

CERN-TH/2001-105
T/01-040
hep-th/0104077

Supersymmetric Flat Directions and Analytic Gauge Invariants

Ph. Brax¹

Theoretical Physics Division, CERN
CH-1211 Geneva 23²

C. A. Savoy³

Service de Physique Théorique, CEA-Saclay F-91191 Gif/Yvette Cedex,
France

Abstract

We review some aspects of the correspondence between analytic gauge invariants and supersymmetric flat directions for vanishing D-terms and propose a criterion to include the F-term constraints.

¹email: philippe.brax@cern.ch

²On leave of absence from Service de Physique Théorique, CEA-Saclay F-91191 Gif/Yvette Cedex, France

³email: savoy@spht.saclay.cea.fr

1 Flat Directions

Flat directions in supersymmetric theories are continuously connected degenerate supersymmetric vacua, modulo the group of gauge symmetry G , *i.e.*, each vacuum corresponds to a G -orbit. The degeneracy of the classical solutions is described in terms of massless fields, the so-called *moduli*. The flat directions, *i.e.*, the moduli space \mathcal{V} , define the low energy regime of the supersymmetric theory. Their analysis has led to the conjecture of dualities [1] between supersymmetric gauge theories as well as a new insight on confinement in some of such models [1,2,3]. Even in the presence of low scale spontaneous supersymmetry breaking, the study of the flat directions in the supersymmetric limit is still instrumental in many applications. The best example are the minimal supersymmetric extensions of the Standard Model, where the flatness of the scalar potential is lifted by the soft supersymmetry breaking operators. In general, the theory possesses other metastable vacua [4], which could invalidate it – or be important – for phenomenological purposes.

The relevance of the analytic gauge invariants in the study of supersymmetric theories was first noticed a long time ago. Indeed, the F -terms in the scalar potential are components of the gradient of the superpotential which is an invariant analytic function. The D -terms are Hermitean functions of the scalar fields, but a beautiful result in algebraic geometry provides the link with holomorphic invariants, at least for the analysis of the vacua [5,6,7,8]. The fundamental mathematical property of the (classical) moduli space (modulo the group of gauge symmetry) is its isomorphism with an algebraic variety of analytic gauge invariant polynomials in the primordial fields called the *chiral ring*. They can be classified in two categories, those which have algebraic constraints among the invariants of the integrity basis, called *syzygies*, and those with a free basis. This classification is also related to the possible patterns of gauge and global symmetry breaking.

Consider a set of chiral superfields ϕ^i whose scalar components z^i belong to a Kahler manifold with Kahler potential K . The scalar potential of the globally supersymmetric theory with gauge group G is the sum $V = \frac{g_a^2}{2} D^{a2} + K^{i\bar{j}} F_i \bar{F}_{\bar{j}}$ where the F terms are the gradients of the gauge invariant analytic superpotential $W(z^i)$, *i.e.* $F_i = \partial_i W$, and the D terms are $D^a = \partial_i K (T^a z)^i$ for the linear action $\delta z^i = \epsilon_a (T^a z)^i$ of the compact gauge group G on the

scalars z^i . The matrices T^a are the generators of G . The supersymmetric vacua \mathcal{V} of the theory are defined by the conditions $F_i = 0$, $D^a = 0$. The gauge invariance of the Kahler potential and the superpotential implies that these solutions form G-orbits $\Omega = \{gz_0, g \in G, z_0 \in \mathcal{V}\}$. The solution z has a little group H_z in G . The G invariance of the orbit through z implies that the little groups $H_{gz} = g^{-1}H_zg$ are conjugate. The little groups are ordered according to the relation: $H \leq H'$ provided H is conjugate to a subgroup of H' . Similarly G-orbits are ordered according to their little groups. The orbits with the largest little groups are called critical.

1.1 D=0

The D -flatness condition is non-analytic, the F -flatness condition is analytic, i.e. involving only the scalars z^i and not $\bar{z}^{\bar{i}}$. The D -flatness conditions is non-trivially related to analytic invariants. Analytic invariants $I(z)$ satisfy $\partial_i I (T^a z)^i = 0$. A necessary and sufficient condition for $D = 0$ is the existence of an analytic invariant I and a non-zero constant c such that

$$\partial_i I = c \partial_i K. \quad (1)$$

These solutions are extrema of I at fixed K . This correspondence was pointed out in [6] and proved in [7], detailed discussions can be found in [8] and [9].

Example 1 – Consider $G = SU(3)$ with three quarks Q_a^α and three anti-quarks $\bar{Q}_{\bar{a}}^{\bar{\alpha}}$, $a = 1 \dots 3$, $\alpha = 1 \dots 3$. The most general invariant which is linear in the basic invariants, see section 2, is $I = \rho_a Q_a^\alpha \bar{Q}_{\bar{a}\alpha} + \mu \epsilon_{\alpha\beta\gamma} \epsilon^{abc} Q_a^\alpha Q_b^\beta Q_c^\gamma + \tilde{\mu} \epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \epsilon^{\bar{a}\bar{b}\bar{c}} \bar{Q}_{\bar{a}}^{\bar{\alpha}} \bar{Q}_{\bar{b}}^{\bar{\beta}} \bar{Q}_{\bar{c}}^{\bar{\gamma}}$ where $M_{a\bar{a}} = Q_a^\alpha \bar{Q}_{\bar{a}\alpha}$, $B = \epsilon_{\alpha\beta\gamma} \epsilon^{abc} Q_a^\alpha Q_b^\beta Q_c^\gamma$ and $\bar{B} = \epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \epsilon^{\bar{a}\bar{b}\bar{c}} \bar{Q}_{\bar{a}}^{\bar{\alpha}} \bar{Q}_{\bar{b}}^{\bar{\beta}} \bar{Q}_{\bar{c}}^{\bar{\gamma}}$. General solutions can be found from (1) and correspond to orbits of the non-compact group $U(3,3)$ [6]. They correspond to combinations of the mesonic orbits: $\partial_a M_{a\bar{a}} = c(Q_a^\alpha)^*$ yields $(Q_1^1)^* = \bar{Q}_{11}$, and the baryonic orbits: $\partial_a B = c(Q_a^\alpha)^*$ yields $Q_1^1 = Q_2^2 = Q_3^3$, all others zero (and analogously for antibaryons), up to a $U(3,3)$ transformation. The critical orbit $Q_1^1 = Q_2^1 = Q_3^1$, all others zero, breaks $SU(3) \rightarrow SU(2)$ and gives $I = 0$ and $D_{U(1)} \neq 0$ for the broken $U(1)$ in $SU(2) \times U(1) \subset SU(3)$. It is not a flat direction and its $U(3,3)$ orbit is not a D-flat direction too.

Example 2 – Consider $G = E_6$ with one representation 27. There is only one invariant $I = d_{abc} z^a z^b z^c$. Two critical orbits have little groups $SO(10)$

and F_4 respectively. For z_0 in the $SO(10)$ critical orbit one finds $I(z_0) = \partial_i I(z_0) = 0$ and $D_{U(1)}(z_0, \bar{z}_0) \neq 0$ since z_0 is charged under the $U(1)$ such that $SO(10) \times U(1) \in E_6$. For z_0 in the orbit with little group F_4 one checks that $D = 0$ with $I(z_0) \neq 0$.

It is important to note that critical orbits with a unique singlet corresponding to a broken $U(1)$ such that $H_{z_0} \times U(1) \subset G$ is maximal are not D -flat, as discussed below. These orbits will be identified with the open orbits of the complexified gauge group G^c .

1.2 $\mathbf{F=0}$

The superpotential is G invariant and since it is an analytic function it is also G^c invariant where G^c is the complexified gauge group $G^c = \{g = e^{i\alpha_a T^a}, \alpha_a \in C\}$. For each solution of $F^i = 0$ the whole complex orbit $\Omega^c = \{gz_0, g \in G^c\}$ is a solution. Note that if z has a (complexified) little group H_z^c , the number of non-trivial conditions $\partial_i W = 0$ is equal to the number of H_z^c singlets. Since $\partial_i W = 0$, the superpotential cannot be chosen to be the invariant I leading to solutions of $D = 0$. For a generic superpotential $W(I^a)$ containing all the basic invariants $I^a(z)$ with degree d_a , $F^i = 0$ implies that

$$\sum_a d_a \frac{\partial W}{\partial I^a} I^a(z_0) = 0. \quad (2)$$

Now for $D = 0$ there is at least one invariant such that $I(z_0) \neq 0$. This implies that directions corresponding to open critical orbits with all $I^a(z_0) = 0$ are F -flat but not D -flat directions. They are characterized by a H_{z_0} singlet in the coset G/H_{z_0} such that $H_{z_0} \times U(1) \subset G$. If z_0 is critical, it is the only H_{z_0} singlet, the action of the complexification of the $U(1)$ is $z_0 \rightarrow \lambda z_0$, $\partial_{z_0} I^a$ is the only potentially non-vanishing gradient direction, and $z_0 \partial_{z_0} I^a = z^i \partial_i I^a = d_a I^a$ vanishes. Therefore $\partial_{z_0} I^a = 0$ for all invariants I^a . Now since (1) is necessary, $D = 0$ can only be satisfied for $z^0 = 0$ which is not in the open orbit but in its closure.

This provides counter-examples to the statement that for each G^c -orbit which gives solutions to $F^i = 0$ one can always find one (G -orbit) solution of $D = 0$. For a generic W and z_0 such that $D = 0$, the $F^i = 0$ conditions lift at least one of the flat directions. If some of the I^a are excluded from W , e.g. with global symmetries, then they can be used to define D -flat directions which are consistent with $F = 0$.

Example 3 – Let us consider a simplified supersymmetric version of the Standard Model, with gauge group $SU(2) \times U(1)$, and the following chiral superfield content: two Higgs $SU(2)$ doublets H_1, H_2 , with $U(1)$ charges $Y = -1, +1$, resp., two lepton doublets L_1, L_2 , with $Y = -1$, and two lepton singlets E_1, E_2 , with $Y = 2$. We also introduce a lepton parity, with $R = -1$ for leptons and $R = 1$ for Higgses. The most general invariant (at most cubic) superpotential is of the form (up to some redefinition), $W = \mu H_1 H_2 + \lambda_1 H_1 L_1 E_1 + \lambda_2 H_1 L_2 E_2$. The following invariants have been excluded because of the R -parity: $L_1 H_2, L_2 H_2, E_1 L_1 L_2$, and $E_2 L_1 L_2$. By taking a linear combinations of these invariants and those in W as the invariant I in equation (1) one finds D-flat solutions. The moduli space is its intersection with the solutions of $\partial_i W = 0$, and has components such that H_2 and the lepton scalars are non-vanishing. This breaks completely the gauge symmetry (in particular the electromagnetic $U(1)$), but also the lepton parity. A similar situation is found when families of quark multiplets are introduced with some kind of baryon or lepton symmetry, giving rise to potentially dangerous charge and colour breaking directions of the moduli space [4]. The flat directions of the minimal supersymmetric extension of the Standard Model are listed in [5].

2 Moduli and Syzygies

To any solution of $D^a = 0$ one can associate a holomorphic gauge invariant satisfying (1). The proof of (1) is obtained by studying the closed orbits of the complexified G^c of the gauge group G and the ring of G^c -invariant analytic polynomials. This ring is finitely generated: one can find an integrity basis, i.e. a set of G -invariant holomorphic homogeneous polynomials $\{I^a(z)\}_{a=1\dots d}$ such that every G^c -invariant polynomial in z can be written as a polynomial in the $I^a(z)$.

Notice that orbits with all $I^a(z) = 0, z \neq 0$, are open, i.e. there exists a one parameter subgroup of G^c with generator T such that $e^{\xi T} \in G^c$ and $e^{\xi T} z \rightarrow 0, \xi \rightarrow \infty$. The compact generator associated to T , i.e. iT , corresponds to a broken $U(1)$ factor in the coset G/H_z with maximal subgroup $H_z \times U(1) \subset G$. In example 2, the $SO(10)$ orbit is open, iT is the $U(1)$ charge, while the F_4 orbit is closed.

The elements of an integrity basis are not always algebraically independent. In general, there exist algebraic relations (called *syzygies*) satisfied by

the fundamental invariants $\mathcal{S}^\alpha(I^a(z)) = 0$. In example 1, there is one such relation $\det M - B\tilde{B} = 0$.

To each closed G^c -orbit corresponds a vector in C^d made out of the values taken by the invariants $\{I^a(z)\}$ along this orbit and satisfying the syzygies. In that sense the algebraic manifold defined by the syzygies is identified with the set of closed G^c -orbits. Notice that the origin $\{I^a = 0\}$ is associated with the unique closed G^c -orbit of $z^i = 0$.

The existence of the syzygies can be related to the index of the matter field representation, μ defined by $\text{tr}(T^a T^b) = \mu\delta^{ab}$. For low indices $\mu < \mu_{adj}$, where μ_{adj} is the index of the adjoint representation, it has been shown that there are no syzygies. For $\mu > \mu_{adj}$ the generic situation is that there are syzygies, with a few exceptional cases with no syzygies.

Equation (1) can be seen as a condition for the points of a closed G^c -orbit to extremize the Kähler potential, i.e. $\partial_i(\lambda_a I^a - K) = 0$ with Lagrange multipliers λ_a . A result of geometric invariant theory [7] states that the points extremizing the Kähler potential on a G^c -orbit form a unique G -orbit and are solutions of $\{D^A = 0\}$. Identifying the points on a same G -orbit, there is a one-to-one correspondence between any two of the following sets: (i) the algebraic manifold \mathcal{M}_I defined by the syzygies; (ii) the closed G^c -orbits; (iii) the solutions of (1) modulo gauge transformations; (iv) the solutions of $D^A = 0$ modulo gauge transformations.

Now imposing the F -flatness condition implies an extra relation $\sum_a d_a \frac{\partial W}{\partial I^a} I^a = 0$ to the syzygies. Geometrically one intersects the hypersurface deduced from the superpotential with the moduli space \mathcal{M}_I . It is a necessary condition. Three case are to be envisaged. If there is no intersection the only solution is $z = 0$, if it exists. If $z = 0$ is not a solution then supersymmetry is spontaneously broken. If the intersection is reduced to one point the D -flat direction is completely lifted. Finally in the generic situation the intersection has at most dimension $\dim\mathcal{M}_I - 1$ corresponding to the lifting of at least one flat direction.

3 Dualities and Confinement

One of the most striking result on supersymmetric gauge theories is the existence of a new type of duality. This duality relates two apparently different theories in the short distance regime that are described by the same effective

theory in the infrared limit. In the same vein the basic question concerning the issue of colour confinement has been tackled and clarified in these non-perturbative approaches for a large class of supersymmetric theories. Effective theories have been written and argued to describe the IR behaviour of some gauge theories in terms of gauge invariant composite chiral superfields [1,2,3]. The non-perturbative quantum effects are fixed by the holomorphy of the superpotential, by the global symmetries of the theory and by several descent links among series of such theories.

A powerful and necessary criterion for the existence of an effective theory describing the IR regime of an asymptotically free gauge theory was stated by 't Hooft [10]: there should be matching of the (formal) anomalies of the global symmetries calculated with either the UV or the IR massless fermions. The analysis of the isomorphism between the moduli space and the chiral ring constrained by the syzygies leads to the following (partially proved) conjecture [11,12] *The 't Hooft conditions are satisfied for supersymmetric gauge theories if and only if the syzygies of the chiral ring of gauge invariants derive from a superpotential $W(I_a)$, through $\mathcal{S}^a = \partial W / \partial I^a = 0$.*

Now by comparing with the general structure of W required by the R -symmetries, one obtains the following condition: *A necessary condition for the matching of the anomalies is: $\mu = \mu_{adj} + k$, ($k = 0, 1, 2$).* This is the confinement condition for theories with $\mu \geq \mu_{adj}$ [13]. There are some subtleties and peculiarities that can be found in the literature.

It is generally assumed that theories with $2 + \mu_{adj} < \mu < 3\mu_{adj}$ have an infrared fixed point [1] where they are described by a superconformal theory. The syzygies are therefore exact quantum relations between the chiral primary fields at the superconformal fixed point. Whereas in “electric” theories the gauge invariants are composite fields and so subject to syzygies, in the magnetic dual theories, some of them appear as elementary fields and do not have *a priori* to satisfy the syzygies. These “magnetic syzygies” appear as the equations of motion from the superpotential generated non perturbatively in the magnetic theories [14].

These dualities amount to an identification between the moduli spaces, *i.e.*, between the flat directions of the dual theories. It is worth noticing that the D-flatness in a theory corresponds to the F-flatness in the dual theory.

- [1] N. Seiberg, Nucl. Phys. B435 (1995) 129.
- [2] T. R. Taylor, G. Veneziano and S. Yankelowiz, Nucl. Phys. B218 (1983) 493.
- [3] N. Seiberg, Phys. Rev. D49 (1994) 6857.
- [4] J. A Casas, A. Leyda and C. Muñoz, Nucl. Phys. B471 (1996) 3.
- [5] T. Gherghetta, C. Kolda, S.P. Martin, Nucl.Phys. B468 (1996) 37.
- [6] F. Buccella, J.P. Derendinger, S. Ferrara and C.A. Savoy, Phys. Lett. 115B (1982) 375.
- [7] C. Procesi and G.W. Schwarz, Phys. Lett. 161B (1985) 117.
- [8] R. Gatto and G. Sartori, Commun. Math. Phys. 109 (1987) 327.
- [9] M.A. Luty and I. Washington Taylor, Phys. Rev. D53 (1996) 3399.
- [10] G. 't Hooft et al., editors, Naturalness, chiral symmetry breaking and spontaneous chiral symmetry breaking, NATO advanced study, Cargese, France, Plenum, 1980.
- [11] G. Dotti and A.V. Manohar, Nucl. Phys. B518 (1998) 575.
- [12] Ph. Brax, C. Grojean and C. A. Savoy Nucl. Phys. B 561 (1999) 77.
- [13] C. Csáki, M. Schmaltz and W. Skiba, Phys. Rev. D55 (1997) 7840.
- [14] K. Intriligator and N. Seiberg, Nucl. Phys. Suppl. 45BC (1996) 1.