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THE EFFECT OF PHASE SLIPPAGE IN MULTICELL CAVITIES on LONGITUDINAL BEAM DYNAMICS

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ABSTRACT

In modern linear accelerators it is often convenient to use multicell cavities which are designed for a constant particle velocity. These cavities are then employed over a certain velocity range until the next cavity type, designed for a higher β, further accelerates the beam. For superconducting cavities this approach has become mandatory due to the high R&D costs of such devices. Using multicell cavities at velocities different from their design value yields phase slippage in the single cells, thereby reducing energy gain and longitudinal focusing. In this paper we derive some simple analytical formulas in order to demonstrate the implications of phase slippage on longitudinal dynamics. It is shown that for large phase slip values the longitudinal focusing is not only reduced by lower transit time factors but also by an additional "slip factor", that can drive particles into unstable regions of parameter space.

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1 Introduction

Multicell cavities with constant cell length are designed for a specific particle velocity β_d. In order to accelerate the beam, each cell has a length of $\beta\lambda/2$, so that a bunch traversing the cavity always sees the accelerating half wave of the electric field. Figure 1 shows an example where the bunch velocity is smaller than the design velocity of the cavity ($\beta < \beta_d$). In the first two cells the bunch is too early with respect to the synchronous phase, and it arrives too late in the subsequent cells. Only in the geometric center of the structure, the bunch is exactly "on phase". For this reason the designation "synchronous phase" ϕ_s no longer applies and is usually replaced by the mean phase of the reference particle ϕ_r.

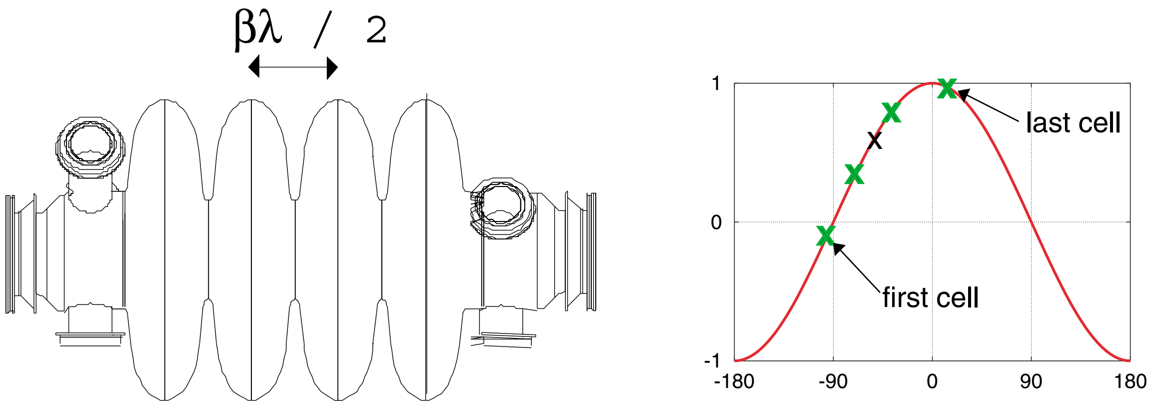

Figure 1: Phase slip in a 4-cell cavity

In the following we study a simplified case, assuming constant electric fields on axis, and neglecting the effects of acceleration. With this model one can understand the implications of phase slippage and derive simple rules for linac designs with multicell cavities. A second purpose of this study is to show that phase slippage has to be taken into account properly when setting up beam dynamics calculations with multiparticle codes.

2 Energy gain

The reduction in energy gain for $\beta \neq \beta_d$ due to phase slippage is given by evaluating the transit time factor integral for the appropriate velocity:

$$E_0 T(\beta) = \int_{-L/2}^{L/2} E(t=0,z) \begin{matrix}\cos\\ \sin\end{matrix} \left(\frac{\omega z}{\beta c_0}\right) \mathrm{d}z \qquad (1)$$

where the cos refers to a structure with even electric field distribution (odd number of cells) and the sin belongs to a structure with odd field distribution (even number of cells). The fields are usually calculated with SUPERFISH [1] or can be approximated by trigonometric expressions. In order to quantify the effect of phase slippage on energy gain, we calculate the transit time factor in multicell π-mode cavities for the simplified case of constant E_z between cell irises. For the single cells we obtain:

$$T_n\left(\frac{\beta}{\beta_d}\right) = \frac{1}{\pi}\frac{\beta}{\beta_d}(-1)^n \left[\cos\left(n\pi\frac{\beta_d}{\beta}\right) - \cos\left([n-1]\pi\frac{\beta_d}{\beta}\right)\right] \qquad n = 1..\frac{N}{2} \qquad N \text{ even} \qquad (2)$$

$$\tilde{T}_n\left(\frac{\beta}{\beta_d}\right) = \frac{1}{\pi}\frac{\beta}{\beta_d}(-1)^n \left[\sin\left(\left[n+\frac{1}{2}\right]\pi\frac{\beta_d}{\beta}\right) - \sin\left(\left[n-\frac{1}{2}\right]\pi\frac{\beta_d}{\beta}\right)\right] \qquad n = 0..\frac{N-1}{2} \qquad N \text{ odd} \qquad (3)$$

[**N - total number of cells, n - cell number, counting from the center of a multicell cavity to the outside (if n_{total} is odd the center cell has the number n=0), β_d - geometrical design beta of the multicell structure, β - actual particle beta.**]

The transit time factor for the whole cavity can now be computed by averaging the single cell numbers. The result for cavities with 2 to 6 cells at different velocities is shown in Figure 2.

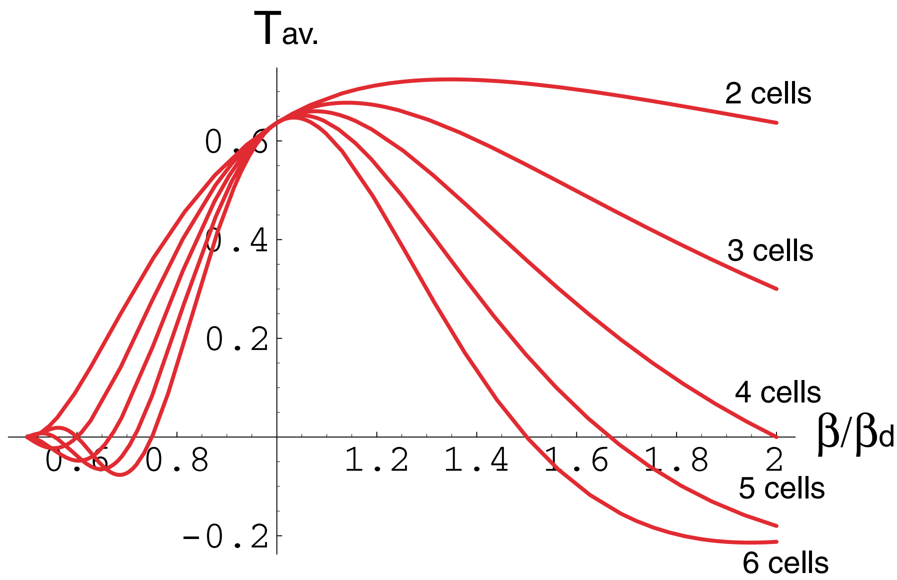

Figure 2: Transit time factor for multicell cavities as a function of normalised β

For $\beta = \beta_d$ all multicell cavities have the same transit time factor $T \approx 2/\pi$. The well known phenomena that low-β multicell cavities reach their maximum performance at $\beta > \beta_d$ can be explained by two competing effects: looking at a single cell the transit time factor rises with higher velocity, however, this rise is limited by the rising phase slip which reduces the accelerating field seen by the reference particle. For realistic field distributions the maximum values for T as well as the optimum ratio of β/β_d can change but the general tendency will be the same. In particular the large bore radii of the cut-off tubes in superconducting cavities change the field pattern in the outer cells, where the higher effective cell lengths increase furthermore the optimum beta. The following table compares the maxima for the simplified constant field cavities with the results from the reduced β cavities of the SPL project.

Table 1: Optimum ratio of β/β_d for multicell cavities

cavity type	**No. cells**	$(\beta/\beta_d)_{opt}$
$E_z = const.$	2	1.35
$E_z = const.$	3	1.14
$E_z = const.$	4	1.08
$\beta = 0.52$ (SPL)	4	1.12
$\beta = 0.7$ (SPL)	4	1.14
$E_z = const.$	5	1.05
$\beta = 0.8$ (SPL)	5	1.08

Although Eq.(1) correctly describes the lowering of E_0T for growing phase slippage it does not fully cover the reduction in longitudinal focusing. For this it is necessary to follow the bunches through every single cell and to extend the standard formulae of longitudinal dynamics.

3 Longitudinal Focusing

In the following it is assumed that: all cavities have either odd or even field symmetry, the electrical center of each cell is identical with the geometrical center, and the phase change by acceleration is negligible. Then the phase slip angles in a multicell cavity are symmetric with respect to the cavity center and are given by:

$$\phi_{sn}(\frac{\beta}{\beta_d}) = \pm\frac{2n-1}{2}\pi\left(\frac{\beta_d}{\beta}-1\right) \qquad n=1..\frac{N_{total}}{2} \qquad N_{total} \text{ is even} \qquad (4)$$

$$\tilde{\phi_{sn}}(\frac{\beta}{\beta_d}) = \pm n\pi\left(\frac{\beta_d}{\beta}-1\right) \qquad n=0..\frac{N_{total}-1}{2} \qquad N_{total} \text{ is odd} \qquad (5)$$

[**The sign is negative for cells which are on the downstream side and positive for cells on the upstream side, n - cell number, counting from the center of a multicell cavity to the outside (if n_{total} is odd the center cell has the number n=0).**]

Assuming continuous acceleration which corresponds to the idea of an average accelerating force, the "single cell" theory (without phase slip and with β_s, $\gamma_s \approx$ const., i.e. Refs. [2],[3],[4]) yields the following equation for particle trajectories in the longitudinal phase space:

$$\frac{\omega}{2mc^3\beta_s^3\gamma_s^3}(\Delta W)^2 + U(\phi) = H_\phi \qquad (6)$$

where each Hamiltonian H_ϕ defines a different trajectory.
The potential well U, which is normalised to $U(\phi_s) = 0$:

$$U(\phi) = qE_0T(\sin\phi - (\phi-\phi_s)\cos\phi_s - \sin\phi_s) \qquad (7)$$

provides the focusing force that keeps particles inside of the stable region, the so called RF bucket[1)]. The limiting trajectory, dividing the stable from the unstable region is called separatrix. Evaluating Eq. (6) at the stationary but unstable point: $[\phi = -\phi_s, \ \Delta W = 0]$ ($\Rightarrow$ potential maximum, see Fig. 4), determines the constant: $H_{\phi,max}$ and yields the separatrix equation:

$$\frac{\omega}{2mc^3\beta_s^3\gamma_s^3}(\Delta W)^2 + qE_0T(\sin\phi - (\phi+\phi_s)\cos\phi_s + \sin\phi_s) = 0 \qquad (8)$$

In the multicell case one can estimate the resulting potential well by extending the idea of averaged forces over all cells of a cavity. This simplification is valid as long as the change of β_s is small along the multicell cavity, an assumption that is usually fulfilled for proton linacs above a certain energy. Looking for instance at a four cell structure with $\beta \neq \beta_d$ one simply computes the mean potential by:

$$\begin{aligned} U(\phi) = \frac{qE_0T_{av}}{4} \Big\{ & \frac{T_2}{T_{av}}\Big[\sin(\phi-\phi_{s2}) - (\phi-\phi_r)\cos(\phi_r-\phi_{s2}) - \sin(\phi_r-\phi_{s2})\Big] \\ + & \frac{T_1}{T_{av}}\Big[\sin(\phi-\phi_{s1}) - (\phi-\phi_r)\cos(\phi_r-\phi_{s1}) - \sin(\phi_r-\phi_{s1})\Big] \\ + & \frac{T_1}{T_{av}}\Big[\sin(\phi+\phi_{s1}) - (\phi-\phi_r)\cos(\phi_r+\phi_{s1}) - \sin(\phi_r+\phi_{s1})\Big] \\ + & \frac{T_2}{T_{av}}\Big[\sin(\phi+\phi_{s2}) - (\phi-\phi_r)\cos(\phi_r+\phi_{s2}) - \sin(\phi_r+\phi_{s2})\Big]\Big\} \end{aligned} \qquad (9)$$

The expression can be simplified to:

$$\begin{aligned} U(\phi) = \frac{qE_0T_{av}}{4}\Big\{\Big[2\frac{T_1}{T_{av}}\cos\phi_{s1} + 2\frac{T_2}{T_{av}}\cos\phi_{s2}\Big] \\ \cdot\Big[\sin\phi - (\phi-\phi_r)\cos\phi_r - \sin\phi_r\Big]\Big\} \end{aligned} \qquad (10)$$

[**ϕ_r is the reference phase (average phase of the synchronous particle), ϕ_{s1} and ϕ_{s2} are the slip angles as defined in (4) and (5), $T_{1,2}$ are the transit time factors in the inner and outer cells, and T_{av} is the average transit time factor for the whole cavity.**]

[1)] particles in the bucket can be "lifted" to higher energies

Comparing Equations (7) and (10), one can see that the phase slip terms are completely decoupled from the phase description, which means that the phase limits for stable particle motion remain the same, while the amplitude of the potential well is now scaled by a slip factor:

$$fs(n=4) = \frac{1}{4}\Big[2\frac{T_2}{T_{av}}\cos\phi_{s2} + 2\frac{T_1}{T_{av}}\cos\phi_{s1}\Big] \tag{11}$$

Using the same principle one can easily derive slip factors for 5 and 6 cell structures:

$$\begin{aligned} fs(n=5) &= \frac{1}{5}\Big[2\frac{T_2}{T_{av}}\cos\tilde{\phi}_{s2} + 2\frac{T_1}{T_{av}}\cos\tilde{\phi}_{s1} + \frac{T_0}{T_{av}}\Big] & (12)\\ fs(n=6) &= \frac{1}{6}\Big[2\frac{T_3}{T_{av}}\cos\phi_{s3} + 2\frac{T_2}{T_{av}}\cos\phi_{s2} + 2\frac{T_1}{T_{av}}\cos\phi_{s1}\Big] & (13)\\ \dots &= \dots \end{aligned}$$

We note that the slip factors yield an additional degradation of the potential well that is not contained in the $(E_0T)_{av}$ integral for the whole cavity. Figure 3 shows the evolution of the slip factor and the total degradation of the potential well (given by $T_{av} \cdot f_s(n)$) for 2-6 cell cavities at different velocities.

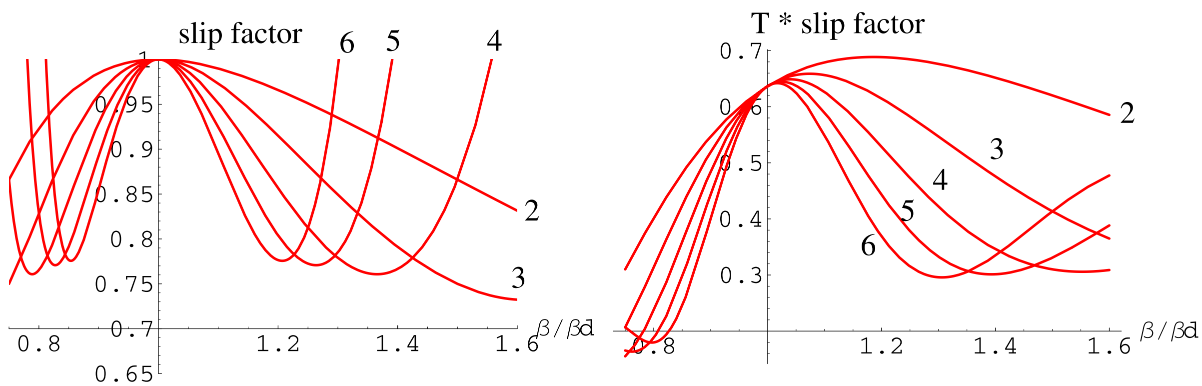

Figure 3: Slip factor and total degradation of the potential well for 2-6 cell cavities at different velocities

The separatrix for the multicell case is obtained by combining Eq. (10) and Eq. (6) and setting $H_\phi = H_{\phi,max}$ i.e. ($\phi = -\phi_r$):

$$\frac{\omega}{2mc^3\beta_r^3\gamma_r^3}(\Delta W)^2 + qE_0T_{av}fs(n)\Big[\sin\phi - (\phi+\phi_r)\cos\phi_r + \sin\phi_r\Big] = 0 \tag{14}$$

The phase slip terms remain decoupled from the phase description, resulting in a bucket that is reduced in energy width but with the same phase width as a "single cell bucket" ($\approx -\phi_r \rightarrow 2\phi_r$). The relative reduction of the maximum energy acceptance is then given by:

$$\sqrt{T_{av}\cdot f_s(n)} \tag{15}$$

and can be computed by taking the square root of the values in Fig. 3. The absolute value of the maximum energy width is defined by:

$$\Delta W(\phi = \phi_r) = \sqrt{\frac{-2qE_0T_{av}mc^3\beta_r^3\gamma_r^3 f_s(n)}{\omega}\Big(\sin\phi_r - \phi_r\cos\phi_r\Big)} \tag{16}$$

We note that even for $\beta/\beta_d >> 1$ the energy acceptance is only reduced down to a certain threshold ($\approx 70\%$ of ΔW at the design velocity β_D), while for $\beta/\beta_d < 1$ the stability limit is reached very quickly (Fig.3).

As an example we show the degrading effect on longitudinal focusing for a multicell cavity with a geometrical β of $\beta = 0.52$. Figure 4 shows the potential well and the separatrix for a 4 and 6 cell structure compared to the single cell case at a kinetic energy of 100 MeV ($\frac{\beta}{\beta_d} = 0.82$).

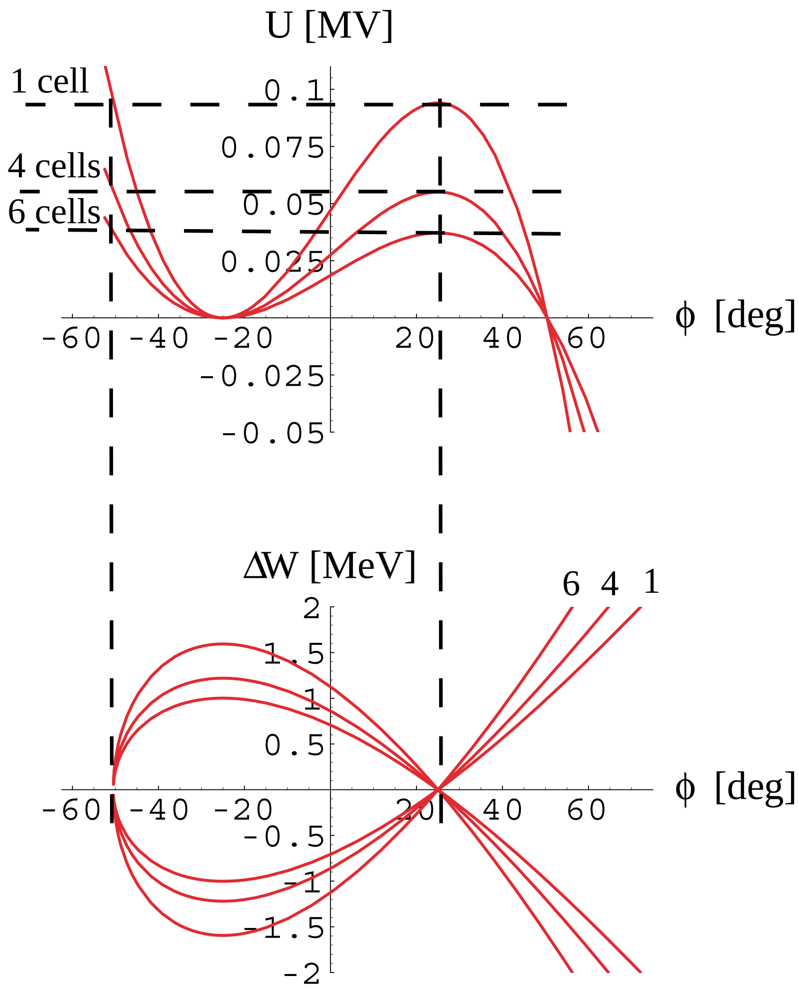

Figure 4: Potential well and separatrix for a 1/4/6 cell $\beta = 0.52$ cavity at 100 MeV, $\phi_r = -25^o$

For the 6 cell case the amplitude of the potential well is $\approx 60\%$ lower than for the single cell case, meaning that also the average longitudinal focusing force is reduced by the same factor.

4 Phase advance

Due to the smaller longitudinal focusing force also the phase advance must change with large slip angles. The smooth phase advance (zero current) per focusing period in the 'single cell' description is defined as [2]:

$$\sigma_{0L}^2 = -\frac{2\pi q N^2 \lambda E_0 T \sin(\phi_s)}{mc^2 \beta \gamma^3} \quad (17)$$

$$\sigma_{0T}^2 = \sigma_\beta^2 - \frac{1}{2}\sigma_{0L}^2 \quad (18)$$

Using again the multicell formalism yields:

$$\sigma_{0L}^2 = -fs(n)\frac{2\pi q N^2 \lambda E_0 T_{av} \sin(\phi_r)}{mc^2 \beta \gamma^3} \quad (19)$$

for the longitudinal phase advance, which is now scaled by $\sqrt{T_{av} \cdot f_s(n)}$. From Fig. 3 one can see that this factor can easily yield a substantial reduction of longitudinal phase advance. At the same time the transverse phase advance is increased. This contradictory variation of transverse and longitudinal phase advance can drive the outer particles into unstable regions of the parameter space and thus produce halo particles. Since the transverse optics can easily compensate for the transverse phase advance variation, the only real limitation comes from the reduction of longitudinal phase advance.

For the above mentioned example of a $\beta = 0.52$ multicell cavity at 100 MeV ($\beta/\beta_d \approx 0.82$) we calculate the reduction in longitudinal zero current phase advance compared to the design velocity: the energy acceptance as well as the longitudinal zero current phase advance are reduced by $\approx 35\%$ for a four cell cavity and by $\approx 45\%$ for a six cell cavity. These values are confirmed by simulations with the 3D envelope code TRACE3D [5] (within 2% accuracy). Simulations with acceleration still show a good agreement with the predicted reduction of phase advance.

5 Conclusions

A simplified model (no acceleration, constant cavity fields) was employed to derive some basic formulae, describing the effect of phase slippage on longitudinal dynamics. We have seen that phase slippage reduces the transit time factor, the longitudinal focusing force and thus the energy width of the RF bucket. The total reduction in energy width is given by the transit time factor times an additional slip factor which accounts for the nonlinear dependence of the focusing force on the slip angle. The maximum phase width, however, is not affected by phase slippage. For velocities below the design velocity β_d the energy acceptance is soon reduced to zero, while for $\beta > \beta_d$ the maximum energy width always stays above a certain threshold ($\approx 70\%$ of the energy width at the design velocity β_d). This indicates that high gradient multicell cavities can be operated well above their design velocity without affecting the stability of the beam. Since the transverse phase advance is also affected by large slip angles, the optics of the machine have to be modified accordingly.

Special care has to be taken, when simulating a multicell structure. It is important that the matching code (usually an r.m.s. envelope code) and the multiparticle code use the same RF gap model and treat the phase slip correctly. Many models use for instance a gap that is represented by a zero length gap and two adjacent drift lengths. Some codes calculate the transit time factors for each cell of a cavity correctly but then use the average phase to calculate the focusing forces in each cell. This simplification works fine as long as the slip angles do not exceed the linear region of the cos curve but becomes increasingly wrong for larger slip angles. If now the matching code and the simulation code use different RF gap models, then every multiparticle simulation starts with an intrinsic mismatch.

6 Acknowledgements

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