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Solenoidal Channels Linear Dynamics

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Abstract

In this paper we study the single particle and beam linear transport through a straight solenoidal channel. We derive the single particle invariants and show their use in an extended Courant-Snyder theory for the solenoidal coupled system. Matching between solenoidal channels and between solenoidal and quadrupolar channels are discussed. We give envelope solutions and benchmark them with some numerical examples.

1 Introduction

For muon colliders [1, 2, 3] and neutrino sources [4] lossless beam transport is an important issue which requires a careful lattice design. Hardware technical limits and facility cost control impose further constrains which make the design challenging. Typically the muon beam exhibits a very large emittance which sets part of the beam in a non paraxial beam dynamics regime. The problem requires a full 6D description: however as a first step in the linac design, one can consider the beam in a linear paraxial regime and use an analytical solution of the single particle dynamics. Afterwards, an the optimization of the channel including more realistic dynamical features can be performed by means of proper tracking computer code [5, 6]. In this work we present the analytical solution of the single particle motion in a straight solenoidal transport channel using the following two approximations: 1) the motion is paraxial; 2) the equations of motion are linear.

2 Solenoidal Channel Single Particle Linear Dynamics

We start with the equations of motion for a solenoidal channel in the linear approximation [7]

$$\begin{array}{l} x'' - Sy' - \frac{1}{2}S'y = 0 \\ y'' + Sx' + \frac{1}{2}S'x = 0 \end{array} \tag{1}$$

where $S = QB_s(s)/p$ (units mks). Here, Q [C] is charge of the particle, B_s [T] is the longitudinal magnetic field, and p [Kg m/s] is the longitudinal momentum. An equivalent expression is $S = 299.79qB_s(s)/(\beta E)$, with q [1] the particle charge state in units of e, β the relativistic factor (only here), and E is the particle energy in units of MeV. With the symbol $'$ we denote the space derivative d/ds. By using Eq. 1, we assume that B_s has no radial dependence from the transverse position of the particle. If this dependence were included, the equations of motion would be nonlinear. Here, we propose a method to solve Eq. 1. General methods [8, 9, 10] have been proposed for the linear coupled particle dynamics. The linear particle dynamics in a solenoidal transport channel is included in them, however we propose here a method only for solenoidal channels which gives simple results.

Following [11, 7] we define the complex variable $z = x + iy$ so that the system of differential equations Eq. 1 become the complex differential equation

$$z'' + iSz' + \frac{i}{2}S'z = 0. \tag{2}$$

In analogy with analytic mechanics, we can represent the particle coordinates in a vectorial notation and the transformation between different frames can be expressed in a vectorial form. This reformulation of the problem allows an easier understanding of the dynamics. Introducing the vector $\mathbf{z} = (z, z')$ the equation of motion Eq. 2 can be written as

$$\mathbf{z}' = \mathsf{C}\mathbf{z}, \quad \text{with} \quad \mathsf{C} = \begin{pmatrix} 0 & 1 \\ -i\frac{S'}{2} & -iS \end{pmatrix}. \tag{3}$$

Following [11, 7] we consider the transformation $w = ze^{i\phi}$, where ϕ is an auxiliary angle that depends on s. Defining $\mathbf{w} = (w, w')$, the relation

$w = ze^{i\phi}$ allows to find the transformation $\lambda : \mathbf{w} \rightarrow \mathbf{z}$. Since λ is a linear transformation, it can be written as

$$\mathbf{z} = \Lambda \mathbf{w}, \quad \text{with} \quad \Lambda = \mathrm{e}^{-\mathrm{i}\phi} \begin{pmatrix} 1 & 0 \\ -i\phi' & 1 \end{pmatrix}. \tag{4}$$

Inserting Eq. 4 in Eq. 3 we obtain the equation

$$\mathbf{w}' = \Lambda^{-1}[-\Lambda' + \mathsf{C}\Lambda]\mathbf{w}. \tag{5}$$

The explicit form of the matrix is

$$\Lambda^{-1}[-\Lambda' + \mathsf{C}\Lambda] = \begin{pmatrix} 0 & 1 \\ \phi'^2 + i\phi'' - i\frac{S'}{2} - S\phi' & i(2\phi' - S) \end{pmatrix}. \tag{6}$$

Choosing $S = 2\phi'$, in the frame $\{w, w'\}$ Eq. 5 become

$$\mathbf{w}' = \mathsf{D}\mathbf{w}, \quad \text{with} \quad \mathsf{D} = \begin{pmatrix} 0 & 1 \\ -\phi'^2 & 0 \end{pmatrix}. \tag{7}$$

Arbitrarily we fix an initial condition for ϕ ϕ such that $\phi(0) = 0$. With this choice at the beginning in $s = 0$ we find $\mathbf{z}_0 = \mathbf{w}_0$, i.e. in the frames $\{z, z'\}$, and $\{w, w'\}$ of the initial coordinates of the particle are identical. The angle ϕ is then given by

$$\phi(s) = \frac{1}{2}\int_0^s S(t)dt. \tag{8}$$

Note that the transformation $w = ze^{i\phi}$ just reformulates the particle dynamics in the Larmor frame, where ϕ is the angle of the rotating frame and ϕ' its angular velocity. The coefficient ϕ'^2 is the square of the solenoid strength which gives to Eq. 7 always a focusing character. Since S is real and w is complex,i.e. $w = w_r + iw_i$, Eq. 7 can be decomposed in two decoupled Hill's differential equations

$$w_r'' + \phi'^2 w_r = 0, \quad w_i'' + \phi'^2 w_i = 0. \tag{9}$$

If the solenoidal channel has periodicity L, we can apply the Courant-Snyder theory [12] at each of the 2 decoupled equations Eqs. 9. For each of these equations the particle coordinates are transported by a one turn map which has the standard form [13]. Recombining the two vectorial representations in a complex vectorial form, the transfer map in $\{w, w'\}$ becomes $\mathbf{w}_1 = \mathsf{M}\mathbf{w}_0$ with

$$\mathsf{M} = \begin{pmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\gamma\sin\psi & \cos\psi - \alpha\sin\psi \end{pmatrix}, \tag{10}$$

where β, α, γ, i.e. the Twiss parameters, are given by

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \frac{S^2}{4}\beta^2 = 1, \qquad \alpha = -\frac{\beta'}{2}, \qquad \psi = \int_s^{s+L} \frac{1}{\beta(t)}dt, \tag{11}$$

and satisfy the periodic conditions $\beta(s) = \beta(s+L)$, and $\alpha(s) = \alpha(s+L)$. The index 0 means entrance section of the channel and the index 1 means the exit section of the channel, more precisely: the map M transport $\mathbf{w}_0$ to $\mathbf{w}_0$, where $\mathbf{w}_0 = \mathbf{w}(s)$, and $\mathbf{w}_1 = \mathbf{w}(s+L)$. Now we can use the transformation Eq. 4 to rewrite the map M in the original frame $\{z, z'\}$. From $\mathbf{w}_1 = \mathsf{M}\mathbf{w}_0$ multiplying by Λ_1 and substituting $\mathbf{w}_0 = \Lambda_0\mathbf{z}_0$, $\mathbf{w}_1 = \Lambda_1\mathbf{z}_1$ we find that in $\{z, z'\}$ M becomes $\mathsf{L} = \Lambda_1\mathsf{M}\Lambda_0^{-1}$ i.e. explicitly

$$\mathsf{L} = e^{-i\phi_1}\begin{pmatrix} \cos\psi + \hat{\alpha}\sin\psi & \hat{\beta}\sin\psi \\ -\hat{\gamma}\sin\psi & \cos\psi - \hat{\alpha}\sin\psi \end{pmatrix}. \tag{12}$$

The hat Twiss parameter are defined as

$$\hat{\beta} = \beta, \qquad \hat{\alpha} = \alpha + i\beta\phi_1', \qquad \hat{\gamma} = \frac{1+\hat{\alpha}^2}{\hat{\beta}}.$$

The solenoidal channel transport one turn map finally reads $\mathbf{z}_1 = \mathsf{L}\mathbf{z}_0$.

2.1 Transformation of the map from $\mathbb{C}^2$ to $\mathbb{R}^4$

In order to transform the map Eq. 12 from the frame $\{z, z'\}$ to the laboratory frame $\{x, x', y, y'\}$ we have to use the definition $z = x + iy$, a matrix $\mathsf{A} \in \mathbb{C}^2 \times \mathbb{C}^2$ is represented in the lab frame phase space by the matrix $\Sigma(\mathsf{A}) \in \mathbb{R}^4 \times \mathbb{R}^4$ of the form

$$\Sigma(\mathsf{A}) = \left(\begin{array}{c|c} \mathrm{Re}\,\mathsf{A} & -\mathrm{Im}\,\mathsf{A} \\ \hline \mathrm{Im}\,\mathsf{A} & \mathrm{Re}\,\mathsf{A} \end{array}\right). \tag{13}$$

We find for Eq. 12 $\Sigma(\mathsf{L})$ is given by the same rule where

$$\mathrm{Re}\,\mathsf{L} = \mathsf{c}_1\,\mathsf{M} + \phi_1'\,\mathsf{s}\begin{pmatrix} \beta\,\mathsf{s}_1 & 0 \\ -2\,\alpha\,\mathsf{s}_1 + \beta\,\phi_1'\,\mathsf{c}_1 & -\beta\,\mathsf{s}_1 \end{pmatrix}, \tag{14}$$

and

$$-\mathrm{Im}\,\mathsf{L} = \mathsf{s}_1\,\mathsf{M} + \phi_1'\,\mathsf{s}\begin{pmatrix} -\beta\,\mathsf{c}_1 & 0 \\ 2\,\alpha\,\mathsf{c}_1 + \beta\,\phi_1'\,\mathsf{s}_1 & \beta\,\mathsf{c}_1 \end{pmatrix}, \tag{15}$$

with $\mathsf{s} = \sin\psi, \mathsf{c} = \cos\psi, \mathsf{s}_1 = \sin\phi_1, \mathsf{c}_1 = \cos\phi_1$, and M given by Eq. 10.

As example of application we consider the channel formed by one solenoid and two small drifts located one at the entrance and one exit of the solenoid. Assuming an hard edge approximation for fringe filed (B_s drops to zero in a thin slice) we find from the equation of β that $S\beta = 2$ with β constant in good approximation and $\alpha = 0$. In this condition we find from Eq. 8 that $\phi = \psi$. The solenoid map for the transport from the center of the small drifts to the next is characterized by $\phi_1' = 0$ so Eqs. 14, 15 become $\mathrm{Re}\,\mathsf{L} = \mathsf{c}\,\mathsf{M}$, and $-\mathrm{Im}\,\mathsf{L} = \mathsf{s}\,\mathsf{M}$ and we recover the classical expression for the transfer map of the solenoid

$$\Sigma(\mathsf{L}) = \left(\begin{array}{c|c} \mathsf{c}\,\mathsf{M} & \mathsf{s}\,\mathsf{M} \\ \hline -\mathsf{s}\,\mathsf{M} & \mathsf{c}\,\mathsf{M} \end{array}\right), \quad \text{with} \quad \mathsf{M} = \begin{pmatrix} \cos\psi & \frac{2}{S}\sin\psi \\ -\frac{S}{2}\sin\psi & \cos\psi \end{pmatrix}. \tag{16}$$

2.2 Simplecticity

The coordinates $\{x, x', y, y'\}$ are not the canonical conjugate coordinates. From the Hamiltonian of the solenoidal channel [11, 14]

$$H = \frac{1}{2}\left(p_x + \frac{S}{2}y\right)^2 + \frac{1}{2}\left(p_y - \frac{S}{2}x\right)^2 \tag{17}$$

we find that the canonical momentums are

$$p_x = x' - \frac{S}{2}y, \qquad p_y = y' + \frac{S}{2}x, \tag{18}$$

hence the transformation from the canonical conjugate coordinates $\{x, p_x, y, p_y\}$ to the standard laboratory coordinates phase space $\{x, x', y, y'\}$ is

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & S/2 & 0 \\ 0 & 0 & 1 & 0 \\ -S/2 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}. \tag{19}$$

Calling $\mathcal{C}$ the transformation matrix in Eq. 19, then $\Sigma(\mathsf{A})$ induced by A is symplectic if $\Sigma(\mathsf{A})$ satisfies $\mathcal{C}^{-1}\Sigma(\mathsf{A})\mathcal{C}\mathsf{J}_4\mathcal{C}^T\Sigma(\mathsf{A})^T\mathcal{C}^{-1\,T} = \mathsf{J}_4$ where

$$\mathsf{J}_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{20}$$

The Simplecticity condition can be rewritten as

$$\Sigma(\mathsf{A})\mathcal{J}_4\Sigma(\mathsf{A})^T = \mathcal{J}_4, \tag{21}$$

where we call $\mathcal{J}_4 = \mathcal{C}\mathsf{J}_4\mathcal{C}^T$. By direct calculation it is straightforward to check that $\Sigma(\mathsf{L})\mathcal{J}_4\Sigma(\mathsf{L})^T = \mathcal{J}_4$ where L is the transport map defined by Eq. 12.

3 Analytical Solution and Invariants

In the previous section we have introduced the frames $\{z, z'\}$,$\{w, w'\}$. We consider now a new transformation of coordinates which changes the matrix M in a more simple form, i.e. a rotation. Following the same technique used for rings [13] we consider the transformation

$$\mathbf{w} = \mathsf{T}\mathbf{u} \quad \text{with} \quad \mathsf{T} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}, \tag{22}$$

and $\mathbf{u} = (u_1, u_2)$. By using Eq. 22 the matrix M expressed in Eq. 10 can be written as

$$\mathsf{M} = \mathsf{T}\mathsf{R}\mathsf{T}^{-1}, \tag{23}$$

where

$$\mathsf{R} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}, \tag{24}$$

and ψ is the phase advance from the beginning of the periodic cell (index 0), to the end of the periodic cell (index 1). In the frame $\{u_1, u_2\}$ the motion becomes a pure rotation, i.e. $\mathbf{u}_1 = \mathsf{R}\mathbf{u}_0$ and in the frame $\{z, z'\}$ the one turn map can be written as

$$\mathsf{L} = \Lambda\mathsf{T}\mathsf{R}\mathsf{T}^{-1}\Lambda^{-1}. \tag{25}$$

Next we derive the invariants and the analytical solution of Eq. 1. In the following discussion the index 0, and 1 will mean simply the initial and the final position, and not beginning and end of the periodic cell. Periodic condition might be dropped as well. In Appendix A we prove that the coordinate transformations $\mathbf{w} = \mathsf{T}\mathbf{u}$, $\mathbf{z} = \Lambda\mathbf{w}$, and the maps $\mathsf{L}, \mathsf{M}, \mathsf{R}$ are related according to the following scheme

$$\begin{array}{ccc} \mathbf{z}_0 & \xrightarrow{\mathsf{L}} & \mathbf{z}_1 \\ \Lambda_0 \uparrow & & \Lambda_1 \uparrow \\ \mathbf{w}_0 & \xrightarrow{\mathsf{M}} & \mathbf{w}_1 \\ \mathsf{T}_0 \uparrow & & \mathsf{T}_1 \uparrow \\ \mathbf{u}_0 & \xrightarrow{R} & \mathbf{u}_1 \end{array} \tag{26}$$

Consequently the transfer map in reads R in $\{u_1, u_2\}$, reads $\mathsf{M} = \mathsf{T}_1\mathsf{R}\mathsf{T}_0^{-1}$ in $\{w, w'\}$, and reads $\mathsf{L} = \mathsf{V}_1\mathsf{R}\mathsf{V}_0^{-1}$ in $\{z, z'\}$. Here, $\mathsf{V} = \Lambda\mathsf{T}$ i.e.

$$\mathsf{V} = e^{-i\phi} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha + i\phi'\beta}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \tag{27}$$

We consider now the transport in the frame $\{u_1, u_2\}$ which can be written as $\mathbf{u}_1 = \mathsf{R}\mathbf{u}_0$. This transformation is equivalent to the following two rotations

$$\begin{pmatrix} u_{1r1} \\ u_{2r1} \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} u_{1r0} \\ u_{2r0} \end{pmatrix} \tag{28}$$

$$\begin{pmatrix} u_{1i1} \\ u_{2i1} \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} u_{1i0} \\ u'_{2i0} \end{pmatrix} \tag{29}$$

where $u_1 = u_{1r1} + iu_{1i1}$, and $u_2 = u_{2r1} + iu_{2i1}$. Eqs. 28, 29 show the existence of the two invariants $u_{1r1}^2 + u_{2r1}^2 = \epsilon_1$ and $u_{1i1}^2 + u_{2i1}^2 = \epsilon_2$ (See also Appendix B, and Appendix C). We can write then $\mathbf{u}_1 = (u_1, u_2)$ in a form which is consistent with the transport relation $\mathbf{u}_1 = \mathsf{R}\mathbf{u}_0$. We find that

$$\begin{aligned} u_1 &= \sqrt{\epsilon_1}\sin(\psi + \delta_1) + i\sqrt{\epsilon_2}\sin(\psi + \delta_2) \\ u_2 &= \sqrt{\epsilon_1}\cos(\psi + \delta_1) + i\sqrt{\epsilon_2}\cos(\psi + \delta_2) \end{aligned} \tag{30}$$

here ϵ_1, ϵ_2 are the invariants, and the initial phases of the particle δ_1, δ_2 are other two invariants. By using V we can write $\mathbf{u} = \mathsf{V}^{-1}\mathbf{z}$ then write the invariants in the $\{z, z'\}$ frame and finally in the lab frame $\{x, x', y, y'\}$. Since the invariants ϵ_1, ϵ_2 exhibits a dependence from $\cos\psi$ and $\sin\psi$, we can try to use combinations of ϵ_1, ϵ_2 in order to obtain an other invariant of simpler form. An appropriate choice is $\epsilon_1 + \epsilon_2$, and we obtain

$$\epsilon_1 + \epsilon_2 = \epsilon_x + \epsilon_y + \beta\phi^{'2}(x^2 + y^2) + 2\beta\phi' L_z \tag{31}$$

where $\epsilon_x = \gamma x^2 + 2\alpha x x' + \beta x^{'2}$, $\epsilon_y = \gamma y^2 + 2\alpha y y' + \beta y^{'2}$ are the usual horizontal and vertical Courant-Snyder invariants, and $L_z = xy' - yx'$ is proportional to the mechanical angular momentum. From Eqs. 30 we find another invariant which is combination of ϵ_1, ϵ_2, in fact

$$u_{1r}u_{2i} - u_{1i}u_{2r} = \sqrt{\epsilon_1\epsilon_2}\sin(\delta_1 - \delta_2) = \phi'(x^2 + y^2) + L_z \tag{32}$$

From Eqs. 31, 32, given the particle coordinates x, x', y, y', and δ_1, δ_2 we can compute ϵ_1, ϵ_2 less an inversion $\epsilon_1 \leftrightarrow \epsilon_2$. The physical interpretation of Eq. 32 is the following: since in Eq. 1 we assume the magnetic field independent from the particle transverse position, the term $\phi' r^2$ is proportional to the magnetic flux through the circular orbit of radius r, that is $\phi' r^2 = (Q\Psi/2\pi)/p$, where Ψ is the magnetic flux. On the other hand L_z can be expressed as $L_z = \gamma m r^2 \dot{\theta}/p$ where here (and only here) γ is the relativistic factor, m is the particle mass, $\dot{\theta}$ is the angular velocity and $\gamma m r^2 \dot{\theta}$ is the mechanical angular momentum. The sum of the mechanical angular momentum with $Q\Psi/2\pi$ is just the canonical angular momentum. Since Eq. 32 expresses an invariant, and p is constant, Eq. 32 is just a reformulation of the well known Bush theorem [15], i.e. of the conservation of the canonical angular momentum $p_\theta = p(\phi' r^2 + L_z)$ for the dynamical systems with cylindrical symmetry.

Since the vector $\mathbf{u}_1 = (u_1, u_2)$ in Eqs. 30 gives the particle coordinates in a generic position during the motion, it can be used to find the particle coordinates in the frame $\{z, z'\}$. By using V we find

$$\mathbf{z}_1 = \mathsf{V}\mathbf{u}_1 \tag{33}$$

and remembering the definitions $z = x + iy, z' = x' + iy'$ and dropping the index 1 we find the solution of Eq. 1

$$
\begin{aligned}
x &= \sqrt{\beta\epsilon_1}\sin(\psi+\delta_1)\cos\phi + \sqrt{\beta\epsilon_2}\sin(\psi+\delta_2)\sin\phi \\
y &= \sqrt{\beta\epsilon_2}\sin(\psi+\delta_2)\cos\phi - \sqrt{\beta\epsilon_1}\sin(\psi+\delta_1)\sin\phi \\
x' &= \sqrt{\tfrac{\epsilon_1}{\beta}}[\cos(\psi+\delta_1)\cos\phi - \alpha\sin(\psi+\delta_1)\cos\phi - \beta\phi'\sin(\psi+\delta_1)\sin\phi] + \\
&+ \sqrt{\tfrac{\epsilon_2}{\beta}}[\cos(\psi+\delta_2)\sin\phi - \alpha\sin(\psi+\delta_2)\sin\phi + \beta\phi'\sin(\psi+\delta_2)\cos\phi] \\
y' &= \sqrt{\tfrac{\epsilon_1}{\beta}}[-\cos(\psi+\delta_1)\sin\phi + \alpha\sin(\psi+\delta_1)\sin\phi - \beta\phi'\sin(\psi+\delta_1)\cos\phi] + \\
&+ \sqrt{\tfrac{\epsilon_2}{\beta}}[\cos(\psi+\delta_2)\cos\phi - \alpha\sin(\psi+\delta_2)\cos\phi - \beta\phi'\sin(\psi+\delta_2)\sin\phi]
\end{aligned}
\tag{34}
$$

Since at the beginning for $s=0$ we have $\phi(0)=0$ the relations between the initial particle coordinates x_0, x'_0, y_0, y'_0 and the invariants $\epsilon_1, \epsilon_2, \delta_1, \delta_2$ are

$$
\begin{array}{l}
x_0 = \sqrt{\beta\epsilon_1}\sin\delta_1 \\
y_0 = \sqrt{\beta\epsilon_2}\sin\delta_2 \\
x'_0 = \sqrt{\epsilon_1}[\cos\delta_1 - \alpha\sin\delta_1]/\sqrt{\beta} + \sqrt{\beta\epsilon_2}\phi'\sin\delta_2 \\
y'_0 = -\sqrt{\beta\epsilon_1}\phi'\sin\delta_1 + \sqrt{\epsilon_2}[\cos\delta_2 - \alpha\sin\delta_2]/\sqrt{\beta}
\end{array}
\tag{35}
$$

From these equations we can compute the value of the two invariants ϵ_1, ϵ_2 by extracting from the expressions of x'_0, y'_0 the quantity $\sqrt{\epsilon_1}\cos\delta_1$, and $\sqrt{\epsilon_2}\cos\delta_2$ and combining them with the expression of $\sqrt{\epsilon_1}\sin\delta_1$, and $\sqrt{\epsilon_2}\sin\delta_2$ in the expressions of x_0, y_0. We find

$$
\begin{array}{l}
\epsilon_1 = \epsilon_{x0} + \phi'^2\beta y_0^2 - 2\alpha\phi' x_0 y_0 - 2\beta\phi' y_0 x'_0 \\
\epsilon_2 = \epsilon_{y0} + \phi'^2\beta x_0^2 + 2\alpha\phi' x_0 y_0 + 2\beta\phi' x_0 y'_0
\end{array}
\tag{36}
$$

Note that the value of the invariants differ from the usual Courant-Snyder form by the presence of a term which is multiplied by ϕ'. Therefore, if at injection there is no magnetic field (i.e. $\phi' = 0$), these invariants coincide with the standard single particle emittances. The initial angles δ_1, δ_2 are instead obtained by combining Eq. 35. We find

$$
\begin{array}{ll}
\cos\delta_1 = \sqrt{\frac{\beta}{\epsilon_1}}(x'_0 + \frac{\alpha}{\beta}x_0 - \phi' y_0) & \sin\delta_1 = \frac{x_0}{\sqrt{\beta\epsilon_1}} \\
\cos\delta_2 = \sqrt{\frac{\beta}{\epsilon_2}}(y'_0 + \frac{\alpha}{\beta}y_0 + \phi' x_0) & \sin\delta_2 = \frac{y_0}{\sqrt{\beta\epsilon_2}}
\end{array}
\tag{37}
$$

4 Beam Transport and Beam Envelopes

In literature the beam transport in presence of linear coupling has been studied [16, 10, 17]. We apply next the solenoidal transport theory here developed.

4.1 Lossless Beam Transport

For lossless beam transport one needs to check along all the transport line if particles exceed a given pipe boundary. We first consider a monochromatic beam, i.e. all the particles have the same longitudinal momentum.

From Eqs. 34 we can compute the radius $r = \sqrt{x^2 + y^2}$ of a particle, we find

$$r^2 = \beta(\epsilon_1 \sin^2(\psi + \delta_1) + \epsilon_2 \sin^2(\psi + \delta_2)). \tag{38}$$

Let us call R the radius of the beam pipe; then, the particle is lost if $r > R$. This condition defines the accepted particles over the transport channel length. Since the sinusoidal terms are always smaller than one, one can estimate

$$r^2 = \beta(\epsilon_1 \sin^2(\psi + \delta_1) + \epsilon_2 \sin^2(\psi + \delta_2)) < \beta(\epsilon_1 + \epsilon_2) \tag{39}$$

and therefore one obtains a necessary (but not sufficient) condition for particle loss which is much simple:

$$\epsilon_1 + \epsilon_2 \leq \frac{R^2}{\beta_{max}} \tag{40}$$

with β_{max} being the maximum β over the channel length. This equation together with Eq. 31 defines the volume in the $4D$ lab frame phase space of the transported particles. All the particles outside this volume might be lost. Since for each particle the quantity $\epsilon_1 + \epsilon_2$ is an invariant, it is possible to predict directly at injection weather a particle is transported up to the end of the channel or not. In fact having the particle distribution at injection, one can evaluate by using Eqs. 36, or Eq. 31 the quantity $\epsilon_1 + \epsilon_2$ for each particle and see if it satisfies the lossless transport condition Eq. 40. This result can be used to design a solenoidal channel.

4.2 Beam Envelopes for an Invariant Matched Distribution

In the previous discussion we considered the condition for a lossless transport throughout the channel. In doing so we assumed that the pipe radius were constant. In practice, there might be reasons to change the beam pipe radius along the channel. One should consider the beam's envelope and see if it is contained into the beam pipe or fix the pipe according to the beam envelope. Unfortunately, it is not possible to predict analytically the envelope for whatever initial distribution.

We first consider a distribution where all the particles have the same invariants ϵ_1, ϵ_2, i.e. we fill of particles the 2D surface embedded in the 4D lab frame phase space characterized by the invariants ϵ_1, ϵ_2. We do not specify in which way particles should be distributed in δ_1, δ_2, but its enough that the invariant surface is filled without holes. We call this distribution invariants matched distribution. For an invariant matched distribution where all particles have the same invariants ϵ_1, ϵ_2, by using Eqs. 34 we find the envelopes

$$\begin{array}{l} X = \sqrt{\beta\epsilon_1}|\cos\phi| + \sqrt{\beta\epsilon_2}|\sin\phi|, \\ Y = \sqrt{\beta\epsilon_2}|\cos\phi| + \sqrt{\beta\epsilon_1}|\sin\phi|, \\ X' = \sqrt{\epsilon_1 A/\beta} + \sqrt{\epsilon_2 B/\beta}, \\ Y' = \sqrt{\epsilon_2 A/\beta} + \sqrt{\epsilon_1 B/\beta}, \end{array} \tag{41}$$

where A, B are the two functions $A = \cos^2\phi + (\alpha\cos\phi + \phi'\beta\sin\phi)^2$, $B = \sin^2\phi + (\alpha\sin\phi - \phi'\beta\cos\phi)^2$. For this distribution the maximum radius of the beam is given by $r_{max} = \sqrt{\beta(\epsilon_1 + \epsilon_2)}$.

Transverse kinetic energy $x^{'2} + y^{'2}$ it is an important quantity as well for designing an ionization cooling channel [18, 19] From Eqs. 34 we find that

$$\begin{aligned} x^{'2} + y^{'2} = \; & [(\alpha\sin(\psi+\delta_1) - \cos(\psi+\delta_1))^2 + (\phi'\beta)^2\sin^2(\psi+\delta_1)]\epsilon_1/\beta + \\ + \; & [(\alpha\sin(\psi+\delta_2) - \cos(\psi+\delta_2))^2 + (\phi'\beta)^2\sin^2(\psi+\delta_2)]\epsilon_2/\beta + \\ + \; & 2\phi'\sqrt{\epsilon_1\epsilon_2}\sin(\delta_2 - \delta_1). \end{aligned} \tag{42}$$

Unfortunately we do not find a simple expression for $(x^{'2} + y^{'2})_{max}$ from

Eq. 42. The result can be given as

$$(x'^2 + y'^2)_{max} = \frac{1}{2}\left[\gamma + \phi'^2\beta + \sqrt{\gamma^2 + (\phi'^2\beta)^2 + 2(\alpha^2 - 1)\phi'^2}\right](\epsilon_1 + \epsilon_2) + \xi, \tag{43}$$

where ξ is a suitable number belonging to the interval $[\,0, 2|\phi'|\sqrt{\epsilon_1\epsilon_2}\,]$. Note that this expression become the usual $\gamma(\epsilon_1 + \epsilon_2)$ in absence of the solenoidal magnetic field.

4.2.1 Numerical Example

We consider a numerical example based on a solenoidal channel used for the ionization cooling in the CERN neutrino factory project [6]. For a solenoid with its center in $s = 0$ we compute the magnetic field $B_s(s)$ by using the formula [20]

$$B_s(s) = B_0\frac{\sqrt{(0.5d)^2 + R_s^2}}{d}\left(\frac{0.5d - s}{\sqrt{(0.5d - s)^2 + R_s^2}} + \frac{0.5d + s}{\sqrt{(0.5d + s)^2 + R_s^2}}\right), \tag{44}$$

where d is the length of the solenoid, R_s is the solenoid radius, and B_0 is the magnetic field in the center of the solenoid at $s = 0$. For the CERN scenario are foreseen solenoids with $R = 0.3$ m, $d = 0.76$ m, and $B_0 = 2$ T. For muons with kinetic energy of $E = 200$ MeV we find that $S = 1$ m^{-1}. By using these parameters we simulated two of the 44MHz cooling solenoidal channels. In Fig. 3c we draw a schematic of the solenoidal channel used for this simulation. We considered a beam with emittances $\epsilon_1 = 1$ cm-rad, and $\epsilon_2 = 2$ cm-rad. In Fig. 1 we generate 30 particles with the same invariant ϵ_1, ϵ_2. Fig. 1a,b show the particles track and in red the envelope obtained with the first two equations of Eqs. 41. In blue we plot the envelope obtained with the equation $r = \sqrt{\beta(\epsilon_1 + \epsilon_2)}$. In Fig. 1c,d we plot for each particle x' and y' respectively. Again we plot in red the envelopes obtained from the last two equations of Eqs. 41. In blue we plot $\sqrt{x'^2 + y'^2}$ obtained from Eq. 43 where we use $\xi = 2|\phi'|\sqrt{\epsilon_1\epsilon_2}$. In Fig. 2a we plot for each particle the transverse kinetic energy $x'^2 + y'^2$ and compare it with Eq. 43 for the two extreme cases: with $\xi = 0$ curve in red, and with $\xi = 2|\phi'|\sqrt{\epsilon_1\epsilon_2}$ curve in green. In Fig. 2b,c,d we plot respectively β, α, and S. In Fig. 3a,b we plot the Larmor angle ϕ and the phase advance ψ.

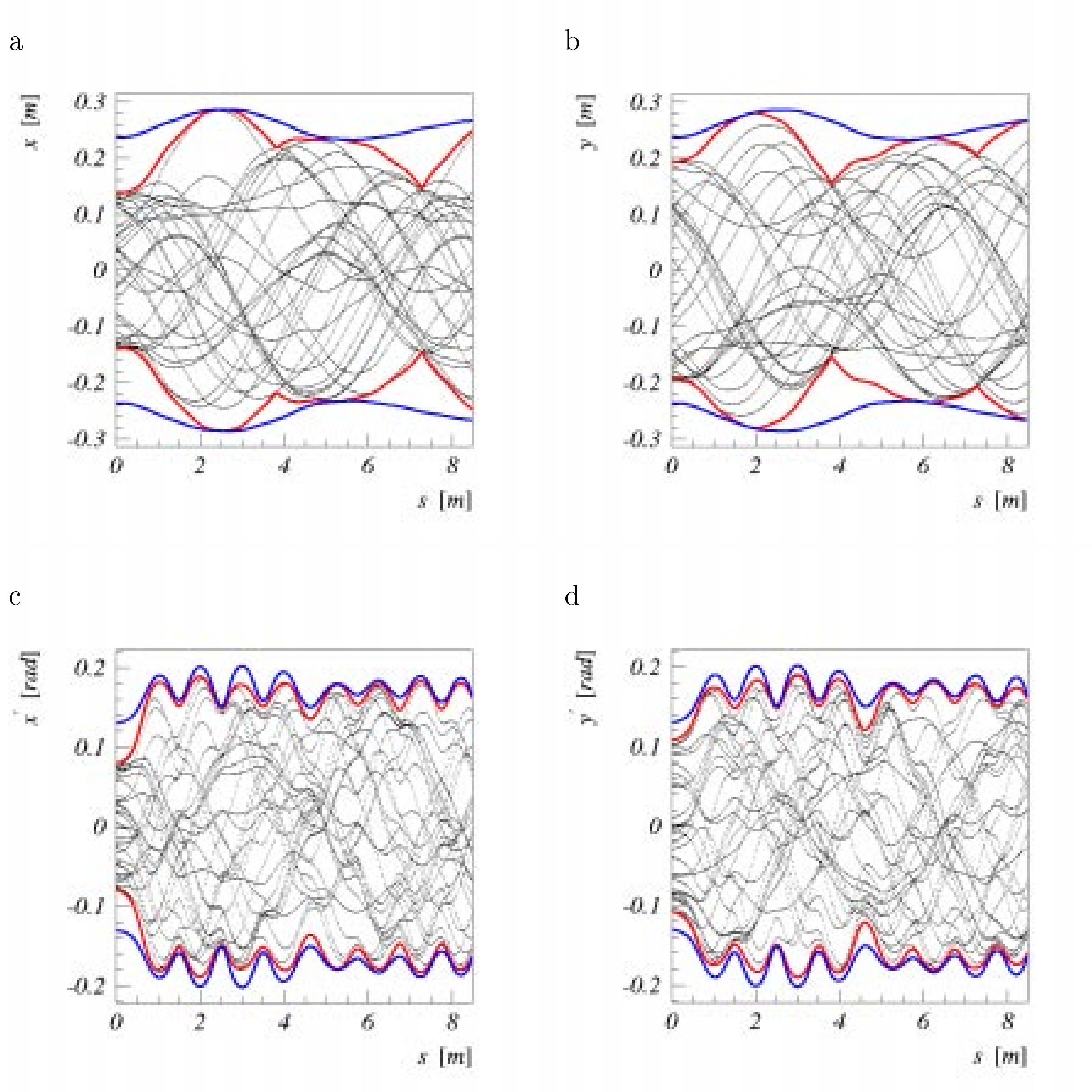

Figure 1: In this figure we plot for the distribution used in Sect. 4.3.1, in a),b) in red the envelope and in blue the maximum radius, and with dotted lines the particle trajectory versus s. In c),d) we plot x', y' versus s. In red we plot the momentum envelope and in blue the maximum momentum envelope.

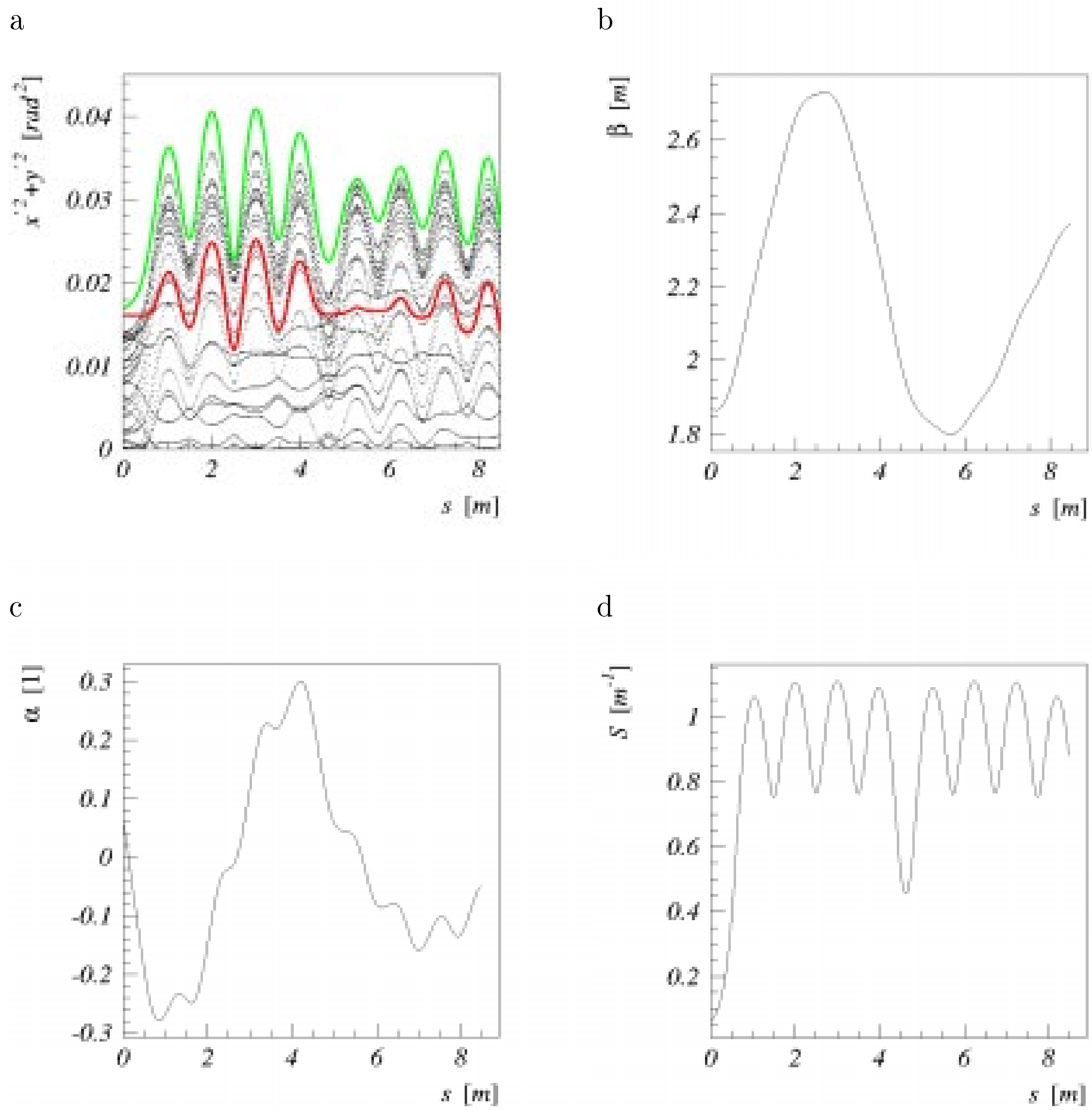

Figure 2: In this figure we plot for the same simulation of Fig. 1: in a) in red and green the theoretical approximations of $(x'^2 + y'^2)_{max}$ and with dotted lines we plot x',and y'. In b),c),d) we plot respectively β, α, S versus s.

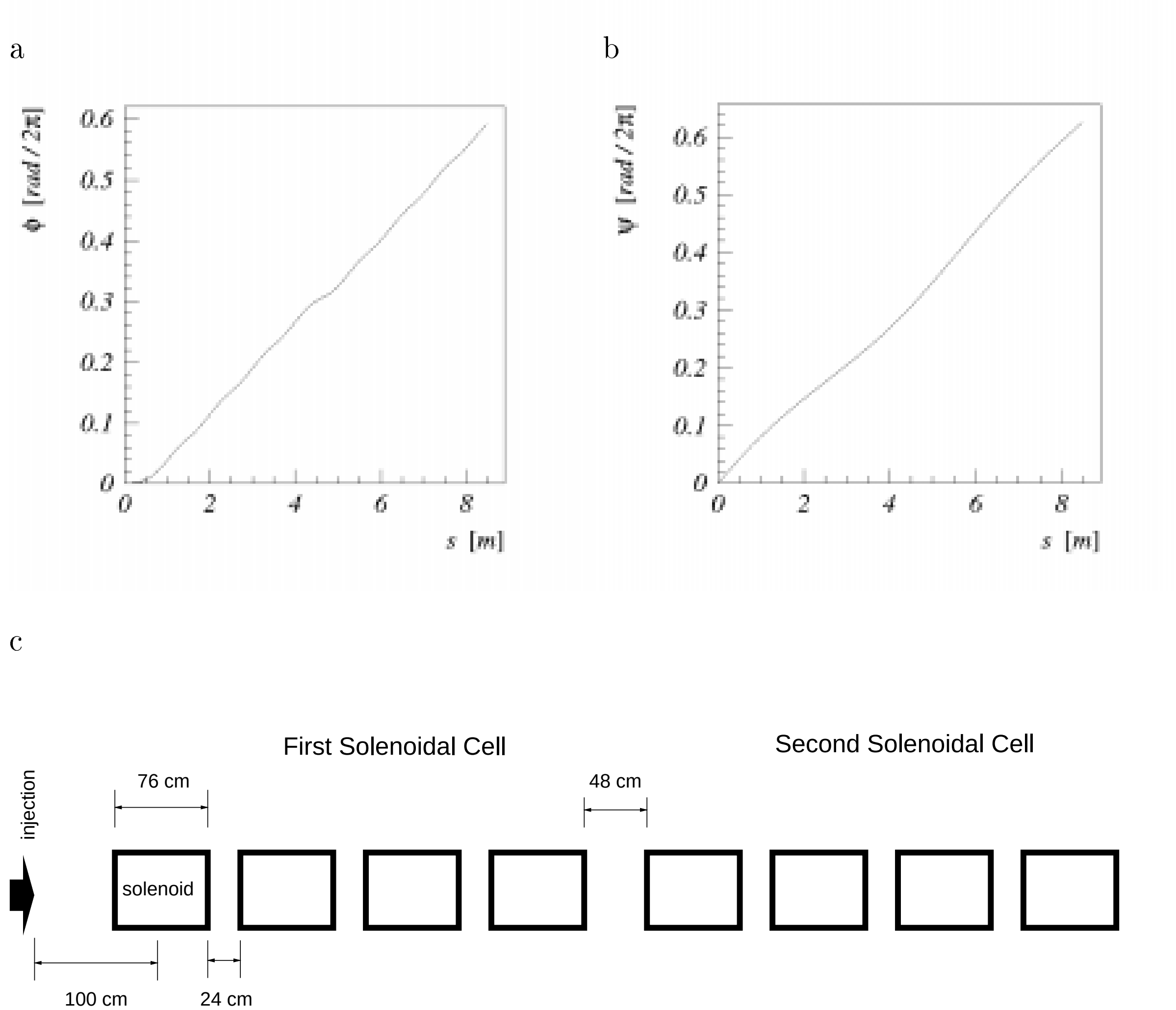

Figure 3: In this figure we plot for the same simulation of Fig. 1: a) ϕ, and b) ψ. In c) we draw a layout of the two solenoidal channel with their dimensions used in the simulation.

4.3 Beam Envelopes for a General Invariant Matched Distribution

A more general invariant matched particle distribution can be written as $f(\epsilon_1/\epsilon_{10}+\epsilon_2/\epsilon_{20})$ where $f(\tau)$ is a function such that $f(\tau)>0$, and $f(\tau)\to 0$ if $\tau \to \infty$ for an example see [15]. This distribution satisfies the time-independent Vlasov equation and therefore it is self consistent. The particle distribution is defined by $\epsilon_{10}, \epsilon_{20}$, and $f(\tau)$. Applying Eqs. 34 to this distribution we find the envelopes

$$\begin{aligned} X &= \sqrt{\beta(\epsilon_{10}\cos^2\phi + \epsilon_{20}\sin^2\phi)}, \\ Y &= \sqrt{\beta(\epsilon_{20}\cos^2\phi + \epsilon_{10}\sin^2\phi)}, \\ X' &= \sqrt{(\epsilon_{10}A + \epsilon_{20}B)/\beta}, \\ Y' &= \sqrt{(\epsilon_{20}A + \epsilon_{10}B)/\beta}. \end{aligned} \tag{45}$$

The maximum radius is $r_{max} = \sqrt{\beta \max\{\epsilon_{10}, \epsilon_{20}\}}$ and we can express $(x'^2 + y'^2)_{max}$ as

$$(x'^2+y'^2)_{max} = \frac{1}{2}\left[\gamma + \phi'^2\beta + \sqrt{\gamma^2 + (\phi'^2\beta)^2 + 2(\alpha^2-1)\phi'^2}\right]\max\{\epsilon_{10},\epsilon_{20}\} + \xi, \tag{46}$$

where ξ is a suitable number belonging to the interval $[\,0, |\phi'|\sqrt{\epsilon_{10}\epsilon_{20}}\,]$.

4.3.1 Numerical Example

We consider here a numerical example based on the same solenoidal channel described in Section 4.2.1, but with the distribution

$$f(\tau) = \begin{cases} 1 & \text{if} \quad 0 < \tau < 1 \\ 0 & \text{if} \quad \tau > 1 \end{cases} \quad . \tag{47}$$

In Fig. 4 we generate 200 particles with the same invariant ϵ_1, ϵ_2. Fig. 4a,b show the particles track and in red the envelope obtained with the first two equations of Eqs. 45. In blue we plot the envelope obtained with the equation $r = \sqrt{\beta \max\{\epsilon_1, \epsilon_2\}}$. In Fig. 4c,d we plot for each particle x' and y' respectively. Again we plot in red the envelopes obtained from the last two equations of Eqs. 45. In blue we plot $\sqrt{x'^2+y'^2}$ obtained from Eq. 46 where we use $\xi = |\phi'|\sqrt{\epsilon_1\epsilon_2}$. In Fig. 5a we plot for each particle the transverse kinetic energy $x'^2+y'^2$ and compare it with Eq. 46 for the two extreme cases: with $\xi=0$ curve in red, and with $\xi=|\phi'|\sqrt{\epsilon_1\epsilon_2}$ curve in green. In Fig. 5b,c,d we plot respectively β,α, and S. In Fig. 6a,b we plot the Larmor angle ϕ and the phase advance ψ.

4.4 Transport through different channels

Let $R_1^2/\beta_{max,1}$ be the maximum emittance accepted in the channel 1, and $R_2^2/\beta_{max,2}$ be the maximum emittance accepted in the channel 2 where R_1, and R_2 are the solenoid radius of the channels 1, and 2. In order not to modify the two beta functions β_1,β_2 a matching between the two optic functions is needed. In this way the beam will be transported from the end of the channel 1 to the beginning of the channel 2 without changing the single particle invariants. It is then enough that $R_1^2/\beta_{max,1}\le R_2^2/\beta_{max,2}$ to guarantee the lossless transport through the channel 2.

5 Chromatic Effect on the Beam Transport

When we consider a beam with a momentum spread, each particle experiences a different solenoidal strength according to its off momentum. Calling $\delta=(p-p_0)/p_0$, for an off momentum particle in a solenoidal channel the Hamiltonian expanded up to the second order reads [11]

$$H=\frac{1}{2}\left(p_x+\frac{S}{2}y\right)^2+\frac{1}{2}\left(p_y-\frac{S}{2}x\right)^2+\delta\left[-\frac{1}{2}S\,y\,p_x-\frac{1}{4}S^2\,y^2+\frac{1}{2}S\,x\,p_y-\frac{1}{4}S^2\,x^2\right], \tag{48}$$

which leads to the equations of motion

$$\begin{array}{l} x''-S(1-\delta)y'-\frac{S'}{2}(1-\delta)y+\frac{S^2}{4}(1-\delta)\,\delta\,x=0,\\ y''+S(1-\delta)x'+\frac{S'}{2}(1-\delta)x+\frac{S^2}{4}(1-\delta)\,\delta\,y=0. \end{array} \tag{49}$$

As first step towards the inclusion of the chromatic effects in the dynamics one can neglect the terms in S^2 so that the equations of motion are identical Eqs. 1 but with a solenoidal strength associated with δ such as $S=S_0/(1+\delta)$, where S_0 is the solenoidal strength for a design reference particle. Taking

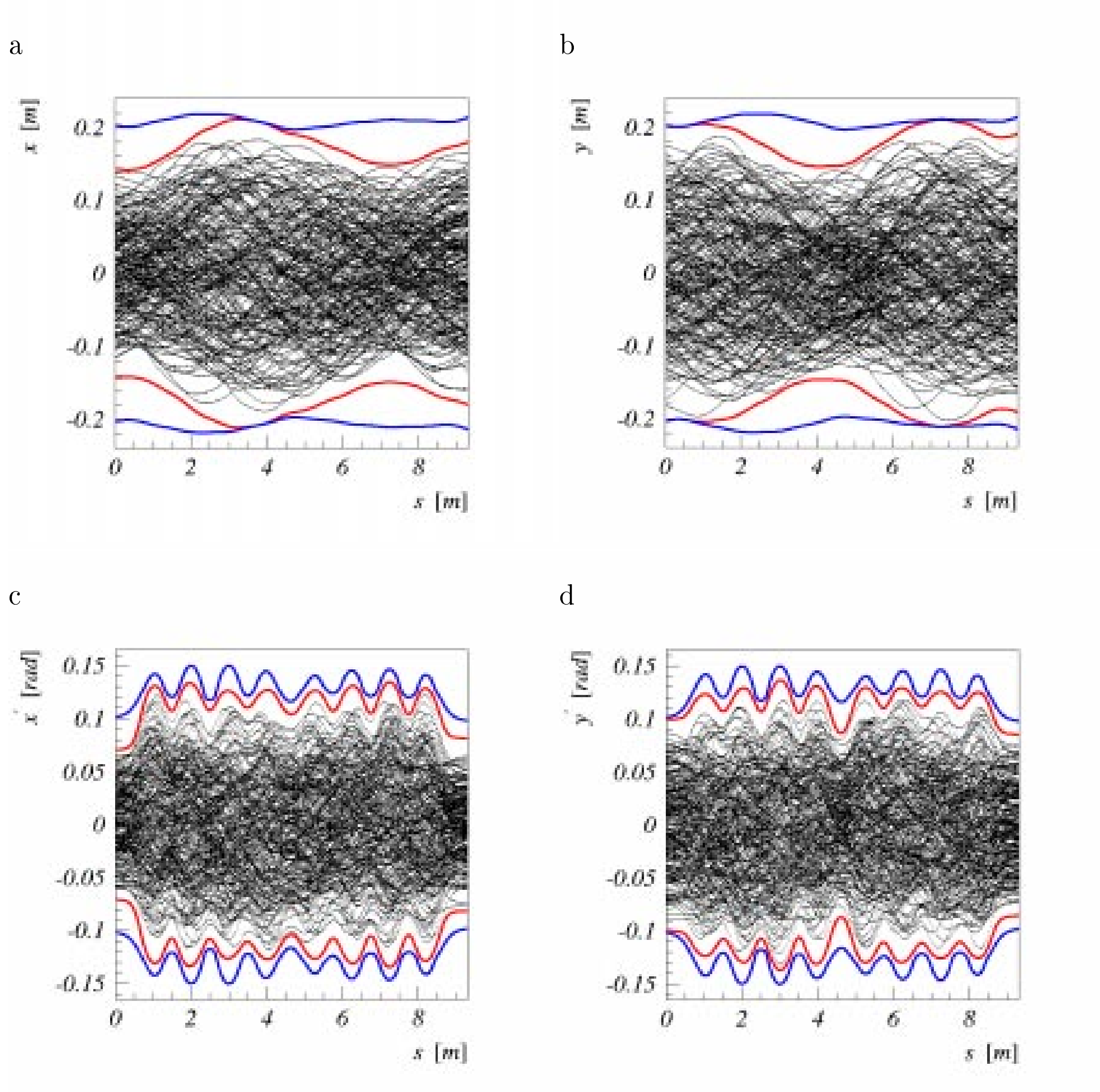

Figure 4: In this figure we plot for $f(\tau) = 1$: in a),b) in red the envelope and in blue the maximum radius and with dotted line the particles trajectories obtained from the tracking for x,y. In c),d) we plot x', y', in red the theoretical momentum envelopes, and in blue the maximum momentum envelope.

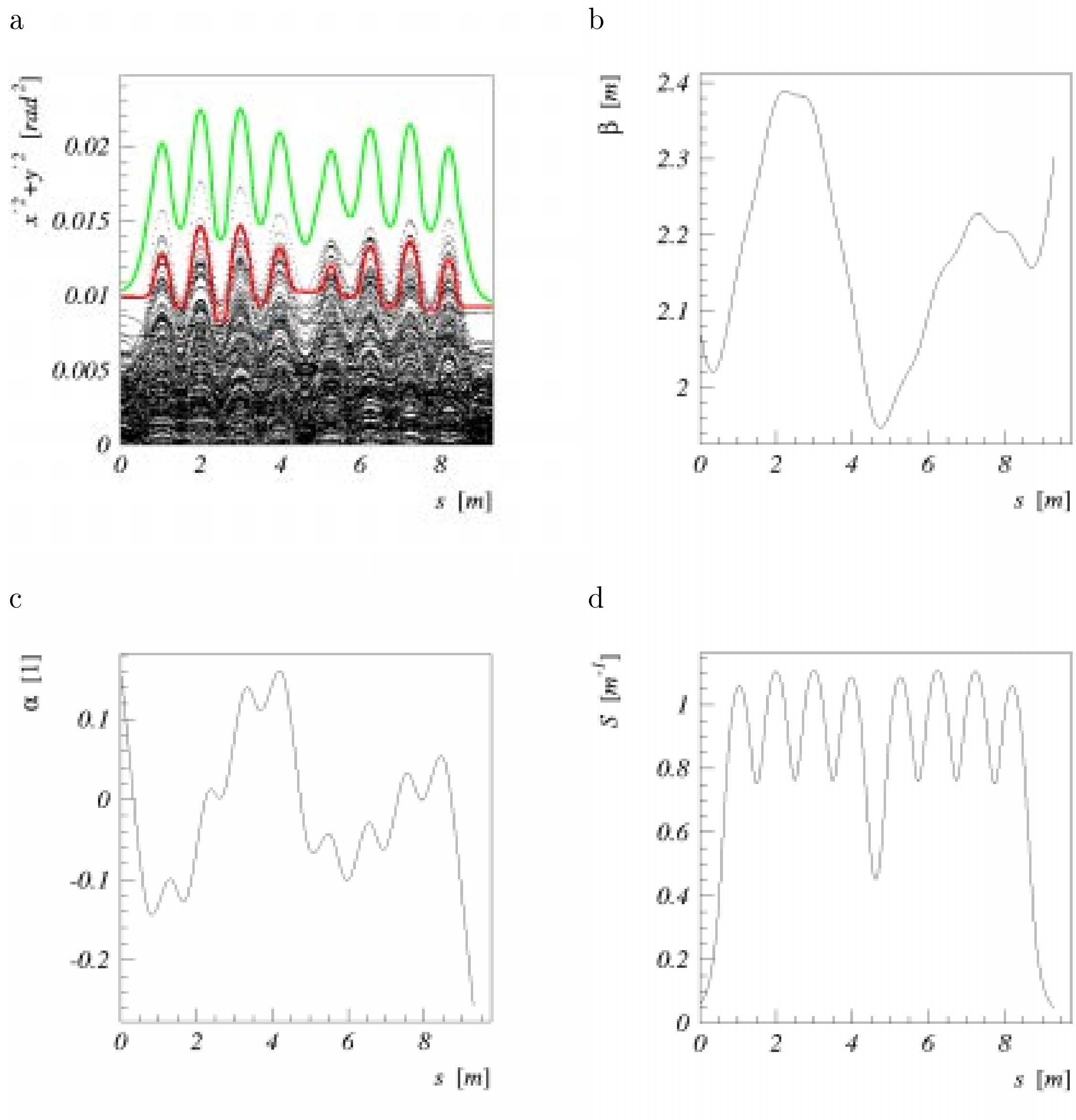

Figure 5: In this figure we plot for the same simulation of Fig. 4: in a) in red and green the theoretical approximations of $(x'^2 + y'^2)_{max}$ and with dotted lines we plot x',and y'. In b),c),d) we plot respectively β, α, S versus s.

a b

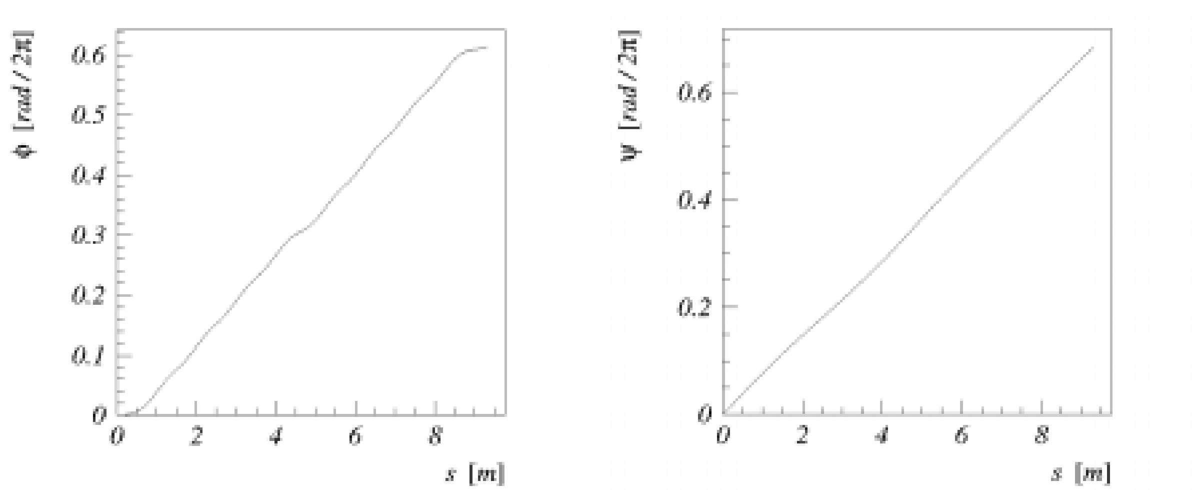

Figure 6: In this figure we plot for the same simulation of Fig. 4: a) ϕ, and b) ψ.

into account of this fact the optic functions should be computed with

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \frac{S_0^2}{4(1+\delta)^2}\beta^2 = 1, \tag{50}$$

where $S_0 = eB/p_0$, and p_0 is the design momentum. If the particle distribution at the entrance of the solenoidal channel is known, then from the value of β_0, α_0 in that position we can find, in both periodic or not periodic case, the value of the invariants for each particle by using Eqs. 36. Since each particle has a different momentum $p = p_0(1+\delta)$, we find that to each δ the lattice exhibits a certain $\beta_{max} = \beta_{max}(\delta)$ over its length. Therefore

$$\begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \\ \delta \end{pmatrix} \xrightarrow{Eq.\ 50} \begin{pmatrix} \beta_{max} \\ \beta_0 \\ \alpha_0 \end{pmatrix} \xrightarrow{Eqs.\ 36} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \longrightarrow r_{max} = \sqrt{\beta_{max}(\epsilon_1+\epsilon_2)}. \tag{51}$$

If $r_{max} > R$ the particle is lost. Eq. 51 defines an algorithm which gives in the 5D space $\{x, x', y, y', \delta\}$ a volume of initial conditions which is accepted by the solenoidal channel (i.e. particles with those initial condition are not lost). The implementation in a computer code of the algorithm defined by Eq. 51 allows a fast optimization method for the design of a solenoidal channel. We remark that this procedure is valid when δ it is not to big so that Eqs. 1 still hold.

6 Quadrupolar Channels and Solenoidal Channels Matching

Consider a general invariant matched distribution in a quadrupolar transport channel. By using the previous definition the distribution is given by $f(\epsilon_x/\epsilon_{x0} + \epsilon_x/\epsilon_{y0})$ where $f(\tau)$ is a function defined in $0 < \tau < 1$ and $\tau = \epsilon_x/\epsilon_{x0} + \epsilon_x/\epsilon_{y0}$. This distribution is self consistent and consequently for each τ the plane $\{\delta_1, \delta_2\}$ is *uniformly filled.* The matching of the quadrupolar channel with the solenoidal one is obtained by means of a matching section which transports the optic functions $\beta_x, \beta_y, \alpha_x, \alpha_y$ from the end of the quadrupolar channel to the beginning of the solenoidal one. Since the solenoidal channel has its own optic functions, the matching is obtained with the condition that the final transported optic functions here named with the index 1 are equal to the optic functions β, α of the solenoidal channel at its entrance, i.e.

$$\beta_{x,1} = \beta_{y,1} = \beta \qquad \alpha_{x,1} = \alpha_{y,1} = \alpha \tag{52}$$

If Eqs. 52 are satisfied, the invariant of each particle are preserved and $\epsilon_1 = \epsilon_{x0}, \epsilon_2 = \epsilon_{y0}$. In the same way one can match the solenoidal channel with the quadrupolar channel. When the beam is transported from the solenoidal channel to the quadrupolar one, again the single particle invariants do not change and each particle will have the same Courant-Snyder invariants it had before to go through the solenoidal channel. At this point one can wonder weather the transported particle distribution is again the same self consistent invariant matched distribution that were injected in the solenoidal channel. From Eq. 34 it is visible that an uniform distribution in $\{\delta_1, \delta_2\}$ (self consistent) is invariant if $\phi = \pi n$ with $n \in \mathbb{N}$.

6.1 Numerical Example

In order to check these conclusions we consider the same solenoidal channel of the numerical example of Section 4.3.1. We consider a solenoid strength of $S = 0.82$ m^{-1} and we inject the beam at 1.5 m from the center of the first solenoid so that $\phi' = 0.001$ m^{-1}. In order to better show the effectiveness of the matching criteria we consider the two cells with central mirror symmetry. The total length of the system is then $L = 10.24$ m. With this assumptions find that the periodic Twiss parameters at injection are $\beta = 17.7$ m, $\alpha = 0.003$. With the solenoid strength of $S = 0.82$ m^{-1} we have a total Larmor phase advance of $\phi(L) = 0.503$ rad$/2\pi$. In Figs. 7,8 we see that in the first and last column the lab frame phase space space are identical as expected from the criteria above stated. In order to show what happen when the matching condition is not satisfied we plot the distribution at $\overline{L} = 9.216$ m which correspond to $\phi(\overline{L}) = 0.496$ rad$/2\pi$. This results are shown in Figs. 7, 8 central column.

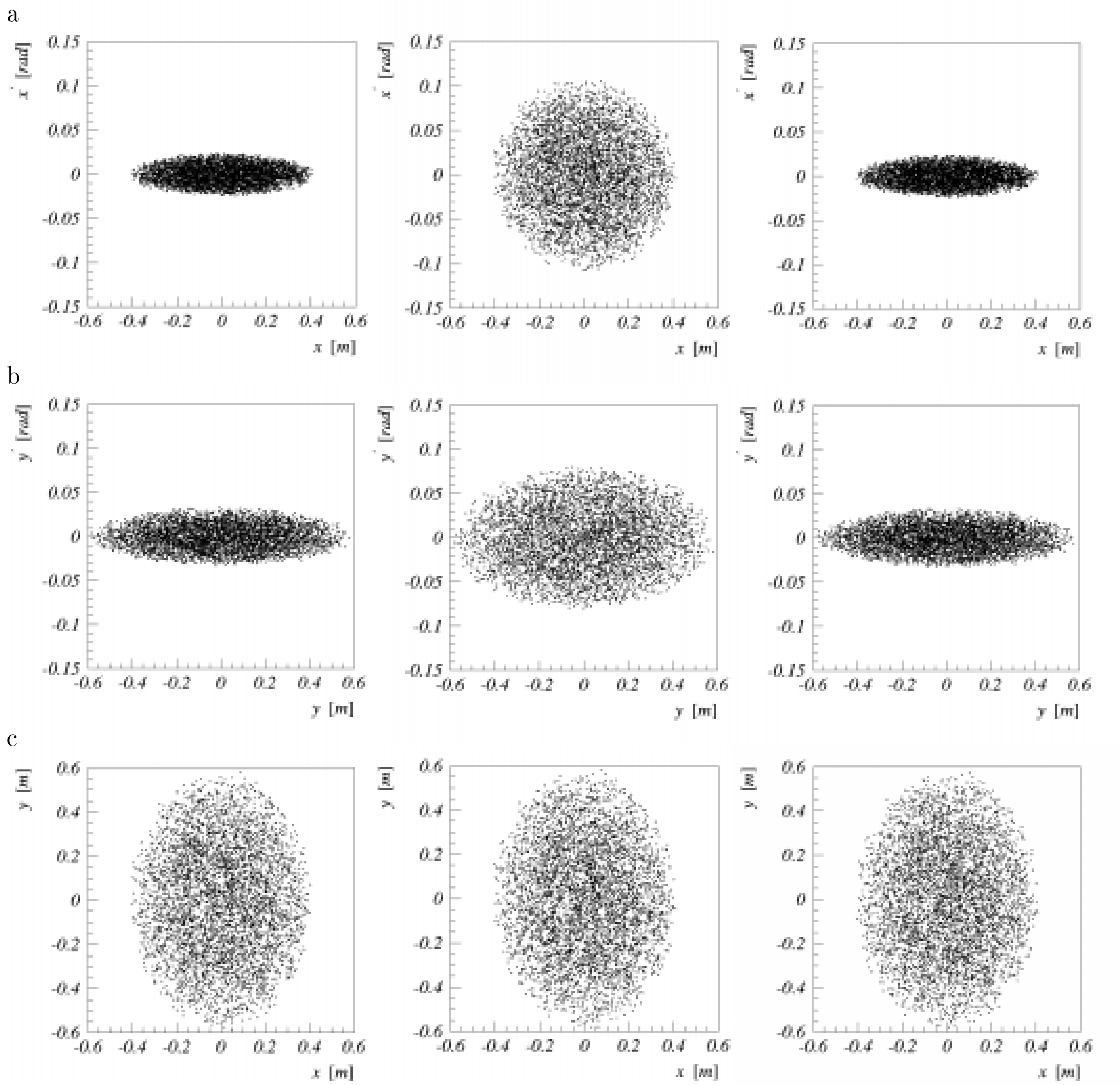

Figure 7: In the right column we plot the initial distribution of 5000 particle for $f(\tau) = 1$. In the central column we show the distribution at $\overline{L} = 9.216$ m where $\phi(\overline{L}) = 0.496$ rad$/2\pi$, and $\phi'(\overline{L}) = 0.0285$ m^{-1}. On the third column we plot the same distribution at $L = 10.24$ m where $\phi(L) = 0.503$ rad$/2\pi$, and $\phi'(L) = 0.001$ m^{-1}.

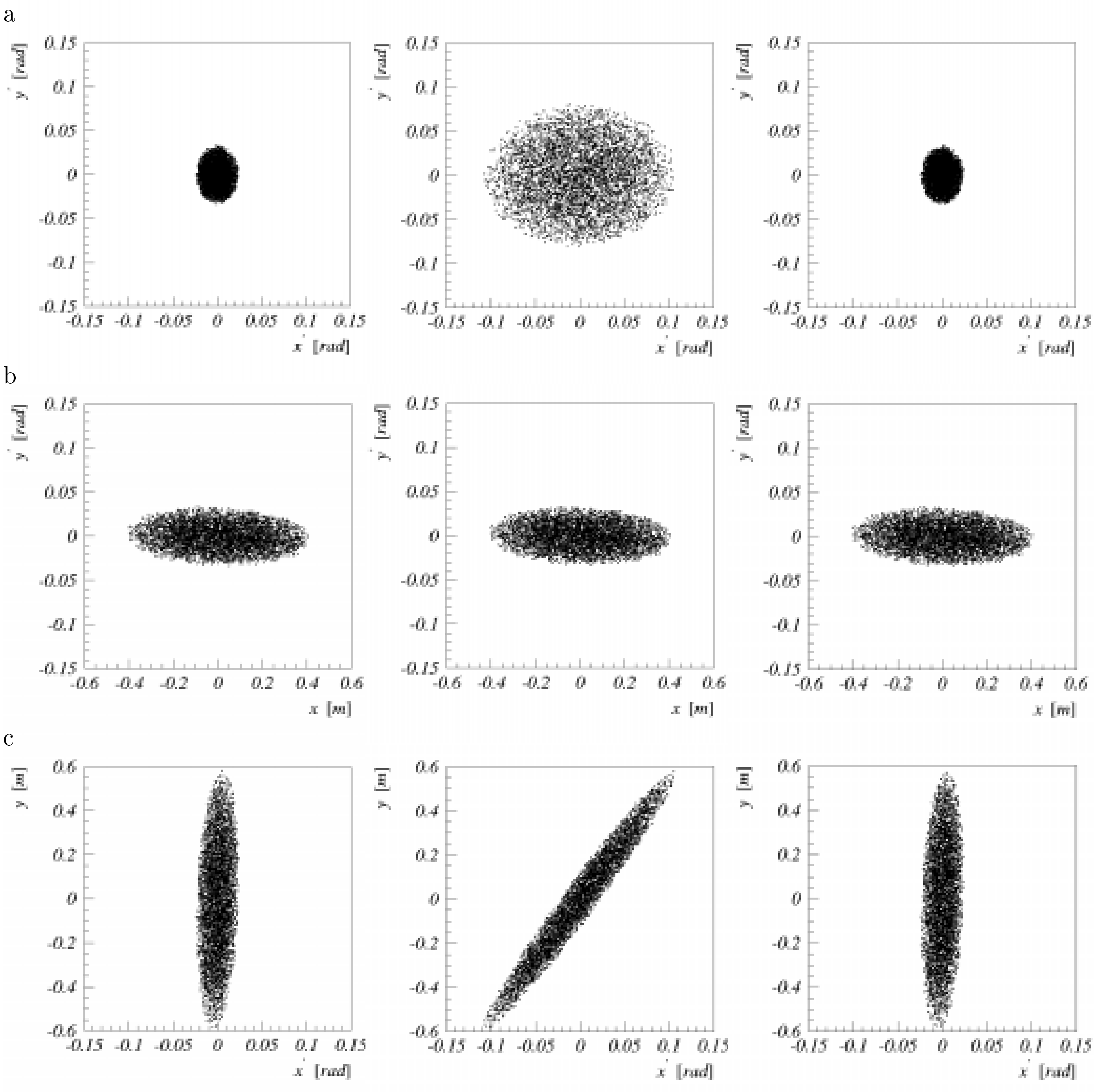

Figure 8: We plot here the other three plans for the same simulation described in Fig. 7, the meaning of the columns is the same.

7 CONCLUSION

We have presented a theory which allow a general description of the single particle linear beam dynamics in a solenoidal channel. We have shown how to compute the single particle invariant and given the explicit solution which link single particle invariant to the Twiss parameters. The knowledge of the invariants allowed us to define matched invariant distributions which can be used to predict the beam envelopes. An algorithm to analyze which part of an initial distribution will be propagated through the channel has been presented, and we give a general rule to perform a matching between solenoidal channels and quadrupolar channels.

8 Acknowledgments

I wish to thank A. Lombardi, and E. Todesco for the corrections and useful discussion.

9 Appendix A

Let us consider

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \frac{S^2}{4}\beta^2 = 1, \qquad \alpha = -\frac{\beta'}{2}, \qquad \psi = \int_0^s \frac{1}{\beta(t)}dt, \tag{53}$$

where we remove the periodic condition over S. Here β is computed starting from arbitrary initial conditions α_0, β_0. We first consider how Eq. 7 will be in the frame $\{u_1, u_2\}$. Applying the transformation Eq. 22 we find

$$\mathbf{u}' = \mathsf{T}^{-1}[-\mathsf{T}' + \mathsf{D}\mathsf{T}]\mathbf{u}. \tag{54}$$

By using the expressions of T, and D (defined in Eq. 7) we find that

$$\mathsf{T}' = \mathsf{D}\mathsf{T} - \frac{1}{\beta}\mathsf{J}_2\mathsf{T}, \quad \text{with} \quad \mathsf{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{55}$$

and substituting this expression in Eq. 54 we find that Eq. 7 in the frame $\{u_1, u_2\}$ becomes

$$\mathbf{u}' = \frac{1}{\beta}\mathsf{J}_2\mathbf{u}. \tag{56}$$

It is straightforward to check that the matrix R defined in Eq. 24 satisfies the equation

$$\mathsf{R}' = \frac{\mathsf{J}_2}{\beta}\mathsf{R}, \tag{57}$$

and therefore the vector $\mathbf{u} = \mathsf{R}\mathbf{u}_0$ is solution of Eq. 56, that is R is the transfer map for the solution of Eq. 56. Now, by using Eq. 55 and Eq. 57, it is straightforward to check that TR satisfies the equation

$$(\mathsf{TR})' = \mathsf{D}(\mathsf{TR}), \tag{58}$$

and therefore $\mathbf{w} = \mathsf{TR}\mathbf{u}_0$ is the solution of Eq. 7. Recalling that in Eq. 6 we imposed the condition $S = 2\phi'$, we have

$$\Lambda' = \mathsf{C}\Lambda - \mathsf{D}\Lambda, \tag{59}$$

and using it together with Eq. 58 one can prove that the matrix $\Lambda\mathsf{TR}$ satisfies the equation

$$(\Lambda\mathsf{TR})' = \mathsf{C}(\Lambda\mathsf{TR}),$$

and consequently $\mathbf{z} = \Lambda\mathsf{TR}\mathbf{u}_0$ is the solution of Eq. 3.

Conclusion: we have shown then that in general on $\{u_1, u_2\}$ the transport is a rotation $\mathbf{u}_1 = \mathsf{R}\mathbf{u}_0$ and that the transformations T, and Λ satisfy the scheme drawn in Eq. 26.

10 Appendix B: 4D Volume and the invariants ϵ_1, ϵ_2

In this appendix we show the relation between the invariants ϵ_1, ϵ_2 and the 4D volume occupied by the surface described by Eqs. 34 when ϵ_1, ϵ_2, and s are fixed. Note that this surface is parameterized by δ_1, δ_2. For convenience we re-write Eqs. 34 in another form. By using Eq. 33, and Eq. 30 we find

$$\begin{pmatrix} z \\ z' \end{pmatrix} = \mathsf{V} \begin{pmatrix} \sqrt{\epsilon_1}\sin(\psi + \delta_1) + i\sqrt{\epsilon_2}\sin(\psi + \delta_2) \\ \sqrt{\epsilon_1}\cos(\psi + \delta_1) + i\sqrt{\epsilon_2}\cos(\psi + \delta_2) \end{pmatrix} \tag{60}$$

and by using the definition $z = x + iy$ we find

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \left(\begin{array}{c|c} \mathrm{Re}\,\mathsf{V} & -\mathrm{Im}\,\mathsf{V} \\ \hline \mathrm{Im}\,\mathsf{V} & \mathrm{Re}\,\mathsf{V} \end{array} \right) \begin{pmatrix} \sqrt{\epsilon_1}\sin(\psi+\delta_1) \\ \sqrt{\epsilon_1}\cos(\psi+\delta_1) \\ \sqrt{\epsilon_2}\sin(\psi+\delta_2) \\ \sqrt{\epsilon_2}\cos(\psi+\delta_2) \end{pmatrix} \tag{61}$$

In this equation the vector $\mathbf{x} = (x, x', y, y')$ is function of $(\epsilon_1, \epsilon_2, \delta_1, \delta_2)$. By changing the independent variables $(\epsilon_1, \epsilon_2, \delta_1, \delta_2)$ in the domain $[0, \epsilon_{1\,max}] \times [0, \epsilon_{2\,max}] \times [0, 2\pi] \times [0, 2\pi]$ the whole 4D volume is explored. The 4D volume depends from $\epsilon_{1\,max}$, and $\epsilon_{2\,max}$. The volume of the invariant surface described by Eq. 61 (i.e. by Eqs. 34) is

$$Volume \; = \int_0^{\epsilon_{1\,max}} \int_0^{\epsilon_{2\,max}} \int_0^{2\pi} \int_0^{2\pi} \left| \frac{\partial(x, x', y, y')}{\partial(\epsilon_1, \epsilon_2, \delta_1, \delta_2)} \right| d\epsilon_1 \, d\epsilon_2 \, d\delta_1 \, d\delta_2 \tag{62}$$

here $\partial(x, x', y, y')/\partial(\epsilon_1, \epsilon_2, \delta_1, \delta_2)$ is the Jacobian of the transformation Eq. 61. The determinant of the Jacobian is the product of the determinant of $\Sigma(\mathsf{V})$ (first matrix in Eq. 61) times the determinant of the Jacobian of the transformation

$$\begin{pmatrix} \sqrt{\epsilon_1}\sin(\psi+\delta_1) \\ \sqrt{\epsilon_1}\cos(\psi+\delta_1) \\ \sqrt{\epsilon_2}\sin(\psi+\delta_2) \\ \sqrt{\epsilon_2}\cos(\psi+\delta_2) \end{pmatrix}. \tag{63}$$

It is straightforward to find that $det[\Sigma(\mathsf{V})] = 1$, and by direct integration we find that

$$Volume = \pi^2 \, \epsilon_{1\,max} \, \epsilon_{2\,max} \tag{64}$$

11 Appendix C: RMS Invariants

It is important to establish the invariants from a particle distribution. In a solenoidal channel the motion of a particle is coupled, but in the Larmor frame we decouple the transverse motion in the two equations

$$w_r'' + {\phi'}^2 w_r = 0, \quad w_i'' + {\phi'}^2 w_i = 0. \tag{65}$$

In each of the two planes (w_r, w_r'), and (w_i, w_i') the motion is linear and there it is well-known how to compute the rms emittances from a particle distribution, in fact

$$\epsilon_{1\,rms}^2 = \overline{w_r^2}\,\overline{w_r'^2} - \overline{w_r\,w_r'}^2, \quad \epsilon_{2\,rms}^2 = \overline{w_i^2}\,\overline{w_i'^2} - \overline{w_i\,w_i'}^2. \tag{66}$$

Eqs. 66 express the invariants of a particle distribution in the Larmor frame. To recover the rms invariants in the laboratory frame it is enough to use the rotation $w = ze^{i\phi}$ with $z = x + i\,y$. We find that such transformation is

$$\begin{array}{ll} w_r = \cos\phi\, x - \sin\phi\, y & w_r' = -\phi' \sin\phi\, x + \cos\phi\, x' - \phi' \cos\phi\, y - \sin\phi\, y' \\ w_i = \cos\phi\, y + \sin\phi\, x & w_i' = -\phi' \sin\phi\, y + \cos\phi\, y' + \phi' \cos\phi\, x + \sin\phi\, x' \end{array} \tag{67}$$

where ϕ is the Larmor's angle given by Eq. 8. By using Eqs. 67 in Eqs. 66 we can compute the rms invariants $\epsilon_{1,rms}, \epsilon_{2,rms}$ from the particle distribution whose coordinates are expressed in the laboratory frame.

As example we considered the invariant matched distribution used in Sect. 4.2.1 in the same solenoidal channel. At each integration step we compute the rms invariants of the particle distribution according to the usual decoupled definition $\epsilon_{x\,rms}^2 = \overline{x^2}\,\overline{x'^2} - \overline{x\,x'}^2, \epsilon_{y\,rms}^2 = \overline{y^2}\,\overline{y'^2} - \overline{y\,y'}^2$, and we also compute the new invariants $\epsilon_{1\,rms}, \epsilon_{2\,rms}$ previously defined. Fig. 9 shows the result of the comparison: only the invariants $\epsilon_{1\,rms}, \epsilon_{2\,rms}$ remain really invariant. As expected, the usual definitions $\epsilon_{x\,rms}, \epsilon_{y\,rms}$ cannot be used as invariants in the coupled system. We conclude with the following remark: let us consider a solenoidal channel such that at its entrance and exit $S = 0$ i.e.there is no magnetic field at entrance/exit, but $S \neq 0$ inside the channel. If at the exit of the solenoidal channel $\phi = \pi n$, with $n \in \mathbb{N}$ then

$$\begin{array}{l} \epsilon_{x\,rms\,entrance} = \epsilon_{1\,rms\,entrance} = \epsilon_{x\,rms\,exit} = \epsilon_{1\,rms\,exit} \\ \epsilon_{y\,rms\,entrance} = \epsilon_{2\,rms\,entrance} = \epsilon_{y\,rms\,exit} = \epsilon_{2\,rms\,exit}, \end{array} \tag{68}$$

if instead at the exit of the solenoidal channel $\phi = \pi(1/2 + n)$, with $n \in \mathbb{N}$ then

$$\begin{array}{ll} \epsilon_{x\,rms\,entrance} = \epsilon_{1\,rms\,entrance} & \epsilon_{y\,rms\,exit} = \epsilon_{1\,rms\,exit} \\ \epsilon_{y\,rms\,entrance} = \epsilon_{2\,rms\,entrance} & \epsilon_{x\,rms\,exit} = \epsilon_{2\,rms\,exit}, \end{array} \tag{69}$$

i.e. the solenoidal channel perform an emittance exchange $\epsilon_{y\,rms} \leftrightarrow \epsilon_{y\,rms}$.

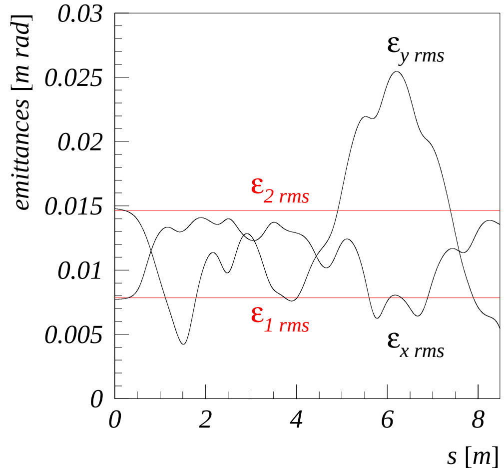

Figure 9: In red the rms invariants $\epsilon_{1\,rms}, \epsilon_{2\,rms}$, in black rms invariants $\epsilon_{x\,rms}, \epsilon_{y\,rms}$.

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