B, β by imposing the obvious requirement that $(\omega, \vec{k}) = (\omega^0, \vec{k}^0)$ for $\xi = 0$. From (10) one obtains

$$k_1(\xi = 0) = k_1^0 = -\frac{B}{\lambda}\frac{\cosh(\beta)}{(1 - B\sinh(\beta))} \tag{11}$$

and thus

$$B = -\frac{\lambda k_1^0}{\cosh(\beta) - \lambda k_1^0 \sinh(\beta)} \tag{12}$$

Using this condition and introducing $A = \tanh(\beta)$ (simply replacing the unknown β with the corresponding unknown A) we can obtain from (10) an expression of $k_1(\xi)$ with only one unknown:

$$k_1(\xi) = k_1^0 \frac{\cosh(\xi) + A\sinh(\xi)}{1 - \lambda k_1^0(A - A\cosh(\xi) - \sinh(\xi))}\ . \tag{13}$$

Corresponding expressions for $\omega(\xi)$,$k_2(\xi)$,$k_3(\xi)$ are easily obtained from Eqs. (6)–(8):

$$\omega(\xi) = \omega^0 - \frac{1}{\lambda}\ln\frac{1}{1 - \lambda k_1^0(A - A\cosh(\xi) - \sinh(\xi))} \tag{14}$$

$$k_2(\xi) = \frac{k_2^0}{1 - \lambda k_1^0(A - A\cosh(\xi) - \sinh(\xi))} \tag{15}$$

$$k_3(\xi) = \frac{k_3^0}{1 - \lambda k_1^0(A - A\cosh(\xi) - \sinh(\xi))} \tag{16}$$

We are therefore left with the task of expressing A in terms of ω^0,k_1^0,k_2^0,k_3^0. The sought condition is easily obtained combining (5) and (13)

$$k_1^0 A - \lambda {k_1^0}^2 = \frac{d}{d\xi} k_1(\xi)\Big|_{\xi=0} = -\frac{\lambda}{2}({k_1^0}^2 - {k_2^0}^2 - {k_3^0}^2) + \frac{1-e^{-2\lambda\omega^0}}{2\lambda} \;, \tag{17}$$

from which it follows that

$$A = \sinh(\lambda\omega^0)\frac{e^{-\lambda\omega^0}}{\lambda k_1^0} + \frac{\lambda}{2k_1^0}\vec{k_0}^2 \tag{18}$$

The equations (13), (14), (15) and (16), with A expressed in terms of ω^0,k_1^0,k_2^0,k_3^0 through (18), describe the exact κ-deformed boost transformations. It is easy to verify that they satisfy[4] the dispersion relation (2). It is also easy to verify that, of course, in the $\lambda \to 0$ limit our transformation rules reduce to ordinary Lorentz boost transformations, and the leading order in λ reproduces the corresponding result obtained in Ref. [3].

III. RANGE OF THE BOOST PARAMETER AND MAXIMUM MOMENTUM

For ordinary Lorentz boosts the ξ parameter can take any real value. In this Section we show that the same property holds for our deformed boosts for $\lambda > 0$, while for $\lambda < 0$ the ξ parameter can vary only within a finite range. We also show that for $\lambda > 0$ our deformed boosts saturate at a maximum value of momentum: $|\vec{k}| = 1/\lambda$.

In preparation for the study of the range of ξ, let us start by analyzing some relevant properties of the integration constant A. For simplicity let us focus on the case $k_2^0 = k_3^0 = 0$ and let us denote k_1^0 simply by k^0; then A takes the form

$$A = \sinh(\lambda\omega^0)\frac{e^{-\lambda\omega^0}}{\lambda k^0} + \frac{\lambda}{2}k^0 \;. \tag{19}$$

Using the deformed dispersion relation (2), which allows to express k^0 as a function of ω^0

$$k^0 = \pm\frac{\sqrt{1+e^{-2\lambda\omega^0} - (2+m^2\lambda^2)e^{-\lambda\omega^0}}}{|\lambda|} \;, \tag{20}$$

and using (3) to express the Casimir eigenvalues m^2 in terms of the physical mass M, we obtain a useful formula for A

$$A = \mathrm{sign}(k)\,\mathrm{sign}(\lambda)\frac{1-\cosh(\lambda M)x}{\sqrt{1+x^2-2\cosh(\lambda M)x}} \tag{21}$$

where we have also introduced $x \equiv e^{-\lambda\omega^0}$.

For $\lambda > 0$, $x \in (0, e^{-\lambda M}]$ and from (21) it follows that for positive k the value of A varies from 1 to $+\infty$ as x varies from 0 to $e^{-\lambda M}$, while for negative k the value of A varies from -1 to $-\infty$ as x varies from 0 to $e^{-\lambda M}$.

For $\lambda < 0$, $x \in [e^{|\lambda| M}, +\infty)$ and from (21) it follows that for positive k the value of A varies from $+\infty$ to 1 as x varies from $e^{|\lambda| M}$ to $+\infty$, while for negative k the value of A varies from $-\infty$ to -1 as x varies from $e^{|\lambda| M}$ to $+\infty$.

Using these properties of A it is easy to establish the range of allowed values of the boost parameter ξ and to establish the main characteristics of the dependence of momentum and energy on ξ.

For positive λ it is possible to vary ξ from 0 toward both $+\infty$ and $-\infty$ without ever encountering any singularities. (For positive λ the denominator $1 - \lambda k(A - A\cosh\xi - \sinh\xi)$ in (13) and (14) never vanishes.) All real values of ξ are therefore allowed for $\lambda > 0$, just as in the case of ordinary Lorentz boosts ($\lambda = 0$). The derivative $dk/d\xi$ vanishes only for one value of ξ (a saddle point for $k(\xi)$) and in that point the momentum vanishes and the energy reaches its minimum value M. In the limit $\xi \to +\infty$ one finds $\omega \to +\infty$ while $k \to 1/\lambda$, while in the limit $\xi \to -\infty$ one finds $\omega \to +\infty$ while $k \to -1/\lambda$. So energy is still unbounded from above, just like in the ordinary Lorentz-boost case, while for positive λ the new boost transformations are such that momentum saturates at $|k| = 1/\lambda$. Our result on the exact transfomation rules therefore extends the maximum-momentum analysis reported in Ref. [3], which was based

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[4]In the new relativistic theory the deformed dispersion relation acquires [3] the status of an observer-independent property, and it must therefore be an invariant of boost transformations. At the mathematical level this is assured [3,10] by the fact that the deformed dispersion relation corresponds, as mentioned, to a Casimir of (1).

on the form of the transformation rules in leading order in λ and found that for positive λ the action of boosts starts saturating as $|k|$ approaches $1/\lambda$ (the exact saturation result we obtained here was of course not within the grasp of the leading-order analysis). The maximum-momentum result that follows from our transformation rules also reflects the property of the κ-Poincaré dispersion relation (2), already emphasized in Ref. [10], that for positive λ connects the infinite-energy limit with the limit $|k| = 1/\lambda$.

The fact that the ξ parameter can take any real value and that the new observer-independent scale λ acquires the intuitive role of inverse of the maximum momentum renders the case $\lambda > 0$ rather attractive for applications in quantum-gravity research.

The situation is significantly different and somewhat less intuitive in the case $\lambda < 0$. For negative λ, increasing ξ from 0 the energy already diverges at

$$\xi_+ = \ln\left(\frac{1-\lambda kA+\sqrt{1-2\lambda kA+\lambda^2k^2}}{-\lambda kA-\lambda k}\right) , \tag{22}$$

and decreasing ξ from 0 the energy already diverges at

$$\xi_- = \ln\left(\frac{1-\lambda kA-\sqrt{1-2\lambda kA+\lambda^2k^2}}{-\lambda kA-\lambda k}\right) . \tag{23}$$

(For negative λ the denominator $1 - \lambda k(A - A\cosh\xi - \sinh\xi)$ in (13) and (14) vanishes at $\xi = \xi_+$ and $\xi = \xi_-$.)

For the momentum one finds that for negative λ the derivative $dk/d\xi$ vanishes only for one value of ξ (again, a saddle point for $k(\xi)$) and in that point the momentum vanishes (and the energy reaches its minimum value M), then to the left and to the right of this saddle point the function $k(\xi)$ approaches singular asymptotes at ξ_- and ξ_+; in fact, $k \to +\infty$ for $\xi \to \xi_+$, while $k \to -\infty$ for $\xi \to \xi_-$.

The fact that it does not predict a maximum momentum and that ξ is confined to the range $\xi_- < \xi < \xi_+$ might render the case $\lambda < 0$ less attractive for physics application, but this is still a very early in the development of the new relativistic theories [3] and our intuition might be changed by future studies.

IV. CONCLUSIONS

The interesting properties of the finite rules of transformation between different inertial observers here obtained provide additional insight in the new relativistic theory proposed in Ref. [3] and further developed in Refs. [10] and [4]. The case $\lambda > 0$ is particularly interesting since it corresponds to boosts that saturate when the momentum gets to $|k| = 1/\lambda$. The fact that these deformed boosts act on small momenta ($|k| \ll 1/\lambda$) in a way that is basically identical to the one of ordinary Lorentz boosts but are then able to saturate at momenta $1/\lambda$ could lead to interesting conceptual and phenomenological developments. It is also reassuring (at least from the limited prospective we presently have on these new relativistic theories) that for this case $\lambda > 0$ the range of the boost parameter is just the same as in ordinary Special Relativity.

Our results for negative λ (no maximum momentum, finite range of the boost parameter) appear to be less encouraging. Although none of the results we obtained can be used to exclude the case $\lambda < 0$ on physical grounds, it appears reasonable to focus future studies of this new relativistic framework on the case with positive λ.

Our analysis also changes the intuition that emerged in some previous studies, also based on κ-Poincaré mathematics. In particular, in Ref. [11] the action of operators obtained by exponentiation of the generators in the Lorentz sector of another κ-Poincaré Hopf algebra was analyzed, encountering several difficulties and finally obtaining a description of these actions that could only be expressed very implicitly (through complicated integrals). The fact that in the κ-Poincaré Hopf albegra here of interest we did not encounter them might indicate that these difficulties are not a general characteristic of κ-Poincaré. It is reasonable to conjecture that this significant difference between the κ-Poincaré Hopf algebra here analyzed and the one analyzed in Ref. [11] be due to the fact, already emphasized in Ref. [8], that, while indeed in the case we considered the exponentiation of the generators does correspond to the action of the adjoint representation, in the case considered in Ref. [8] by exponentiating the generators in the Lorentz sector one does not obtain the action of the adjoint representation. Another possible element for the understanding of this different behaviour of different κ-Poincaré Hopf algebra appears to be provided by the type of duality emphasized in Ref. [6], which is enjoyed by the example here considered, but is not present in other κ-Poincaré Hopf algebras.

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