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We study neutrino oscillations and the level-crossing probability $P_{\rm LSZ} = \exp(-\gamma_n \mathcal{F}_n \pi/2)$ in power-law like potential profiles $A(r) \propto r^n$. After showing that the resonance point coincides only for a linear profile with the point of maximal violation of adiabaticity, we point out that the "adiabaticity" parameter γ_n can be calculated at an arbitrary point if the correction function $\mathcal{F}_n$ is rescaled appropriately. We present a new representation for the level-crossing probability, $P_{\rm LSZ} = \exp(-\kappa_n \mathcal{G}_n)$, which allows a simple numerical evaluation of $P_{\rm LSZ}$ in both the resonant and non-resonant cases and where $\mathcal{G}_n$ contains the full dependence of $P_{\rm LSZ}$ on the mixing angle ϑ. As an application we consider the case $n = -3$ important for oscillations of supernova neutrinos.

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Introduction—The analytical study of non-adiabatic neutrino oscillations has a long history. Soon after the discovery of "resonant" neutrino flavor conversions by Mikheev and Smirnov [1], the leading non-adiabatic effects were calculated for a linear potential profile in Ref. [2]. They were obtained in the form of the Landau–Stückelberg–Zener (LSZ) crossing probability [3]

$$P_{\rm LSZ} = \exp\left(-\frac{\pi\gamma}{2}\right) . \qquad (1)$$

The adiabaticity parameter γ for a linear profile is

$$\gamma = \frac{\Delta S^2}{2EC\,|{\rm d}\ln A/{\rm d}r|_0} , \qquad (2)$$

where E is the energy, $\Delta = m_2^2 - m_1^2 > 0$, $S = \sin(2\vartheta)$, $C = \cos(2\vartheta)$, $\vartheta \in [0 : \pi/2]$ is the (vacuum) mixing angle and $A = 2EV = 2\sqrt{2}G_F N_e E$ is the induced mass squared term for the electron neutrino. The parameter γ has to be evaluated at the so-called resonance point r_0, i.e. the point where the mixing angle in matter is $\vartheta_{\rm m} = \pi/4$. For a linear profile, adiabaticity is maximally violated for $\vartheta_{\rm m} = \pi/4$ and, therefore, the probability that a neutrino jumps from one branch of the dispersion relation to the other is indeed maximal at r_0.

Later, Kuo and Pantaleone derived in Ref. [4] the LSZ crossing probability for an arbitrary power-law like profile, $A \propto r^n$. They found that also in this case the dependence on the neutrino masses and energies can be factored out, while the effect of a non-linear profile can be encoded into a correction function $\mathcal{F}_n$,

$$P_{\rm LSZ} = \exp\left(-\frac{\pi\gamma_n}{2}\,\mathcal{F}_n(\vartheta)\right) , \qquad (3)$$

which only depends on ϑ and n. The "adiabaticity" parameter γ_n has to be evaluated still at r_0 although, as we will show, it does not coincide with the point of maximal violation of adiabaticity (pmva) for $n \neq 1$. An unsatisfactory feature of Eq. (3) is its restricted range of applicability: the resonance condition $\vartheta_{\rm m} = \pi/4$ has a solution only if $\vartheta < \pi/4$ for neutrinos or if $\vartheta > \pi/4$ for anti-neutrinos, respectively. Therefore, it is not possible to calculate analytically, e.g., the survival probability of supernova anti-neutrinos in the quasi-adiabatic regime assuming a normal mass hierarchy.

The purpose of this Letter is twofold. First, we discuss the physical significance of the resonance point r_0. We show that the product $\gamma_n\mathcal{F}_n$ can be evaluated at an arbitrary point and conclude that the "resonance" point r_0 has, for a general profile $n \neq 1$, no particular physical meaning: neither does it describe the point of maximal violation of adiabaticity nor is it necessary to calculate the "adiabaticity" parameter γ_n at r_0. Second, we propose a new representation for $P_{\rm LSZ}$ that is valid for all ϑ and allows an easy numerical evaluation. As an application, we consider the case $n = -3$ which is important for oscillations of supernova neutrinos.

Resonance point versus point of maximal violation of adiabaticity—We use as starting point for our discussion the evolution equation for the medium states $\tilde\psi$ first given in Ref. [5],

$$\frac{\rm d}{{\rm d}r}\begin{pmatrix}\tilde\psi_1\\ \tilde\psi_2\end{pmatrix} = \begin{pmatrix}{\rm i}\Delta_{\rm m}/(4E) & -\vartheta_{\rm m}' \\ \vartheta_{\rm m}' & -{\rm i}\Delta_{\rm m}/(4E)\end{pmatrix}\begin{pmatrix}\tilde\psi_1\\ \tilde\psi_2\end{pmatrix} \qquad (4)$$

Here,

$$\Delta_{\rm m} = \sqrt{(A-\Delta C)^2 + (\Delta S)^2} \qquad (5)$$

denotes the difference between the effective mass of the two neutrino states in matter, $\vartheta_{\rm m}$ is the mixing angle in matter with

$$\tan 2\vartheta_{\rm m} = \frac{\Delta S}{\Delta C - A} \qquad (6)$$

and $\vartheta_{\rm m}' = {\rm d}\vartheta_{\rm m}/{\rm d}r$.

The evolution of the neutrino state is adiabatic at a given r, if the diagonal terms are large compared to the off-diagonal ones, $|\Delta_{\rm m}| \gg |4E\vartheta'_{\rm m}|$. Thus the point where adiabaticity is maximally violated is given by the minimum of $\Delta_{\rm m}/\vartheta'_{\rm m}$. Differentiating [6]

$$\frac{\Delta_{\rm m}}{\vartheta'_{\rm m}} = \frac{2\Delta^2 S^2}{\sin^3 2\vartheta_{\rm m}} \frac{1}{{\rm d}A/{\rm d}r} \tag{7}$$

for the case of a power-law profile, $A(r) \propto r^n$, we find the minimum at

$$\cot(2\vartheta_{\rm m} - 2\vartheta) + 2\cot(2\vartheta_{\rm m}) - \frac{1}{n}\left[\cot(2\vartheta_{\rm m} - 2\vartheta) - \cot(2\vartheta_{\rm m})\right] = 0\,. \tag{8}$$

For $n = 1$, the pmva is at $\vartheta_{\rm m} = \pi/4$ for all ϑ. Thus, in the region where the resonance point is well-defined, the pmva and r_0 coincide. In the general case, $n \neq 1$, the pmva however agrees with the resonance point only for $\vartheta = 0$. Finally, we recover the result of Ref. [6] for an exponential profile in the limit $n \to \pm\infty$.

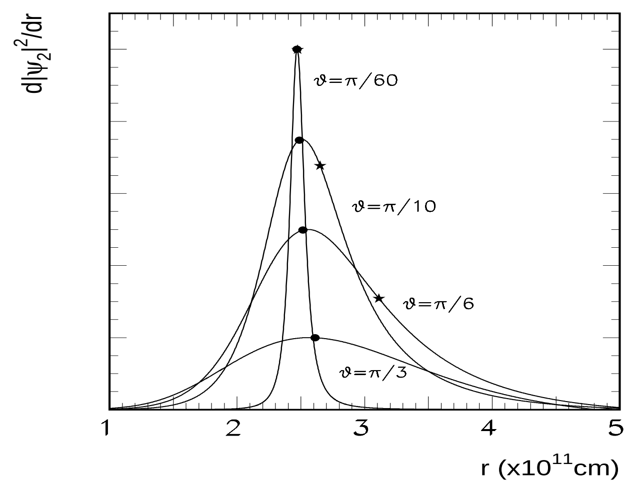

FIG. 1. Change of the survival probability ${\rm d}p(r)/{\rm d}r$ of a neutrino produced at $r = 0$ as ν_2 together with the point of maximal violation of adiabaticity (dots) and the resonance point (stars) for a power-law profile $A \propto r^{-3}$. The height of the different curves is rescaled.

In Fig. 1, we show the change of the survival probability ${\rm d}p(r)/{\rm d}r = {\rm d}|\psi_2(r)|^2/{\rm d}r$ of a neutrino produced at $r = 0$ as ν_2 as function of r together with the point of maximal violation of adiabaticity predicted by Eq. (8) and the resonance point for a power-law profile $A \propto r^{-3}$. It can be clearly seen that Eq. (8) accurately describes the most probable position of the level crossing, while the resonance point predicts a transition in less and less dense regions until for $\vartheta = \pi/4$ the concept of a resonant transition breaks down completely. If the true profile is only approximately a power-law, its exponent should be determined therefore by the region around the pmva, not by the region around the resonance point.

The correction functions $\mathcal{F}_n$—We now recall briefly the calculation of the leading term to the crossing probability within the WKB formalism [4]. In the ultrarelativistic limit and omitting an overall phase, the WKB formula results in

$$\ln P_{\rm LSZ} = -\frac{1}{E}\,\Im \int_{x_1(A_1)}^{x_2(A_2)} {\rm d}x\, \Delta_{\rm m}\,, \tag{9}$$

where $A_2 = \Delta e^{2{\rm i}\vartheta}$ is the branch point of $\Delta_{\rm m}$ in the upper complex x plane. We identify the physical coordinate $r \in [0:\infty]$ with the positive part of the real x axis, i.e. we consider a neutrino state produced at small but positive x propagating to $x = \infty$. Then, a convenient choice for A_1 is to use the real part of A_2 for $C > 0$, i.e. the "resonance" point $A_1 = \Delta C$. However, we stress that this choice has technical reasons and makes sense only for $C > 0$: consider for instance the simplest case $n = 1$. Then both the integration path chosen and the branch cut are for $C < 0$ in the half-plane $\Re(x) < 0$. The physical interpretation is therefore that an anti-neutrino state created at small but negative x propagates to $-\infty$. This case is however equivalent to a neutrino state propagating with $C > 0$ in the right half-plane and therefore contains no new information. Thus, we expect the correction functions $\mathcal{F}_n$ obtained with the integration path from ΔC to $\Delta e^{2{\rm i}\vartheta}$ to be functions with period $\pi/4$ and to be valid only in the resonant region.

After the substitution $A = \Delta(B + C)$,

$$\ln P_{\rm LSZ} = -\frac{\Delta}{E}\,\Im \int_0^{{\rm i}S} {\rm d}B\, \frac{{\rm d}x}{{\rm d}B}\, (B^2 + S^2)^{1/2}\,, \tag{10}$$

one has to expand the Jacobian ${\rm d}x/{\rm d}B$ in a power series in order to solve the B integrals. Kuo and Pantaleone chose as expansion point the "resonance" point $B_0 = 0$, because it leads to the simplest result. It is this choice, arbitrary from a physical point of view, that leads to the evaluation of the parameter γ_n at $B_0 = 0$. In general, a change of B_0 in the definition of γ_n will be compensated by such a change in the correction function $\mathcal{F}_n$ that the physical observable $P_{\rm LSZ}$ is independent of B_0 [7].

The final result for the correction functions given by Kuo and Pantaleone was

$$\mathcal{F}_n(\vartheta, 0) = 2\sum_{m=0}^{\infty} \binom{1/n - 1}{2m} \binom{1/2}{m+1} (\tan(2\vartheta))^{2m}. \tag{11}$$

This series for $\mathcal{F}_n(\vartheta, 0)$ converges only for $\vartheta < \pi/8$ and is therefore not suited, even in the resonant region, for numerical evaluation in the phenomenologically most interesting case of maximal or nearly-maximal mixing. Representing the series as a hypergeometric function,

$$\mathcal{F}_n(\vartheta, 0) = {}_2F_1\left(\frac{n-1}{2n}, \frac{2n-1}{2n}; 2; -\tan^2(2\vartheta)\right)\,, \tag{12}$$

one can use, however, the Euler integral representation [8] of ${}_2F_1$ as the analytical continuation for all $\vartheta \in [0:\pi/4]$.

We now discuss the non-resonant case, $\vartheta \in [\frac{\pi}{4} : \frac{\pi}{2}]$. The expression $\gamma_n(0)\mathcal{F}_n(0,\vartheta)$ becomes ill-defined in this region for several reasons: the assumption $C \geq 0$ used in the derivation for $\mathcal{F}_n(\vartheta, 0)$ is not fulfilled, and the "resonance" condition $B = 0$ or $A = \Delta C$ has no solution for $\vartheta \in [\pi/4 : \pi/2]$. Moreover, the integration path used in Eq. (10) can be for $C < 0$ in the unphysical region $x < 0$. However, we know from the results above that the crossing probability is non-zero also in the non-resonant case. Also, the WKB method should work independently of the sign of C as long as the evolution of the neutrino state is not strongly non-adiabatic.

We now show that it is not necessary to evaluate $\gamma_n \mathcal{F}_n$ at the resonance point $B_0 = 0$. Requiring the invariance of $P_{\rm LSZ}$ against variations of B_0,

$$\mathcal{F}_n \frac{\partial \gamma_n}{\partial B_0} + \gamma_n \frac{\partial \mathcal{F}_n}{\partial B_0} = 0\,, \tag{13}$$

we obtain with

$$\frac{\partial \gamma_n}{\partial B_0} = \frac{\gamma}{n(B_0 + C)} \tag{14}$$

a differential equation for $\mathrm{d}\mathcal{F}/\mathrm{d}B_0$. Its solution

$$\mathcal{F}_n(\vartheta, B_0) = \left(\frac{C}{B_0 + C}\right)^{1/n} \mathcal{F}_n(\vartheta, 0) \tag{15}$$

allows us to rescale the correction functions obtained for $B_0 = 0$ to arbitrary B_0. Therefore, we can calculate $P_{\rm LSZ}$ now also for the non-resonant region in the two cases in which the function $\mathcal{F}_n$ is known for all ϑ.

For a $1/r$ profile, $A(r) = 2EV_0(R_0/r)$, the correction function is [4]

$$\mathcal{F}_{-1}(0, \vartheta) = \frac{\left(1 - \tan^2(\vartheta)\right)^2}{1 + \tan^2(\vartheta)}\,. \tag{16}$$

If we want to calculate the level-crossing probability in both the resonant and non-resonant cases, we have to choose $B_0 \geq 1$, e.g.

$$\mathcal{F}_{-1}(1, \vartheta) = \frac{1 + C}{C} \mathcal{F}_{-1}(0, \vartheta)\,; \tag{17}$$

the crossing probability follows as

$$P_{\rm LSZ} = \exp\left\{-2\pi\, R_0 V_0\, \sin^2(\vartheta)\right\}\,. \tag{18}$$

As an example, we compare in Fig. 2 the results of a numerical solution of the Schrödinger equation (4) with the analytical calculation of the $P_{\rm LSZ}$ using the rescaled F_{-1} function [9]. The agreement between the different methods is excellent.

In the limit $n \to \pm\infty$, which corresponds to an exponential potential profile, the scale factor $[C/(B_0 + C)]^{1/n}$ goes to 1 and the correction function becomes independent of B_0, as it should be. The resulting crossing probability is

$$P_{\rm LSZ} = \exp\left\{-\frac{\pi \Delta R_0}{E}\, \sin^2(\vartheta)\right\}\,. \tag{19}$$

In Refs. [10], this expression was derived by solving Schrödinger's equation directly. In these works, it was assumed that the obtained expression is valid only in the resonant region and only recently was it pointed out that it is valid for all ϑ [11]. Note also that $\ln P_{\rm LSZ}$ has the same, very simple dependence on ϑ, for both $n = -1$ and $n = \pm\infty$.

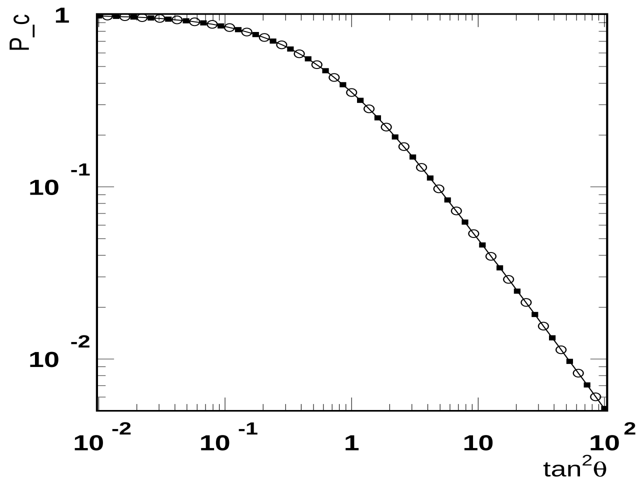

FIG. 2. Crossing probability P_c for $A \propto 1/r$ with $R_0 V_0 = 0.2$: num (squares) and $\mathcal{G}_{-1}$ (circles).

The correction functions $\mathcal{G}_n$—We start directly from Eq. (9), but use now as integration path in the complex x plane the part of a circle centered at zero starting at $A_1 = \Delta$ and going to the end of the branch cut at $A_2 = \Delta e^{2i\vartheta}$. Substituting $x = R_0(\Delta/A_0)^{1/n} e^{i\varphi}$ in the case of a potential $A = A_0(r/R_0)^n$, we can factor out the ϑ dependence of $P_{\rm LSZ}$ into functions $\mathcal{G}_n$,

$$\ln P_{\rm LSZ} = -\kappa_n \mathcal{G}_n(\vartheta)\,, \tag{20}$$

where

$$\kappa_n = \left(\frac{\Delta}{E}\right) \left(\frac{\Delta}{A_0}\right)^{1/n} R_0 \tag{21}$$

is independent of ϑ and

$$\mathcal{G}_n(\vartheta) = \left| \Re \int_0^{2\vartheta/n} \mathrm{d}\varphi\, e^{i\varphi} \left[\left(e^{in\varphi} - C\right)^2 + S^2\right]^{1/2} \right|\,. \tag{22}$$

The functions $\mathcal{G}_n$ are well suited for numerical evaluation and always correspond to a neutrino state propagating in the physical part of the x plane, $x > 0$. Therefore, they have, in contrast to the $\mathcal{F}_n$ functions, the period $\pi/2$ and are valid for all ϑ. In Fig. 2, the results of this new representation are compared with those obtained above for the case $n = -1$.

Oscillations of supernova neutrinos—The potential profile $A(r)$ in supernova (SN) envelopes can be approximated by a power law with $n \approx -3$, and $V(r) = 1.5 \times 10^{-9}$ eV $(10^9 \text{ cm}/r)^{-3}$ [12]. Since only $\bar{\nu}_e$ were detected from SN 1987A and also in the case of a future galactic SN the $\bar{\nu}_e$ flux will dominate the observed neutrino signal, an analytical expression for P_c valid in the non-resonant part of the mixing space is especially useful [7]. The probability for a $\bar{\nu}_e$ to arrive at the surface of the Earth can be written as an incoherent sum of probabilities [13],

$$P_{\bar{e}\bar{e}} = (1 - P_c) \cos^2 \vartheta + P_c \sin^2 \vartheta \, . \qquad (23)$$

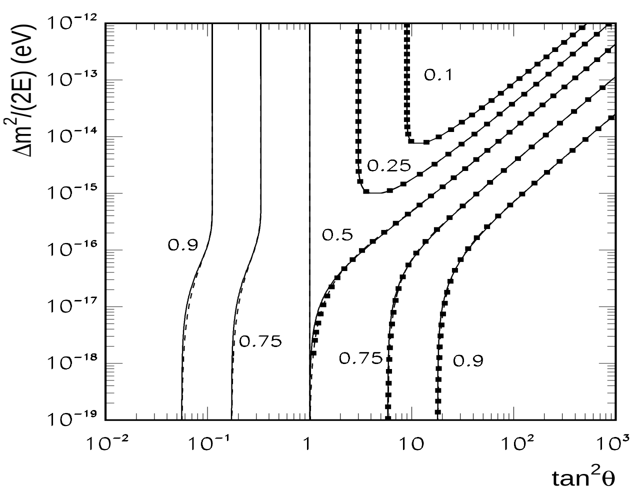

FIG. 3. Contours of constant survival probability $P_{\bar{e}\bar{e}}$, numerically (solid lines), with $\mathcal{G}_{-3}$ (dashed lines) and $\mathcal{F}_{-3}$ (squares, only for $\vartheta > \pi/4$), for $A \propto r^{-3}$ as given in the text.

In Fig. 3, we compare the results of a numerical solution of the Schrödinger equation (4) with the analytical calculation of $P_{\bar{e}\bar{e}}$ using the $\mathcal{G}_{-3}$ and the $\mathcal{F}_{-3}$ function. The latter is shown only for its range of applicability, $\vartheta > \pi/4$. The agreement between the two methods using the WKB approach is again (for $\vartheta > \pi/4$) excellent. Generally, these two methods agree also very well with the results of the numerical solution of the Schrödinger equation; there are only small deviations in the regions where the contours change their slope.

Summary—We have discussed non-adiabatic neutrino oscillations in general power-law potentials $A \propto r^n$. We have found that the conventional splitting of $\ln P_{\rm LSZ}$ in an "adiabaticity" parameter γ_n evaluated at the resonance point and a correction function $\mathcal{F}_n$ is misleading for $n \neq 1$: neither does the level-crossing probability have a maximum at r_0 nor does this splitting allow the calculation of $P_{\rm LSZ}$ in the non-resonant region. We have proposed a new representation for $P_{\rm LSZ}$ that avoids these problems and is hopefully useful for the investigations of oscillations of supernova neutrinos.

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