Three-Neutrino MSW Effect and the LNW Mass Matrix

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Abstract

We review recent work on analytical solutions to the MSW equations for three neutrino flavours, for exponential and linear potentials. An application to a particular mass matrix is also discussed. The three neutrino masses are determined, respectively, to be 0.001–0.004, and roughly 0.01 and 0.05 eV.

1 Introduction

In this paper we shall review some recent results on analytical solutions of the Mikheyev–Smirnov–Wolfenstein (MSW) effect [1] for the propagation of three neutrino flavours. For two model densities which are relevant to the neutrino propagation in the sun, such results have been obtained. These are the exponential density [2] and the linear density [3]. (The case of a constant density, which is of some relevance for propagation through the Earth, has also been studied recently [4].)

For the exponential density, the solutions for the three neutrino wave functions [2] can be expressed in terms of generalized hypergeometric functions, $_2F_2$ and $_3F_1$. For the linear density, the solutions can be expressed as a Fourier transform of a rather simple expression, which, in the case $N =$ 2 (two flavours) reduces to the well-known parabolic cylinder functions or confluent hypergeometric functions [5].

We also briefly discuss the application of the results to the Lehmann–Newton–Wu (LNW) mass matrix [6, 7].

2 Exponential density

Neutrino propagation through a medium where the electron neutrino (denoted $\phi_1(r)$) interacts differently from the others, is governed by the equation

$$
i\frac{\mathrm{d}}{\mathrm{d}r}
\begin{bmatrix}\phi_1(r)\\ \phi_2(r)\\ \phi_3(r)\end{bmatrix}
=
\left(
\begin{bmatrix}D(r) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}
\right.
$$
$$
\left.
+\frac{1}{2p}
\begin{bmatrix}M_{11}^2 & M_{12}^2 & M_{13}^2\\ M_{21}^2 & M_{22}^2 & M_{23}^2\\ M_{31}^2 & M_{32}^2 & M_{33}^2\end{bmatrix}
\right)
\begin{bmatrix}\phi_1(r)\\ \phi_2(r)\\ \phi_3(r)\end{bmatrix},
\quad (1)
$$

where the mass matrix is real and symmetric, $M^2_{ji} = M^2_{ij} \equiv (M^2)_{ij}$. Here, $D(r) = \sqrt{2}G_F N_e(r)$, with G_F the Fermi constant and $N_e(r)$ the solar electron density.

For the sun, the density [8] is well approximated by an exponential,

$$N_e(r) = N_e(0)\,e^{-r/r_0}, \quad r_0 \simeq 0.1 \times R_\odot. \quad (2)$$

It is convenient to introduce a new radial variable: $u = r/r_0 + u_0$, and perform a rotation on the second and third components,

$$\begin{bmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix} \frac{r_0}{2p} \begin{bmatrix} M^2_{22} & M^2_{23} \\ M^2_{32} & M^2_{33} \end{bmatrix}$$
$$\times \begin{bmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{bmatrix} = \begin{bmatrix} \omega_2 & 0 \\ 0 & \omega_3 \end{bmatrix}. \quad (3)$$

Eq. (1) then takes the form

$$i\frac{\mathrm{d}}{\mathrm{d}u} \begin{bmatrix} \psi_1(u) \\ \psi_2(u) \\ \psi_3(u) \end{bmatrix} = \begin{bmatrix} \omega_1 + e^{-u} & \chi_2 & \chi_3 \\ \chi_2 & \omega_2 & 0 \\ \chi_3 & 0 & \omega_3 \end{bmatrix} \begin{bmatrix} \psi_1(u) \\ \psi_2(u) \\ \psi_3(u) \end{bmatrix} \quad (4)$$

Let μ_1, μ_2 and μ_3 be the eigenvalues of the 3×3 matrix

$$\begin{bmatrix} \omega_1 & \chi_2 & \chi_3 \\ \chi_2 & \omega_2 & 0 \\ \chi_3 & 0 & \omega_3 \end{bmatrix}. \quad (5)$$

These μ_j are the squares of the neutrino masses multiplied by $r_0/(2p)$. Together with ω_1 and ω_2 they control the evolution of the ψ_i.

We again introduce a new variable, $z = ie^{-u}$. Then, the solutions to Eq. (4) can be expressed in terms of solutions to

$$\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z} - i\mu_1 \right) \left(z\frac{\mathrm{d}}{\mathrm{d}z} - i\mu_2 \right) \left(z\frac{\mathrm{d}}{\mathrm{d}z} - i\mu_3 \right) \right.$$
$$\left. - z \left(z\frac{\mathrm{d}}{\mathrm{d}z} - i\omega_2 \right) \left(z\frac{\mathrm{d}}{\mathrm{d}z} - i\omega_3 \right) \right]\psi = 0, \quad (6)$$

namely generalized hypergeometric functions [9]:

$$\psi^{(1)} = e^{-i\mu_1 u}$$
$$\times\ {}_2F_2 \left[\begin{matrix} -i(\omega_2 - \mu_1), & -i(\omega_3 - \mu_1) \\ 1 - i(\mu_2 - \mu_1), & 1 - i(\mu_3 - \mu_1) \end{matrix} \middle|\, ie^{-u} \right]$$

$$\psi^{(2)} = e^{-i\mu_2 u}$$
$$\times\ {}_2F_2 \left[\begin{matrix} -i(\omega_2 - \mu_2), & -i(\omega_3 - \mu_2) \\ 1 - i(\mu_1 - \mu_2), & 1 - i(\mu_3 - \mu_2) \end{matrix} \middle|\, ie^{-u} \right]$$

$$\psi^{(3)} = e^{-i\mu_3 u}$$
$$\times\ {}_2F_2 \left[\begin{matrix} -i(\omega_2 - \mu_3), & -i(\omega_3 - \mu_3) \\ 1 - i(\mu_1 - \mu_3), & 1 - i(\mu_2 - \mu_3) \end{matrix} \middle|\, ie^{-u} \right]$$
$$(7)$$

The ${}_2F_2$ can be defined in terms of the series expansions

$$_2F_2 \begin{bmatrix} \alpha_1, & \alpha_2 \\ \rho_1, & \rho_2 \end{bmatrix} z = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k}{(\rho_1)_k (\rho_2)_k} \frac{z^k}{k!} \quad (8)$$

where $(\alpha)_k$ is a Pochhammer symbol, $(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1)$. The solutions to Eq. (4) are thus

$$\psi_i = C_1 \psi_i^{(1)} + C_2 \psi_i^{(2)} + C_3 \psi_i^{(3)}, \quad (9)$$

where the constants C_j are determined by the boundary conditions: $\psi_1(r = 0) = 1$, $\psi_2(r = 0) = 0$, $\psi_3(r = 0) = 0$.

For neutrino masses and energies of physical interest, the parameters and arguments of the ${}_2F_2$ become too large for the series expansion to be useful. A practical procedure is to adopt a stationary-phase approximation for the ${}_2F_2$ of $\psi_i^{(3)}$, and express the others by ${}_3F_1$ functions. This procedure is outlined below.

Consider the ordinary differential equation for ${}_2F_2$ in the form

$$\left[\left(z\frac{d}{dz} + \beta_1 \right) \left(z\frac{d}{dz} + \beta_2 \right) \left(z\frac{d}{dz} + \beta_3 \right) \right.$$
$$\left. - z \left(z\frac{d}{dz} + \alpha_1 \right) \left(z\frac{d}{dz} + \alpha_2 \right) \right]f = 0. \quad (10)$$

This can be converted to an ordinary differential equation for $_3F_1$ by substituting $\hat{z} = z^{-1}$:

$$\left[\left(\hat{z}\frac{d}{d\hat{z}} - \alpha_1\right)\left(\hat{z}\frac{d}{d\hat{z}} - \alpha_2\right)\right.$$
$$+ \hat{z}\left(\hat{z}\frac{d}{d\hat{z}} - \beta_1\right)\left(\hat{z}\frac{d}{d\hat{z}} - \beta_2\right)$$
$$\left.\left(\hat{z}\frac{d}{d\hat{z}} - \beta_3\right)\right]f = 0. \quad (11)$$

Two solutions are of the form

$$z^{\text{power}}{}_3F_1\left[\begin{matrix}a_1, & a_2, & a_3 \\ & b & \end{matrix}\right| - z^{-1}\right]. \quad (12)$$

The full solutions (9) can then be constructed schematically (leaving out powers) as follows:

$$\psi_i = A_i\, _2F_2(1) + B_i\, _3F_1(1) + C_i\, _3F_1(2). \quad (13)$$

The series expansion for $_3F_1$ has zero radius of convergence. However, it can be expressed in terms of an integral involving the familiar hypergeometric function $_2F_1$,

$$_3F_1\left[\begin{matrix}a_1, & a_2, & a_3 \\ & b & \end{matrix}\right| - x^{-1}\right]$$
$$= \frac{1}{\Gamma(a_1)}x^{a_1}\int_0^\infty dt\, e^{-xt}\, t^{a_1-1}$$
$$\times\, _2F_1(a_2, a_3; b; -t)$$

For large parameters and argument, the Pochhammer contour $\mathcal{P}$ is useful [2]:

$$_2F_1(a, b; c; -t)$$
$$= \frac{-\Gamma(c)e^{-i\pi c}}{4\Gamma(b)\Gamma(c-b)\sin\pi b\sin\pi(c-b)}$$
$$\times \int_{\mathcal{P}} s^{b-1}(1-s)^{c-b-1}(1+ts)^{-a}ds$$

This approach leads to an accurate and efficient evaluation of the three neutrino wave functions in terms of stationary phase approximations to the $_2F_2$ and $_3F_1$ functions.

3 Linear density

The terminology "linear electron density" is used to mean that $N_e(x)$ is a linear function of x.

3.1 Two generations

The case of two states and a linear potential has been studied extensively, starting with Landau and Zener in the 1930s and applied to neutrino mixing in the 1980s [5]. After a suitable scaling and shift of the variable, one has

$$i\frac{d}{dt}\begin{bmatrix}\psi_1(t) \\ \psi_2(t)\end{bmatrix} = \begin{bmatrix}-t & a_2 \\ a_2 & 0\end{bmatrix}\begin{bmatrix}\psi_1(t) \\ \psi_2(t)\end{bmatrix}, \quad (14)$$

or more explicitly:

$$i\frac{d}{dt}\psi_1(t) = -t\psi_1(t) + a_2\psi_2(t),$$
$$i\frac{d}{dt}\psi_2(t) = a_2\psi_1(t). \quad (15)$$

Elimination of ψ_2 gives

$$\frac{d^2\psi_1(t)}{dt^2} - it\frac{d\psi_1(t)}{dt} + (a_2^2 - i)\psi_1(t) = 0. \quad (16)$$

The first-derivative term can be removed by taking

$$\psi_1(t) = e^{it^2/4}\phi_1(t). \quad (17)$$

Then the equation for $\phi_1(t)$ is

$$\frac{d^2\phi_1(t)}{dt^2} + (\tfrac{1}{4}t^2 + a_2^2 - \tfrac{1}{2}i)\phi_1(t) = 0. \quad (18)$$

Two linearly independent solutions of this equation are the parabolic cylinder functions

$$D_\rho(\pm e^{i\pi/4}\, t) \quad (19)$$

where $\rho = -ia_2^2 - 1$.

Parabolic cylinder functions are special cases of the confluent hypergeometric function,

$$D_\rho(z) = 2^{(\rho-1)/2}\, e^{-z^2/4}\, z\, \Psi(\tfrac{1}{2} - \tfrac{1}{2}\rho, \tfrac{3}{2}; \tfrac{1}{2}z^2) \quad (20)$$

In terms of confluent hypergeometric functions Ψ and Φ:

$$\psi_1(t) = t\Big[C\,\Phi(1+\tfrac{1}{2}ia_2^2,\tfrac{3}{2};\tfrac{1}{2}it^2) \\ +C'\,\Psi(1+\tfrac{1}{2}ia_2^2,\tfrac{3}{2};\tfrac{1}{2}it^2)\Big] \quad (21)$$

Unfortunately, it is not clear how to generalize this approach to $N \geq 3$.

Let us therefore consider an alternative solution to the $N = 2$ case [3]. We start by writing

$$F(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\zeta t}\,\psi_1(t). \qquad (22)$$

Then it follows from Eq. (16) that $F(\zeta)$ satisfies the first-order differential equation

$$-\zeta^2 F(\zeta) - \frac{d}{d\zeta}[-i\zeta F(\zeta)] + (a_2^2 - i)F(\zeta) = 0 \qquad (23)$$

or

$$\frac{1}{F(\zeta)}\frac{dF(\zeta)}{d\zeta} = \frac{i}{\zeta}(a_2^2 - \zeta^2). \qquad (24)$$

Integrating over ζ, one finds

$$F(\zeta) = \text{const.}\, e^{-i\zeta^2/2}\,\zeta^{ia_2^2}. \qquad (25)$$

With the notation [cf. Eq. (30)]

$$b_1 = -\infty \quad \text{and} \quad b_{N+1} = +\infty, \qquad (26)$$

the two solutions can be written as

$$\psi_1^{(n)}(t) = \int_{b_n}^{b_{n+1}} d\zeta\, e^{-i\zeta t}\, e^{-i\zeta^2/2}\, |\zeta|^{ia_2^2} \quad (27)$$

for $n = 1, 2$. It can be shown that they are confluent hypergeometric functions of the correct parameters and argument [i.e., identical to Eq. (21)].

3.2   General N

This second approach outlined above has the advantage that it can be generalized to an arbitrary number of neutrino flavors [3]. We

start out by writing the equation analogous to (1) in dimensionless standard form

$$i\frac{d}{dt}\psi(t) = A(t)\psi(x), \qquad (28)$$

where

$$\psi(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \\ \vdots \\ \psi_N(t) \end{bmatrix} \qquad (29)$$

and

$$A(t) = \begin{bmatrix} -t & a_2 & a_3 & \dots & a_N \\ a_2 & b_2 & 0 & \dots & 0 \\ a_3 & 0 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_N & 0 & 0 & \dots & b_N \end{bmatrix}. \qquad (30)$$

There are two kinds of equations:

$$i\frac{d\psi_1(t)}{dt} = -t\,\psi_1(t) + \sum_{j=2}^{N} a_j\psi_j(t) \qquad (31)$$

and, for $k = 2, 3, 4 \dots N$:

$$\left(i\frac{d}{dt} - b_k\right)\psi_k(t) = a_k\psi_1(t). \qquad (32)$$

One finds

$$\psi(t) = \sum_{n=1}^{N} C_n\,\psi^{(n)}(t), \qquad (33)$$

where [3]

$$\psi^{(n)}(t) = \int_{b_n}^{b_{n+1}} d\zeta\, e^{-i\zeta t}\, e^{-i\zeta^2/2} \\ \times \left(\prod_{j=2}^{N} |\zeta - b_j|^{ia_j^2}\right) \begin{bmatrix} 1 \\ \frac{a_2}{\zeta - b_2} \\ \frac{a_3}{\zeta - b_3} \\ \vdots \\ \frac{a_{N-1}}{\zeta - b_{N-1}} \\ \frac{a_N}{\zeta - b_N} \end{bmatrix} \qquad (34)$$

4

While these solutions are reasonably simple, a numerical Fourier transform is required. As $t \to \pm\infty$, explicit expressions can be written out for the different $\psi^{(n)}$. However, these can not directly be used for imposing the boundary conditions, since they would correspond to negative density at $t \to -\infty$.

4 The LNW mass matrix

For quark mixing, it was found [6] that a particular, simple texture for the d (d, s, b) and u (u, c, t) quark mass matrices leads to an acceptable CKM matrix [10]. This same mass matrix has been applied to the case of three neutrinos [7], and rather good fits to the atmospheric [11] and solar [12–16] neutrino data have been obtained.

The mass matrix is assumed to have the form

$$M = \begin{pmatrix} 0 & d & 0 \\ d & c & b \\ 0 & b & a \end{pmatrix} \qquad (35)$$

with $b^2 = 8c^2$. The eigenvalues are given by m_1, m_2, and m_3, with $m_1 \le m_3$.

In order to outline the diagonalization, whereby $M = R M_{\text{diag}} R^{\text{T}}$, let us introduce the notation

$$
\begin{aligned}
S_1 &\equiv m_3 - m_2 + m_1, \\
&= a + c \\
-S_2 &\equiv m_3 m_2 - m_3 m_1 + m_2 m_1, \\
&= 8c^2 + d^2 - ac \\
-S_3 &\equiv m_1 m_2 m_3 \\
&= a d^2.
\end{aligned} \qquad (36)
$$

Then, a cubic equation for the parameter a can be written as

$$9a^3 - 17 S_1 a^2 + (8 S_1^2 + S_2) a - S_3 = 0. \qquad (37)$$

A physical solution requires a real and positive. This is equivalent to having three real solutions for a. One of these is negative and two are positive. At any point inside the allowed domain in the m_1/m_3–m_2/m_3 plane, there are thus two allowed solutions, denoted Solutions 1 and 2.

Let us consider first the atmospheric neutrino data. The Super-Kamiokande results [11] give $\Delta m^2 \simeq (2-3) \times 10^{-3}$ eV, with $\sin^2(2\theta) \simeq 1$. The survival of muon neutrinos is given by

$$
\begin{aligned}
P_{\nu_\mu \to \nu_\mu}(t) = 1 - 4 &\left[U_{\mu 1}^2 U_{\mu 2}^2 \sin^2 \left(\frac{\Delta m_{21}^2 t}{4p} \right) \right. \\
&+ U_{\mu 1}^2 U_{\mu 3}^2 \sin^2 \left(\frac{\Delta m_{31}^2 t}{4p} \right) \\
&\left. + U_{\mu 2}^2 U_{\mu 3}^2 \sin^2 \left(\frac{\Delta m_{32}^2 t}{4p} \right) \right], \quad (38)
\end{aligned}
$$

where U is the neutrino mixing matrix. In the limit of $\Delta m_{21}^2 t / 4p \ll 1$ this simplifies, and invoking further unitarity, one finds

$$P_{\nu_\mu \to \nu_\tau}(t) \simeq 4 U_{\mu 3}^2 U_{\tau 3}^2 \sin^2 \left(\frac{\Delta m_{32}^2 t}{4p} \right), \quad (39)$$

which suggests that one needs $|U_{\mu 3} U_{\tau 3}| = \mathcal{O}(1)$. This can be achieved within the model (for both Solutions 1 and 2), for $m_1 \ll m_3$, with also m_2 small compared with m_3. Furthermore, the data suggest that the scale m_3 must be such that $m_3^2 \simeq (2-3) \times 10^{-3}$ eV.

Fits to atmospheric data confirm this qualitative analysis. Invoking also the solar Cl, Ga and Super-Kamiokande neutrino data [12–16], one finds that both Solutions 1 and 2 give good fits for $m_1 \ll m_3$, with m_2 also small as compared with m_3. Forming a χ^2 from these different atmospheric and solar survival probabilities, we found good fits [7], with m_3 of the order of 0.05 eV, m_2 about 0.01 eV, and $m_1 \sim 0.001$–0.004 eV. In terms of the more conventional two-flavour analyses for the solar-neutrino sector, these fits roughly correspond to the large-mixing-angle solution.

5   Summary

We have reviewed analytic work on the solutions to the MSW equations for three neutrino flavours. Such results are very valuable for a fast scanning over the parameters of some given model for the mass matrix.

Also, we have more briefly reviewed the LNW mass matrix, as applied to the neutrino data. This is a very constrained model that in the quark sector describes the CKM matrix, and in the neutrino sector gives the mixing in terms of the mass eigenvalues.

The solar neutrino data has also been studied within the same model, using numerical integration methods (no $_2F_2$'s) [17]. An additional fit was then found at $m_1 \simeq 2.8 \times 10^{-6}$ eV, corresponding to the small-mixing-angle solutions. However, this point is disfavoured by the atmospheric neutrino data, and by the electron recoil spectrum.

Acknowledgments

References

[1] L. Wolfenstein, Phys. Rev. **D 17** (1978) 2369, *ibid.* **20** (1979) 2634;
S. P. Mikheyev and A. Yu. Smirnov, Yad. Fiz. **42** (1985) 1441 [Sov. J. Nucl. Phys. **42** (1985) 913], Nuovo Cimento **9C** (1986) 17.

[2] P. Osland and T. T. Wu, Phys. Rev. D **62** (2000) 013008 [hep-ph/9912540].

[3] H. Lehmann, P. Osland and T. T. Wu, hep-ph/0006213.

[4] T. Ohlsson and H. Snellman, J. Math. Phys. **41** (2000) 2768 [hep-ph/9910546].

[5] See, for example, L. D. Landau, Phys. Z. Sovjetunion, **2** (1932) 46;
C. Zener, Proc. Roy. Soc. (London), **A137** (1932) 696;
W. C. Haxton, Phys. Rev. Lett. **57** (1987) 1271;
S. J. Parke, Phys. Rev. Lett. **57** (1986) 1275; and
S. T. Petcov, Phys. Lett. B **191** (1986) 299.

[6] H. Lehmann, C. Newton and T. T. Wu, Phys. Lett. B **384** (1996) 249.

[7] P. Osland and T. T. Wu, hep-ph/0006185.

[8] J. N. Bahcall, S. Basu and M. H. Pinsonneault, Phys. Letters **B 433**, 1 (1998);
see also J. N. Bahcall's homepage, http://www.sns.ias.edu/~jnb.

[9] Bateman Manuscript Project, *Higher Transcendental Functions*, A. Erdélyi, ed. (McGraw-Hill, New York, 1953) vol. I.

[10] N. Cabibbo, Phys. Rev. Lett. **10** (1963) 531;
M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49** (1973) 652

[11] Y. Fukuda *et al.* [Super-Kamiokande Collaboration], Phys. Rev. Lett. **81** (1998) 1562.

[12] Y. Fukuda *et al.* [Super-Kamiokande Collaboration], Phys. Rev. Lett. **81** (1998) 1158, Erratum, ibid. **81** (1998) 4279; ibid. **82** (1999) 1810.

[13] Y. Fukuda *et al.* [Super-Kamiokande Collaboration], Phys. Rev. Lett. **82** (1999) 2430.

[14] R. Davis, Jr., in *Frontiers of Neutrino Astrophysics*, edited by Y. Suzuki and K. Nakamura (Universal Academy Press, Inc., Tokyo, 1993), p. 47; B. T. Cleveland *et al.*, Nucl. Phys. (Proc. Suppl.) **B 38**, 47 (1995); Astrophys. J. **496** (1998) 505.

[15] J. N. Abdurashitov *et al.* [SAGE Collaboration], Phys. Rev. Lett. **83** (1999) 4686, Phys. Rev. **C60** (1999) 055801.

[16] W. Hampel *et al.* [GALLEX Collaboration], Phys. Lett. **B447** (1999) 127.

[17] P. Osland and G. Vigdel, Phys. Lett. B **488** (2000) 329 [hep-ph/0006343].