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On fermion masses, gradient flows and potential in supersymmetric theories

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Abstract: In any low energy effective supergravity theory general formulae exist which allow one to discuss fermion masses, the scalar potential and breaking of symmetries in a model independent set up. A particular role in this discussion is played by Killing vectors and Killing prepotentials. We outline these relations in general and specify then in the context of $N = 1$ and $N = 2$ supergravities in four dimensions. Useful relations of gauged quaternionic geometry underlying hypermultiplets dynamics are discussed.

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Contents

1. Introduction

Supersymmetry Ward identities play a crucial role in disentangling general properties of effective supersymmetric lagrangians arising from some more fundamental theory at the Planck scale where gravity is strongly coupled.

A particular role is played by supersymmetry relations on the scalar potential [1, 2] and on supersymmetry preserving vacua which are at the basis for the discussion. of partial supersymmetry breaking [3]–[7] and of BPS configurations and non perturbative string or M theory vacua [6, 8, 9, 10, 11]. Some of these relations where studied long ago, [2], but more recently a more careful analysis of supersymmetry preserving configurations has played a crucial role in the study of the so called “attractor mechanism” [12] for charged “black holes” in four and five dimensions [13] as well as for the study of supergravity flows [14, 15] related to the so called “renormalization group flow” [16, 17] in the framework of the AdS/CFT correspondence [18].

Very recently these relations have been applied to a variety of interrelated problems such as domain walls in five dimensional supergravity [19]–[24], and supergravity instantons [25] responsible for non perturbative corrections to the hypermultiplet moduli space in Calabi–Yau compactifications [26].

In the present note we make some general consideration on scalar potentials, fermion masses and killing prepotentials in a generic supersymmetric theory encompassing any low-energy effective lagrangian of a more fundamental theory which at low energy incorporates a theory of gravity with N-extended supersymmetry.

Much of the information comes from the analysis of simple terms in the supersymmetry variation of the effective action, namely terms with one fermion and one boson (or its first derivative).

It is shown that general formulae for fermion masses and scalar potential exist which are simply related to the fermion shifts of the supersymmetry transformation laws; in particular the $N = 1$ and $N = 2$ structures of the matter coupled supergravities [27]–[31], can be recovered in a simplified form. To illustrate the general procedure we limit ourselves to the four-dimensional case, but it is straightforward to see that our considerations can be extended also to higher dimensions.

In the particular case of $N = 2$ supergravity with arbitrary gauge interactions turned on an important role is played by gauged quaternionic [32]–[34] and special geometry [30, 35, 36] which was discussed in full detail some time ago [34, 31]. Here we are able to find new relations between Killing prepotentials which allow us to show that some gradient flow relations due to supersymmetry are merely due to some simple properties of special and quaternionic geometry in presence of gauged isometries. These relations purely depend on the geometrical data of the theory, including gauging of isometries of the scalar manifold.[1] This note is organized as follows: in section 2 we set up the formalism and derive some basic relations between scalar potential, fermionic shifts and fermion mass-matrices.

In section 3 and 4 we specify these relations to $N = 1$ and $N = 2$ theories and recast some results in a model independent set up.

Section 4 is particularly relevant because it deals with $N = 2$ supergravity with general interactions of vector multiplets and hypermultiplets turned on [31]. Here some interesting relations emerge due to the special structure of coupled special and quaternionic geometries in presence of gauge isometries.

One of the amusing results, already noted in some special cases, is that the (non derivative) part of the spin 1/2-shifts can be written in terms of the (covariant) derivative of some scalar functions [20, 24, 25, 38] (the hypermultiplet and vector multiplet prepotentials) exactly as in the case of charged (abelian) black-hole configurations [12], where the "central charge" matrix is here replaced by the SU(2) valued prepotential matrix.

An appendix with the basic relations of gauged quaternionic geometry is included.

[1] A short account of the geometrical approach to Supergravity in the $N = 2$ case is given in [31] (see especially the appendices of the second paper); a more general reference is [37, volume 2].

JHEP05(2001)034

2. The formalism: entangling supersymmetry with geometry

We write down the generic form of $4D$ N -extended supergravity theory up to 4-fermion terms in the following way:

$$
\begin{aligned}
(\det V)^{-1}\mathcal{L} = & -\frac{1}{2}\mathcal{R} + \mathrm{i}\left(\bar{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{F}^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\Sigma\mu\nu}\right) + \hat{P}^{IA}_{\mu}\hat{P}^{\mu}_{IA} + \\
& + \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\left(\bar{\psi}^{A}_{\mu}\gamma_{\sigma}D_{\nu}\psi_{A\lambda} - \bar{\psi}_{A\mu}\gamma_{\sigma}D_{\nu}\psi^{A}_{\lambda}\right) + \mathrm{i}\frac{1}{2}\left(\bar{\lambda}^{I}\gamma^{\mu}\nabla_{\mu}\lambda_{I} + \bar{\lambda}_{I}\gamma^{\mu}\nabla_{\mu}\lambda^{I}\right) - \\
& - \hat{P}^{IA}_{\mu}\bar{\lambda}_{I}\gamma^{\nu}\gamma^{\mu}\psi_{A\nu} - \hat{P}_{IA\mu}\bar{\lambda}^{I}\gamma^{\nu}\gamma^{\mu}\psi^{A}_{\nu} + \\
& + \mathcal{F}^{\Lambda}_{\mu\nu}\mathcal{N}_{\Lambda\Sigma}\left(L^{\Sigma}_{AB}\bar{\psi}^{\mu A}\psi^{\nu B} + L^{\Sigma}_{I}\bar{\psi}^{\mu A}\gamma^{\nu}\lambda^{I}_{A} + L^{\Sigma}_{IJ}\bar{\lambda}^{I}\gamma^{\mu\nu}\lambda^{J} + \mathrm{h.c.}\right) + \\
& + 2\bar{S}^{AB}\bar{\psi}_{\mu A}\gamma^{\mu\nu}\psi_{\nu B} + 2S_{AB}\bar{\psi}^{A}_{\mu}\gamma^{\mu\nu}\psi^{B}_{\nu} + \mathrm{i}N^{A}_{I}\bar{\lambda}^{I}\gamma^{\mu}\psi_{\mu A} + \mathrm{i}N^{I}_{A}\bar{\lambda}_{I}\gamma^{\mu}\psi^{A}_{\mu} + \\
& + \mathcal{M}^{IJ}\bar{\lambda}_{I}\lambda_{J} + \mathcal{M}_{IJ}\bar{\lambda}^{I}\lambda^{J} - \mathcal{V}(q)
\end{aligned}
\tag{2.1}
$$

where q^u,λ^I,$\psi_{A\mu}$ are the scalar fields, the spin 1/2 fermions, and $\psi_{A\mu}$ the gravitino fields. The vector field–strengths $\mathcal{F}^{\Lambda\pm}_{\mu\nu}$ are defined in equation (3.7) below. The labels of the fields are as follows; we indicate by $A, B, \ldots$ the indices of the fundamental representation of the R-symmetry group $\mathrm{SU}(N) \otimes \mathrm{U}(1)$, their lower (upper) position indicating their left (right) chirality. The indices I on the spin 1/2 fermions,besides to enumerate the fields, are a condensed notation which encompasses various possibilities; if the fermions belong to vector multiplets we have to set $I \to IA$ since they also transform under R-symmetry; if they refer to fermions of the gravitational multiplet they are a set of three $\mathrm{SU}(N)$ antisymmetric indices: $I \to [ABC]$. In the case of n_H hypermultiplets $I \to \alpha$ where α is in the fundamental of $\mathrm{Sp}(2n_H)$.

The matrices entering the lagrangian are all dependent on the scalar fieds q^i. $\mathcal{N}_{\Lambda\Sigma}$ is the kinetic symmetric matrix of the vector field-strengths, with Λ, Σ indices in the symplectic representation under which they transform; S_{AB}, N^I_A, M^{IJ}, together with their hermitian conjugates $\bar{S}^{AB}$, N^A_I, M_{IJ},are matrices of order g in the gauge coupling constant while the scalar potential $\mathcal{V}(q)$ is of order g^2. Note that $\bar{S}^{AB}$, M^{IJ} are the mass matrices of the gravitino and the spin 1/2 fermions. Finally $\hat{P}^{IA}$ are the the gauged vielbein 1-forms of the scalar manifold defined as

$$
\hat{P}^{IA}_{\mu} = P^{IA}_{u}\left(\partial_{\mu}q^{u} + gA^{\Lambda}_{\mu}k^{u}_{\Lambda}\right) \equiv P^{IA}_{u}\nabla_{\mu}q^{u}\,, \tag{2.2}
$$

where P^{IA}_i is the ordinary vielbein of the scalar manifold. Also in this case the index I of the vielbein must be given the same interpretation as explained in the case of the spin 1/2 fields. Moreover for any boson field v carrying $\mathrm{SU}(N)$ indices we have that lower and upper indices are related by complex conjugation,namely:

$$
(v_{AB\ldots})^{*} \sim \bar{v}^{AB\ldots}\,. \tag{2.3}
$$

When $N > 2$, so that the scalar manifold is a coset G/H, the gauged vielbein 1-form can be rewritten

$$\hat{P}^{IA} = \left(L^{-1}\, dL\right)^{IA} + gL^{-1}{}^{I}_{\Gamma} A^{\Lambda}(T_{\Lambda})^{\Gamma}_{\Pi} L^{\Pi A} \tag{2.4}$$

due to the general relation

$$P^{IA}_{u} k^{u}_{\Lambda} \equiv \left(L^{-1}\partial_{u} L\right)^{IA} k^{u}_{\Lambda} = L^{-1}{}^{I}_{\Gamma}(T_{\Lambda})^{\Gamma}_{\Pi} L^{\Pi A}\,, \tag{2.5}$$

where T_{Λ} are the generators of the gauge group.

We now write down the relevant terms of the supersymmetry transformation laws of the various fields in order to perform the supersymmetry variation of the lagrangian; this will allow us to identify the differential equations for the fermionic shifts and other important relations between geometrical quantities mentioned in the introduction. We have:

$$\begin{aligned}
\delta\psi_{A\mu} &= \mathcal{D}_{\mu}\varepsilon_{A} + \cdots + S_{AB}\gamma_{\mu}\varepsilon^{B} && (2.6)\\
\delta\lambda^{I} &= \mathrm{i}\hat{P}^{IA}_{i}\varepsilon_{A}\nabla_{\mu} q^{i} + \cdots + N^{IA}\varepsilon_{A} && (2.7)\\
\delta V^{a}_{\mu} &= -\mathrm{i}\overline{\psi}_{\mathrm{A}}\gamma_{\mu}\varepsilon^{\mathrm{A}} + \text{h.c.} && (2.8)\\
\delta A^{\Lambda}_{\mu} &= 2f^{\Lambda[AB]}\overline{\psi}_{A\mu}\varepsilon_{B} + \mathrm{i}f^{\Lambda IA}\bar{\lambda}^{I}\gamma_{\mu}\varepsilon_{A} + \text{h.c.} && (2.9)\\
\delta q^{u} P^{IA}_{u} &= \bar{\lambda}^{I}\varepsilon_{A}\,, && (2.10)
\end{aligned}$$

where $f^{\Lambda[AB]}$ and $f^{\Lambda IA}$ are symplectic sections on the scalar manifold. We are going to explore the invariance of (2.1) (up to a total derivative) for terms of the form $f\,\varepsilon\,B$ where f is a fermion and B is a function of the scalar fields.

We have to look to two kinds of terms:

- terms with one derivative
- terms with no derivatives

In the first case we can choose $f\,\partial\varepsilon\,q$ and $f\,\varepsilon\,\partial q$ as independent variations [1] since all these terms are independent. It is a simple exercise, first carried out in [1], to see that the terms containing the derivative of the supersymmetry parameter just fix the couplings $2\,\bar{S}^{AB}\overline{\psi}^{A}_{\mu}\gamma^{\mu\nu}\psi^{B}_{\nu} + \mathrm{i}N^{A}_{I}\bar{\lambda}^{I}\gamma^{\mu}\psi_{\mu A} + \text{h.c.}$ of the lagrangian in terms of the shifts proportional to g of the equations (2.6), (2.7).

The terms with no derivatives of the form $f\,\varepsilon\,B$ instead give rise to two important relations [1]. The first one is due specifically to the terms $\psi_{A}\gamma_{\mu}\varepsilon^{A}\,B(q)$ which determine the scalar potential $V(q)$ in terms of the squared modulus of the shifts (2.6), (2.7); one finds:

$$\delta^{A}_{B}\, V(q) = -12\bar{S}^{AC}\, S_{CB} + N^{A}_{I}\, N^{I}_{B}\,. \tag{2.11}$$

The second one is due to terms of the form $\lambda\varepsilon B(q)$ and their h.c. that give rise to a formula from which the Goldstone theorem for supergravity can be derived:

$$\frac{\partial V}{\partial q^i} P^{iIA} = -4N^I_B\,\bar{S}^{BA} + 2\mathcal{M}^{IJ}\,N^A_J\,. \tag{2.12}$$

Tracing equations (2.11) with respect to A, B and differentiating with respect to q^u, by comparison with equation (2.12), it follows that there must be some relation between N^A_I and ∂S_{AB} as well as between $\mathcal{M}^{IJ}$ and ∂N^J_A. We shall refer to these relations as "gradient flows" for the fermionic shifts. These gradient flows can be obtained in the simplest way by looking at the terms $f\,\varepsilon\,\partial q$ of the first item (terms with one derivative) which have not been yet considered.

Let us first consider the equations derived when considering terms of the form $\psi\,\varepsilon\,\partial q$. There are two independent structures proportional to the currents with $\delta^{\mu\nu}$ and $\gamma^{\mu\nu}$,respectively. The $\delta^{\mu\nu}$ current gives the equation:

$$k^u_\Lambda\, f^{\Lambda\,[AB]} + N^{[A}_I\, P^{B]I\,iu} = 0\,, \tag{2.13}$$

where one has to take into account the contribution coming from the kinetic term of the scalars due to the definition of $\hat{P}^{IA}_u$ which contributes through δA^Λ_μ given in equation (2.9). Equation (2.13) relates the Killing vector k^u_Λ $(\delta q^u = \xi^\Lambda\, k^u_\Lambda)$ to the spin 1/2 shifts N^A_I.

The terms proportional to the $\gamma^{\mu\nu}$ current yield the gradient flow:

$$D_u S^{AB} = N^{(A}_I\, P^{B)I}_u\,. \tag{2.14}$$

Considering next the equation coming from the terms $\lambda\varepsilon\partial q$ we find the gradient flow of the spin 1/2 shifts:

$$\nabla_u N^A_I = g_{uv}\, k^v_\Lambda\, f^{\Lambda A}_I + 2\, P_{IB\,u}\, S^{BA} + 2\,\mathcal{M}_{IJ}\, P^{JA}_u\,. \tag{2.15}$$

Alternatively eqs. (2.13), (2.14) can be cast in the following form:

$$D_u S^{AB} = P^{AI}_u N^B_I - k_{u\,\Lambda}\, f^{\Lambda\,[AB]} \tag{2.16}$$

which is analogous to eq. (2.15).

We note that the previous results determine the full fermionic mass matrix $\mathcal{M}_{IJ}$ through eq. (2.15).

If there are multiplets with no scalars as it happens in the $N = 1$ case then the fermionic mass matrix for the fermions of such multiplets is obtained by looking in the variation of the lagrangian to extra terms of the form $\lambda\varepsilon\,\mathcal{F}$, $\mathcal{F}$ being the field-strength of the vector replacing in this case the ∂q factor: indeed if λ has no scalar partner it must certainly have a vector partner and the mass matrix of the fermions can be obtained by the aforementioned variation.

JHEP05(2001)034

In a different fashion also behave multiplets where the fermions are the only partner of scalar fields (Wess-Zumino multiplets in $N = 1$ and hypermultiplets in $N = 2$) because in those cases $f^{\Lambda[AB]}$ and $f^{\Lambda IA}$ do not exist in the eq. (2.9) and therefore they do not enter in the determination of $\nabla_u N^A_I$. Under these circumstances $\nabla_u N^A_I$ and $\mathcal{M}_{IJ}$ can be expressed through eqs. (2.15), (2.16) purely in terms of the gravitino mass matrix S_{AB} and its derivatives [27].

3. The $N = 1$ case

In order to apply our formulae to the $N = 1$ case we recall that the scalar manifold is in this case a Kähler-Hodge manifold [32] and that the R- symmetry reduces simply to U(1). It is convenient in this case to use as "vielbeins" the differential of the complex coordinates $dz^i, d\bar{z}^{i^\star}$ where $z^i(x)$ are the complex scalar fields parametrizing the Kähler-Hodge manifold of (complex) dimension n_C. Therefore in the present case we have to set $q^u \to (z^i, \bar{z}^{i^\star})$. The spin 1/2 fermions are either in chiral or in vector multiplets. So the index I runs over the number $n_V + n_C$ of vector and chiral multiplets, $I = 1, \dots, n_V + n_C$. It is convenient to assign the index Λ, the same as for the vectors, to the fermions of the vector multiplets and we will denote them as λ^Λ, $\Lambda = 1, \dots, n_V$; the fermions of the chiral multiplets will instead be denoted by $\chi^i, \chi^{i^\star}$ in the case of left-handed or right-handed spinors, respectively. Since the gravitino and the gaugino fermions have no SU(N) indices their chirality will be denoted by a lower or an upper dot for left-handed or right handed fermions respectively, namely ($\psi_\bullet$, $\psi^\bullet$); ($\lambda^\Lambda_\bullet$, $\lambda^{\bullet\Lambda}$). Moreover we have two metrics, namely the Kähler metric $g_{ij^\star}$ of the scalar manifold and the metric $\mathcal{N}_{\Lambda\Sigma}$ of the vector kinetic term with symplectic indices Λ, Σ. Using these conventions we have the following supersymmetry transformation laws for the fields [27, 29]:

$$\delta\psi_{\bullet\mu} = \mathcal{D}_\mu \varepsilon_\bullet + \cdots + \mathrm{i} L(z, \bar{z}) \gamma_\mu \varepsilon^\bullet \tag{3.1}$$
$$\delta\chi^i = \mathrm{i} \nabla_\mu z^i \gamma^\mu \varepsilon_\bullet + \cdots + N^i \varepsilon_\bullet \tag{3.2}$$
$$\delta\lambda^\Lambda_\bullet = \mathcal{F}^{(-)\Lambda}_{\mu\nu} \gamma^{\mu\nu} \varepsilon_\bullet + \cdots + \mathrm{i} D^\Lambda \varepsilon_\bullet \tag{3.3}$$
$$\delta V^a_\mu = -\mathrm{i} \psi_\bullet \gamma_\mu \varepsilon^\bullet + \mathrm{h.c.} \tag{3.4}$$
$$\delta A^\Lambda_\mu = \mathrm{i} \frac{1}{2} \bar{\lambda}^\Lambda_\bullet \gamma_\mu \varepsilon^\bullet + \mathrm{h.c.} \tag{3.5}$$
$$\delta z^i = \bar{\chi}^i \varepsilon_\bullet \,, \tag{3.6}$$

where we have defined

$$\begin{aligned}
\mathcal{F}^{(\mp)\Lambda}_{\mu\nu} &= \frac{1}{2} \left(\mathcal{F}^\Lambda_{\mu\nu} \mp \mathrm{i} \star \mathcal{F}^\Lambda{}_{\mu\nu} \right) \\
\star \mathcal{F}^\Lambda{}_{\mu\nu} &\equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\rho\sigma\Lambda} \\
\star \mathcal{F}^{\Lambda(\pm)}{}_{\mu\nu} &= \mp \mathrm{i} \mathcal{F}^{\Lambda(\pm)}{}_{\mu\nu}
\end{aligned} \tag{3.7}$$

and the dots mean 3-fermion terms (irrelevant for our purposes).

The $N = 1$ supergravity lagrangian invariant under the transformations (3.1)–(3.6)(up to 4-fermion terms) is:

$$
\begin{aligned}
(\det \mathrm{V})^{-1}\mathcal{L} = & -\frac{1}{2}\mathcal{R} + \mathrm{i}\left(\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{F}^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\Sigma\mu\nu}\right) + g_{ij^\star}\nabla_\mu z^i \nabla^\mu z^{j^\star} + \\
& + \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\left(\bar{\psi}^\bullet_\mu \gamma_\sigma \nabla_\nu \psi_{\bullet\lambda} - \bar{\psi}_{\bullet\mu}\gamma_\sigma \nabla_\nu \psi^\bullet_\lambda\right) - \\
& - \frac{1}{8}\left(\mathcal{N}_{\Lambda\Sigma}\bar{\lambda}^{\bullet\Lambda}\gamma^\mu \nabla_\mu \lambda^\Sigma_\bullet - \overline{\mathcal{N}}_{\Lambda\Sigma}\bar{\lambda}^\Lambda_\bullet \gamma^\mu \nabla_\mu \lambda^{\bullet\Sigma}\right) - \\
& - \mathrm{i}\frac{1}{2}g_{ij^\star}\left(\bar{\chi}^i \gamma^\mu \nabla_\mu \chi^{j^\star} + \bar{\chi}^{j^\star}\gamma^\mu \nabla_\mu \chi^i\right) + \\
& + g_{ij^\star}\left(\bar{\psi}^\bullet_\nu \gamma^\nu \gamma^\mu \chi^i \nabla^\mu z^{\bar{j}} + \bar{\psi}_{\bullet\nu}\gamma^\nu\gamma^\mu \chi^{\bar{j}}\nabla_\mu z^i\right) - \\
& - 2\,\mathrm{i}\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\left(\mathcal{F}^{-\Lambda}_{\mu\nu}\bar{\lambda}^\Sigma_\bullet \gamma^\mu \psi^{\bullet\nu} + \mathcal{F}^{+\Lambda}_{\mu\nu}\bar{\lambda}^{\bullet\Sigma}\gamma^\mu \psi^\nu_\bullet\right) + \\
& + \frac{\mathrm{i}}{4}\left(\partial_i \overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\bar{\chi}^i \gamma^{\mu\nu}\lambda^\Sigma_\bullet - (\partial_{i^\star}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\bar{\chi}^{i^\star}\gamma^{\mu\nu}\lambda^{\bullet\Sigma}\right) + \\
& + 2L\bar{\psi}^\bullet_\mu \gamma^{\mu\nu}\psi^\bullet_\nu + 2\bar{L}\bar{\psi}_{\bullet\mu}\gamma^{\mu\nu}\psi_{\bullet\nu} + \mathrm{i}g_{ij^\star}\left(\overline{N}^{\bar{j}}\bar{\chi}^i \gamma^\mu \psi^\bullet_\mu + N^i \bar{\chi}^{j^\star}\gamma^\mu \psi_{\bullet\mu}\right) + \\
& + \frac{1}{2}P_\Lambda \left(\bar{\lambda}^{\bullet\Lambda}\gamma^\mu \psi_{\bullet\mu} - \bar{\lambda}^\Lambda_\bullet \gamma^\mu \psi^{\bullet|}_\mu\right) + \mathcal{M}_{ij}\bar{\chi}^i \chi^j + \overline{\mathcal{M}}_{i^\star j^\star}\bar{\chi}^{i^\star}\chi^{j^\star} + \\
& + \mathcal{M}_{\Lambda\Sigma}\bar{\lambda}^\Lambda_\bullet \lambda^\Sigma_\bullet + \overline{\mathcal{M}}_{\Lambda\Sigma}\bar{\lambda}^{\Lambda\bullet}\lambda^{\Sigma\bullet} + \mathcal{M}_{\Lambda i}\bar{\lambda}^\Lambda_\bullet \chi^i + \overline{\mathcal{M}}_{\Lambda i^\star}\bar{\lambda}^{\Lambda\bullet}\chi^{i^\star} - \mathcal{V}(z,\bar{z})
\end{aligned}
\tag{3.8}
$$

and the kinetic matrix $\overline{\mathcal{N}}_{\Lambda\Sigma}$ turns out to be a holomorphic function of z^i: $\overline{\mathcal{N}}_{\Lambda\Sigma} = \overline{\mathcal{N}}_{\Lambda\Sigma}(z^i) \to \mathcal{N}_{\Lambda\Sigma} = \mathcal{N}_{\Lambda\Sigma}(\bar{z}^{i^\star})$. Note that since the scalar manifold is a Kähler-Hodge manifold all the fields and the bosonic sections have a definite U(1) weight p under U(1). We have

$$
\begin{aligned}
p(V^a_\mu) &= p(A^\Lambda) = p(z^i) = p(g_{ij^\star}) = p(\mathcal{N}_{\Lambda\Sigma}) = p(D^\Lambda) = p(P_\Lambda) = p(\mathcal{V}) = 0 \\
p(\psi_\bullet) &= p(\chi^{i^\star}) = p(\lambda^\Lambda_\bullet) = p(\varepsilon_\bullet) = \frac{1}{2} \\
p(\psi^\bullet) &= p(\chi^i) = p(\lambda^{\Lambda\bullet}) = p(\varepsilon^\bullet) = -\frac{1}{2} \\
p(L) &= p(\mathcal{M}_{ij}) = p(\overline{\mathcal{M}}_{\Lambda\Sigma}) = 1 \\
p(\bar{L}) &= p(\overline{\mathcal{M}}_{i^\star j^\star}) = p(\mathcal{M}_{\Lambda\Sigma}) = -1\,.
\end{aligned}
\tag{3.9}
$$

Accordingly, when a covariant derivative acts on a field Φ of weight p it is also U(1) covariant (besides possibly Lorentz, gauge and scalar manifold coordinate symmetries) according to the following definitions:

$$
\nabla_i \Phi = (\partial_i + \frac{1}{2}p\partial_i \mathcal{K})\Phi\,; \qquad \nabla_{i^*}\Phi = (\partial_{i^*} - \frac{1}{2}p\partial_{i^*}\mathcal{K})\Phi\,, \tag{3.10}
$$

where $\mathcal{K}(z,\bar{z})$ is the Kähler potential.

A covariantly holomorphic section of is defined by the equation: $\nabla_{i^*}\Phi = 0$.

Supersymmetry implies that all the quantities entering the transformation laws and the lagrangian can be expressed in terms of the following geometric quantities: the covariantly holomorphic gravitino mass-matrix $L(z,\bar{z})$, the Killing vector real prepotential $P_\Lambda(z,\bar{z})$ the Kähler potential and the holomorphic matrix $\overline{\mathcal{N}}_{\Lambda\Sigma}(z)$.

Indeed we have the following relations:[2]

$$L(z,\bar z) = W(z) e^{\frac{1}{2}\mathcal{K}(z,\bar z)} \tag{3.11}$$
$$\nabla_{i^\star} L(z,\bar z) = 0 \tag{3.12}$$
$$N^i = 2\mathrm{i} g^{ij^\star} \nabla_{j^\star} \bar L \tag{3.13}$$
$$D^\Lambda = 2\,\mathrm{Im}(\mathcal{N}_{\Lambda\Sigma})^{-1} P_\Sigma \tag{3.14}$$
$$\mathcal{M}_{ij} = \nabla_i \nabla_j L \tag{3.15}$$
$$\mathcal{M}_{\Lambda\Sigma} = \frac{1}{8} N^i \partial_i \overline{\mathcal{N}}_{\Lambda\Sigma} \tag{3.16}$$
$$M_{\Lambda i} = -\mathrm{i}\frac{1}{4}\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\partial_i D^\Sigma - \frac{1}{2} k_\Lambda^{j^\star} g_{ij^\star} \tag{3.17}$$
$$\mathcal{V} = 4\left(-3L\bar L + g^{ij^\star}\nabla_i L \nabla_{j^\star}\bar L + \frac{1}{16}\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} D^\Lambda D^\Sigma\right), \tag{3.18}$$

where the Killing vector is defined in terms of the real prepotential P_Λ as follows:

$$k^i_\Lambda = \mathrm{i} g^{ij^\star} \partial_{j^\star} P_\Lambda\,. \tag{3.19}$$

Note that eqs. (3.14) and (3.17) imply:

$$M_{\Lambda i} D^\Lambda = -\mathrm{i}\frac{1}{4}\partial_i (\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} D^\Lambda D^\Sigma)\,. \tag{3.20}$$

In Kähler geometry $k^i_\Lambda = k^i_\Lambda(z)$ is holomorphic and satisfies the following relations:

$$\nabla_i k_{j\Lambda} = 0 \tag{3.21}$$
$$\nabla_i k_{j^\star\Lambda} = \nabla_{j^\star} k_{i\Lambda}\,. \tag{3.22}$$

Finally the gradient flows are:

$$\begin{aligned} \nabla_i N^j &= 2\,\delta_i^j \\ \nabla_i \bar N^{j^\star} g_{kj^\star} &= 2\mathcal{M}_{ik} \\ \nabla_i L &= \frac{1}{2} g_{ij^\star} \bar N^{j^\star} \\ \nabla_{i^\star} L &= 0\,. \end{aligned} \tag{3.23}$$

4. The $N = 2$ case

For the $N = 2$ supergravity the scalar manifold is a product manifold [30, 34, 45]

$$\mathcal{M}^{(\mathrm{scalar})} = \mathcal{M}^{(\mathrm{vec})} \otimes \mathcal{M}^{(\mathrm{hyper})} \tag{4.1}$$

since we have two kinds of matter multiplets, the vector multiplets and the hypermultiplets. The geometry of $\mathcal{M}^{(\mathrm{vec})}$ is described by the *Special Kähler geometry* [30, 35, 36, 43] while the geometry of $\mathcal{M}^{(\mathrm{hyper})}$ is described by *Quaternionic geometry* [39]–

[2]For constant scalar background unbroken supersymmetry requires $\nabla_i L = 0$, i.e. the "norm" $\|L\|^2 = L\bar L$ to be extremized on the Kähler-Hodge manifold. This is the $N = 1$ example of "attractor equation" [9, 12].

[42], [32, 30, 34, 44]. A full account of *Special Kähler geometry* is given in refs. [31]. As far as *Quaternionic geometry* is concerned, we have set an appendix to the present paper since the account given in the reference [31] and also in [32, 34] do not comprise several new important identities that we present in the appendix.

With respect to the general case we now have

$$\Lambda = 1, \ldots, n_V\,; \qquad A, B = 1, 2\,; \qquad i = 1, \ldots, 4n_H + 2n_V\,; \qquad I = 1, \ldots n_H + n_V\,. \tag{4.2}$$

We will use the same notations and conventions as in ref. [31] where the complete theory of the $N = 2$ supergravity has been fully worked out in a geometrical setting.

Let us now shortly describe how our general framework particularizes to the present case.

As in the case of $N = 1$ we denote the complex scalars parametrizing $\mathcal{M}^{(vec)}$ by $z^i, \bar{z}^{\bar{i}}$, while the scalars parametrizing $\mathcal{M}^{(hyper)}$ will be denoted by q^u. As already noted in the previous section, when the index I runs over the vector multiplets it must be substituted by IB in all the formulae relevant to the vector multiplet, since the fermions λ^{IA} are in the fundamental of the R-symmetry group U(2). Furthermore if we use coordinate indices as in the $N = 1$ case so that the vielbeins of $\mathcal{M}^{(vec)}$ are simply $dz^i, d\bar{z}^{\bar{i}}$ we have to perform the following substitutions:

$$\begin{aligned} P^{IA}_u dq^u &\to P^{I\,BA}_i dz^i = -\epsilon^{AB} dz^i \\ P^{I^\star A}_i dq^u &\to P^{I^\star\,BA}_{i^\star} d\bar{z}^{i^\star} = -\epsilon^{AB} d\bar{z}^{i^\star} \end{aligned} \tag{4.3}$$

In particular, the general objects $f^{\Lambda[AB]}$, $f^{\Lambda IA}$ introduced in equation (2.9) become in our case:

$$f^{\Lambda[AB]} = \epsilon^{AB}\overline{L}^{\Lambda}\,; \qquad f^{\Lambda A}_I \to f^{\Lambda A}_{IB} = \delta^A_B\, \nabla_{i^\star} \bar{L}^{\Lambda}\,, \tag{4.4}$$

where $L^{\Lambda}(z, \bar{z})$ and its "magnetic" counterpart $M_{\Lambda}(z, \bar{z}) = \mathcal{N}_{\Lambda\Sigma}\, L^{\Sigma}$ actually form a $2n_V$ dimensional covariantly holomorphic section $V = (L^{\Lambda}, M_{\Lambda})$ of a flat symplectic bundle.

When the index I runs over the hypermultiplets we will rename them as follows: $(I, J) \to (\alpha, \beta)$ and since there are no vectors in the hypermultiplets we have $f^{\Lambda A}_{\alpha} = 0$

The vielbeins of the quaternionic manifold $\mathcal{M}^{(hyper)}$ will be denoted by $\mathcal{U}^{\alpha A} \equiv \mathcal{U}^{\alpha A}_u dq^u$ where $\alpha = 1, \ldots, 2n_H$ is an index labelling the fundamental representation of $\mathrm{Sp}(2n_H)$. The inverse matrix vielbein is $\mathcal{U}^u_{\alpha A}$. We raise and lower the indices $\alpha, \beta, \ldots$ and $A, B, \ldots$ with the symplectic matrices $C_{\alpha\beta}$ and ϵ_{AB} according to the following conventions

$$\begin{aligned} \epsilon^{AB}\,\epsilon_{BC} &= -\delta^A_C\,; & \epsilon^{AB} &= -\epsilon^{BA} \\ \mathbb{C}^{\alpha\beta}\,\mathbb{C}_{\beta\gamma} &= -\,\delta^{\alpha}_{\gamma}\,; & \mathbb{C}^{\alpha\beta} &= -\mathbb{C}^{\beta\alpha}\,. \end{aligned} \tag{4.5}$$

For any SU(2) vector P_A we have:

$$\epsilon_{AB}\, P^B = P_A\,; \qquad \epsilon^{AB}\, P_B = -P^A \tag{4.6}$$

and equivalently for Sp($2n$) vectors P_α:

$$\mathbb{C}_{\alpha\beta}\, P^\beta = P_\alpha\,; \qquad \mathbb{C}^{\alpha\beta}\, P_\beta = -P^\alpha\,. \tag{4.7}$$

Since we have a product manifold the generic Killing vector k^i_Λ splits into

$$k^i_\Lambda \to (k^i_\Lambda,\, k^{\bar{i}}_\Lambda);\, k^u_\Lambda \tag{4.8}$$

and they can be determined in terms of the prepotentials of special Kähler and Quaternionic geometry as follows:

$$\begin{aligned} k^i_\Lambda(z) &= \mathrm{i} g^{ij^\star}\partial_{\bar{j}}\, P_\Lambda(z,\,\bar{z}) \\ 2k^u_\Lambda\, \Omega^x_{uv} &= -\nabla_v\, P^x_\Lambda(q)\,, \end{aligned} \tag{4.9}$$

where Ω^x_{uv} are the SU(2)-valued components of the quaternionic curvature strictly related to the three complex structures existing on a quaternionic manifold. Note that as a consequence of quaternionic geometry (see appendix) the quaternionic prepotentials satisfy the following "harmonic" equation:[3].

$$\nabla_u\, \nabla^u P^x_\Lambda \;=\; -4n_H\, P^x_\Lambda\,. \tag{4.10}$$

For the purpose of making the paper self contained we report now the lagrangian and the transformation laws of the $N=2$ lagrangian as given in reference [31].We limit ourselves to report the lagrangian up to 4-fermion terms and the supersymmetry transformation laws up to 3-fermion terms since this is sufficient for our treatment. We have:

N=2 Supergravity lagrangian.

$$\begin{aligned} (\mathrm{det}V)^{-1}\,\mathcal{L} = &-\frac{1}{2}R + g_{ij^\star}\nabla^\mu z^i\nabla_\mu \bar{z}^{j^\star} + h_{uv}\nabla_\mu q^u\nabla^\mu q^v + \\ &+\frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\left(\bar{\psi}^A_\mu\gamma_\sigma\rho_{A\nu\lambda} - \bar{\psi}_{A\mu}\gamma_\sigma\rho^A_{\nu\lambda}\right) - \\ &-\frac{\mathrm{i}}{2}g_{ij^\star}\left(\bar{\lambda}^{iA}\gamma^\mu\nabla_\mu\lambda^{j^\star}_A + \bar{\lambda}^{j^\star}_A\gamma^\mu\nabla_\mu\lambda^{iA}\right) - \mathrm{i}\left(\bar{\zeta}^\alpha\gamma^\mu\nabla_\mu\zeta_\alpha + \bar{\zeta}_\alpha\gamma^\mu\nabla_\mu\zeta^\alpha\right) + \\ &+\mathrm{i}\left(\bar{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{F}^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\Sigma\mu\nu}\right) + \Big\{ - g_{ij^\star}\nabla_\mu\bar{z}^{j^\star}\,\bar{\psi}^\mu_A\lambda^{iA} - \\ &-2\mathcal{U}^{A\alpha}_u\nabla_\mu q^u\bar{\psi}^\mu_A\zeta_\alpha + g_{ij^\star}\nabla_\mu\bar{z}^{j^\star}\,\bar{\lambda}^{iA}\gamma^{\mu\nu}\psi_{A\nu} + 2\mathcal{U}^{\alpha A}_u\nabla_\mu q^u\bar{\zeta}_\alpha\gamma^{\mu\nu}\psi_{A\nu} + \\ &+\mathrm{h.c.}\Big\} + \Big\{\mathcal{F}^{-\Lambda}_{\mu\nu}\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\Big[4L^\Sigma\bar{\psi}^{A\mu}\psi^{B\nu}\epsilon_{AB} - 4\mathrm{i}\bar{f}^\Sigma_{i^\star}\bar{\lambda}^{i^\star}_A\gamma^\nu\psi^\mu_B\epsilon^{AB} \\ &+\frac{1}{2}\nabla_i f^\Sigma_j\bar{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB}\epsilon_{AB} - L^\Sigma\bar{\zeta}_\alpha\gamma^{\mu\nu}\zeta_\beta C^{\alpha\beta}\Big] + \mathrm{h.c.}\Big\} + \\ &+g\Big[2S_{AB}\bar{\psi}^A_\mu\gamma^{\mu\nu}\psi^B_\nu + \mathrm{i}g_{ij^\star}W^{iAB}\bar{\lambda}^{j^\star}_A\gamma_\mu\psi^\mu_B + 2\mathrm{i}N^A_\alpha\bar{\zeta}^\alpha\gamma_\mu\psi^\mu_A + \\ &+\mathcal{M}^{\alpha\beta}\bar{\zeta}_\alpha\zeta_\beta + \mathcal{M}^\alpha{}_{iB}\bar{\zeta}_\alpha\lambda^{iB} + \mathcal{M}_{ij\,AB}\bar{\lambda}^{iA}\lambda^{jB} + \mathrm{h.c.}\Big] - \mathcal{V}(z,\bar{z},q)\,, \end{aligned} \tag{4.11}$$

[3]Here and in the following we have set $\lambda = 1$ where λ is the scale (defined in the appendix) of $\mathcal{M}^{(hyper)}$. This is required by four dimensional supersymmetry of the lagrangian (see [34, 31])

where we have set $\mathcal{F}^{\pm\Lambda}_{\mu\nu} = \frac{1}{2}(\mathcal{F}^{\Lambda}_{\mu\nu} \pm \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda}_{\rho\sigma})$, $\mathcal{F}^{\Lambda}_{\mu\nu}$ being the field-strengths of the vectors A^{Λ}_{μ}. Furthermore $L^{\Lambda}(z,\,\bar{z})$ are the covariantly holomorphic sections of the special Geometry, $f^{\Lambda}_i \equiv \nabla_i L^{\Lambda}$ and the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ is constructed in terms of L^{Λ} and its magnetic dual according to reference [31]. The normalization of the kinetic term for the quaternions depends on the scale λ of the quaternionic manifold for which we have chosen the value $\lambda = -1$,(see footnote 3). Finally the mass matrices of the spin 1/2 fermions $\mathcal{M}^{\alpha\beta}$, $\mathcal{M}_{AB\,ij}$, $\mathcal{M}^{\alpha}_{iA}$ (and their hermitian conjugates) and the scalar potential $\mathcal{V}$ they are given by:[4]

$$\mathcal{M}^{\alpha\beta} = -\mathcal{U}^{\alpha A}_{u}\mathcal{U}^{\beta B}_{v}\,\varepsilon_{AB}\,\nabla^{[u}k^{v]}_{\Lambda}\,L^{\Lambda} \tag{4.12}$$

$$\mathcal{M}^{\alpha}{}_{iB} = -4\,\mathcal{U}^{\alpha}_{Bu}\,k^{u}_{\Lambda}\,f^{\Lambda}_{i} \tag{4.13}$$

$$\mathcal{M}_{AB\;ik} = \epsilon_{AB}\,g_{l^\star[i}f^{\Lambda}_{k]}k^{l^\star}_{\Lambda} - \frac{1}{2}\mathrm{i}P_{\Lambda AB}\,\nabla_i f^{\Lambda}_{k} \tag{4.14}$$

$$\mathcal{V}(z,\bar{z},q) = g^2\Big[\Big(g_{ij^\star}k^{i}_{\Lambda}k^{j^\star}_{\Sigma} + 4h_{uv}k^{u}_{\Lambda}k^{v}_{\Sigma}\Big)\bar{L}^{\Lambda}L^{\Sigma} + g^{ij^\star}f^{\Lambda}_{i}f^{\Sigma}_{j^\star}\mathcal{P}^{x}_{\Lambda}\mathcal{P}^{x}_{\Sigma} - 3\bar{L}^{\Lambda}L^{\Sigma}\mathcal{P}^{x}_{\Lambda}\mathcal{P}^{x}_{\Sigma}\Big]\,. \tag{4.15}$$

The supersymmetry transformation laws leaving invariant (4.11) are:

Supergravity transformation rules of the (left-handed) Fermi fields:

$$\delta\psi_{A\mu} = \mathcal{D}_{\mu}\,\epsilon_A + \left(\mathrm{i}gS_{AB}\eta_{\mu\nu} + \epsilon_{AB}T^{-}_{\mu\nu}\right)\gamma^{\nu}\epsilon^{B} \tag{4.16}$$

$$\delta\lambda^{iA} = \mathrm{i}\nabla_{\mu}\,z^i\gamma^{\mu}\epsilon^{A} + G^{-i}_{\mu\nu}\gamma^{\mu\nu}\epsilon_B\epsilon^{AB} + gW^{iAB}\epsilon_B \tag{4.17}$$

$$\delta\zeta_{\alpha} = \mathrm{i}\,\mathcal{U}^{B\beta}_{u}\nabla_{\mu}q^{u}\gamma^{\mu}\epsilon^{A}\epsilon_{AB}\,C_{\alpha\beta} + gN^{A}_{\alpha}\,\epsilon_A \tag{4.18}$$

where $T^{-}_{\mu\nu} = 2\mathrm{i}\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}L^{\Sigma}F^{\Lambda-}_{\mu\nu}$ and $G^{i-}_{\mu\nu} = -g^{ij^\star}\bar{f}^{\Gamma}_{j^\star}\,\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}F^{\Lambda-}_{\mu\nu}$.

Supergravity transformation rules of the Bose fields:

$$\delta\,V^{a}_{\mu} = -\mathrm{i}\,\bar{\psi}_{A\mu}\,\gamma^{a}\,\epsilon^{A} - \mathrm{i}\,\bar{\psi}^{A}_{\mu}\,\gamma^{a}\,\epsilon_{A} \tag{4.19}$$

$$\begin{aligned}\delta\,A^{\Lambda}_{\mu} &= 2\bar{L}^{\Lambda}\bar{\psi}_{A\mu}\epsilon_B\epsilon^{AB} + 2L^{\Lambda}\bar{\psi}^{A}_{\mu}\epsilon^{B}\epsilon_{AB} + \\ &\quad +\mathrm{i}\,f^{\Lambda}_{i}\,\bar{\lambda}^{iA}\gamma_{\mu}\epsilon^{B}\,\epsilon_{AB} + \mathrm{i}\,\bar{f}^{\Lambda}_{i^\star}\,\bar{\lambda}^{i^\star}_{A}\gamma_{\mu}\epsilon_B\,\epsilon^{AB}\end{aligned} \tag{4.20}$$

$$\delta\,z^{i} = \bar{\lambda}^{iA}\epsilon_A \tag{4.21}$$

$$\delta\,z^{i^\star} = \bar{\lambda}^{i^\star}_{A}\epsilon^{A} \tag{4.22}$$

$$\delta\,q^{u} = \mathcal{U}^{u}_{\alpha A}\left(\bar{\zeta}^{\alpha}\epsilon^{A} + C^{\alpha\beta}\epsilon^{AB}\bar{\zeta}_{\beta}\epsilon_B\right). \tag{4.23}$$

The gauge shifts for the three kinds of fermions, gravitinos $\psi_{A\mu}$,(A=1,2), gauginos $\lambda^{iA}\,(i=1,\ldots n_V)$ and hyperinos $\zeta^{\alpha}, (\alpha = 1,\ldots,n_H)$ appearing both in the lagrangian and in the supersymmetry transformation laws are given by [31]:

$$S_{AB} = \mathrm{i}\frac{1}{2}P_{AB\,\Lambda}\,L^{\Lambda} \equiv \mathrm{i}\frac{1}{2}P_{AB} \equiv \mathrm{i}\frac{1}{2}P^{x}\sigma^{x}_{AB} \tag{4.24}$$

$$W^{i\,AB} = \mathrm{i}P^{AB}_{\Lambda}\,g^{ij^\star}f^{\Lambda}_{j^\star} + \epsilon^{AB}k^{i}_{\Lambda}\overline{L}^{\Lambda} \equiv \mathrm{i}\nabla^{i}P^{AB} + \epsilon^{AB}k^{i} \tag{4.25}$$

$$N^{A}_{\alpha} = 2\,\mathcal{U}^{A}_{\alpha u}\,k^{u}_{\Lambda}\overline{L}^{\Lambda} \equiv 2\,\mathcal{U}^{A}_{\alpha u}\,k^{u}\,, \tag{4.26}$$

[4] There are misprints in the equation (4.14) as given in reference [31] which have been corrected.

where:[5]

$$P_{AB\Lambda}L^\Lambda \;=\; P^x_\Lambda L^\Lambda\,\sigma^x_{AB} \equiv P^x\,\sigma^x_{AB}\,, \qquad (x=1,2,3) \tag{4.27}$$
$$(P_{\Lambda AB}L^\Lambda)^\star \;=\; -P^{AB}_\Lambda \bar{L}^\Lambda \;=\; -P^x_\Lambda \bar{L}^\Lambda \sigma^{xAB} \equiv -P^x \sigma^{xAB} \tag{4.28}$$

and we have further defined

$$k^i_\Lambda \bar{L}^\Lambda \;=\; k^i \tag{4.29}$$
$$k^u_\Lambda \bar{L}^\Lambda = k^u \tag{4.30}$$
$$P_{AB\Lambda}\,L^\Lambda = P_{AB} \tag{4.31}$$
$$P^{AB}_\Lambda\, g^{ij^\star}\, f^\Lambda_{j^\star} = \nabla_{j^\star}\bar{L}^\Lambda P^{AB}_\Lambda\, g^{ij^\star} = \nabla^i P^{AB}\,. \tag{4.32}$$

Taking into account the definitions (4.29), (4.30) we see that the $\delta\lambda^{iA}$ and $\delta\chi_\alpha$ shifts are all covariant derivatives of the quaternionic and Kähler prepotentials:[6]

$$W^{i\,AB} = \mathrm{i}(\nabla^i P^{AB} + \epsilon^{AB}\nabla^i P_\Lambda \bar{L}^\Lambda) \tag{4.33}$$
$$N^A_\alpha = -\frac{1}{3}\mathcal{U}^A_{\alpha u}\,\Omega^{xuv}\,\nabla_v P^x \;=\; 2\,\mathcal{U}^A_{\alpha u}\,k^u_\Lambda\,. \tag{4.34}$$

The gradient flow equations (2.15)and (2.16) adapted to the present case are:

$$\nabla_k\, W^{i\,AB} = 2\,\delta^i_k\, S^{AB} - \epsilon^{AB}\, g^{ij^\star}\, k_{k\,\Lambda}\, f^\Lambda_{j^\star} \tag{4.35}$$
$$\nabla_{k^\star}\, W^{i\,AB} = -g^{ij^\star}\mathcal{M}^{AB}_{k^\star j^\star} + \frac{1}{2}\epsilon^{AB} g^{ij^\star}\left(f^\Lambda_{j^\star}k_{\Lambda k^\star} + f^\Lambda_{k^\star}k_{\Lambda j^\star}\right) \tag{4.36}$$
$$\nabla_u\, W^{j^\star}_{AB} = -\frac{1}{2} g^{ij^\star}\,\mathcal{M}^\alpha_{i(B}\mathcal{U}_{A)\alpha u} \tag{4.37}$$
$$\nabla_i N^\alpha_A = \frac{1}{2}\,\mathcal{M}^\alpha_{iA} \tag{4.38}$$
$$\mathcal{U}^{u\,B\alpha}\,\nabla_u\, N^{A\beta} = 4\,C^{\alpha\beta}\, S^{AB} + \epsilon^{AB}\,\mathcal{M}^{\alpha\,\beta} \tag{4.39}$$
$$\nabla_u S_{AB} = -\frac{1}{2}\mathcal{U}_{\alpha u(A}\,N^\alpha_{B)} \tag{4.40}$$
$$\nabla_i S_{AB} = \frac{1}{2}\,W^{j^\star}_{(AB)}\, g_{ij^\star} \tag{4.41}$$
$$\nabla_{i^\star}\, S_{AB} = 0\,, \tag{4.42}$$

where we have set $W^{j^\star}_{AB} \equiv (W^{i\,AB})^\star$ and $\mathcal{M}^{AB}_{k^\star j^\star} \equiv (\mathcal{M}_{AB\;kj})^\star$.

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[5] We use Pauli matrices with both indices in the upper or lower position so that they are symmetric.Note that $(\sigma^x_{AB})^\star = -\sigma^{x\,AB}$.

[6] Note that $\nabla^i P_\Lambda \bar{L}^\Lambda$ cannot be written as a total derivative since $P_\Lambda \bar{L}^\Lambda = 0$, so no gauge invariant prepotential exists for vector multiplets as in the $N=1$ case (see (3.14)). This should no come as a surprise since a prepotential having the interpretation of "superpotential" should be related to the gravitino mass and it is S_{AB} (quaternionic prepotential) and not the Hodge-Kähler prepotential P_Λ which enters in it. Indeed no gauge invariant scalar exists for vector multiplets that could enter in the spin 3/2 mass term.

Note that in the manipulations performed from (4.33) to (4.35) one has to use the gauged special geometry identities [34]:

$$P_\Lambda L^\Lambda = P_\Lambda \bar{L}^\Lambda = 0 \tag{4.43}$$

$$k^i_\Lambda L^\Lambda = k^{i\star}_\Lambda \bar{L}^\Lambda = 0\,. \tag{4.44}$$

We also note that $\mathcal{M}^{\alpha\beta}$ can be written in terms of the (traceless part of the) anticommutator of two covariant derivatives:[7]

$$\mathcal{M}_{\alpha\beta} = -\mathrm{i}\frac{1}{6}\mathcal{U}^u_{A(\alpha}\mathcal{U}^v_{\beta)B}\nabla_{(u}\nabla_{v)}P^{AB}\,. \tag{4.45}$$

This is a consequence of the basic relations of gauged quaternionic geometry given in appendix. Equation (4.45) is analogous to what happened for chiral multiplet in the $N = 1$ case, see equation (3.15). We also note that equation (4.39) is equivalent to equation (A.39) of the appendix where we put $\lambda = -1$.

Using the identity (2.11) adapted to the present case:

$$\delta^A_B\overline{V} = -12(S^{MA})^\star S_{MB} + g_{ij^\star}W^{iMA}W^{j^\star}_{MB} + 2N^A_\alpha N^\alpha_B \tag{4.46}$$

one may compute the scalar potential which turns out to be the one given in eq. (4.15) [34, 31]. Using equations (4.40), (4.41), (4.42) the scalar potential (4.15) can be also rewritten in the following way:

$$\mathcal{V} = -6S_{AB}(S^{AB})^\star + 2g^{ij^\star}\nabla_i S_{AB}\nabla_{j^\star}(S^{AB})^\star + 4\nabla_u S_{AB}\nabla^u(S^{AB})^\star + g_{ij^\star}k^i k^{j^\star}\,. \tag{4.47}$$

Finally we note that the last term in equations (4.33), (4.25) has a similar structure as the $N = 2$ central charge [47], but the SU(2) valued prepotential P^{AB} adds a symmetric part to $W^{i\,AB}$. However unbroken $N = 2$ supersymmetry is still controlled by a gradient flow equation [20, 25, 24] which is equivalent to $k^i = 0$ and to extremize in the moduli space P^x.[8] We note in particular that if $P^x = 0$ the supersymmetry flow has always vanishing potential (no AdS vacua). On the other hand if $k^i = 0$ the supersymmetry flow is controlled by the "superpotential" $P^x P^x$ whose extrema in the full moduli space (at $P^x \neq 0$) imply $\delta\lambda^{iA} = \delta\zeta_\alpha = 0$. For abelian gauging $k^i = k^i_\Lambda = 0$ and in absence of hypermultiplets, for constant P^x_Λ, we retrieve the situation discussed in the literarature [48, 15, 20, 24].

We note that the supersymmetric flow of the hypermultiplets (at points where the scalars have vanishing velocity) implies a vanishing value of the Killing vectors: $k^u_\Lambda L^\Lambda = 0$. Since the covariant holomorphic section $L^\Lambda = L^\Lambda(z,\bar{z})$ is complex, this

[7] Note that in eq. (4.45) the symmetric part of the anticommutator is automatically traceless because $\mathcal{U}^{u(A}_\alpha\mathcal{U}^{B)v}_\beta h_{uv} = 0$

[8] These conditions are generally too restrictive if only $N = 1$ supersymmetry is preserved. For instance, in the example of ref. [4] N^A_α does not vanish.

implies that the values $q^u = q^u_0$ for which $k^u_\Lambda L^\Lambda = 0$ are the fixed points[9] of the group generated by the two (real) Lie algebra elements $k^u_\Lambda \operatorname{Im} L^\Lambda, k^u_\Lambda \operatorname{Re} L^\Lambda$. This group depends on $\operatorname{Im} L^\Lambda$ and $\operatorname{Re} L^\Lambda$. If

$$f^\Delta{}_{\Lambda\Sigma} \operatorname{Im} L^\Lambda \operatorname{Re} L^\Sigma = 0 \tag{4.48}$$

the two-dimensional gauge group is abelian. If either $\operatorname{Im} L^\Lambda$ or $\operatorname{Re} L^\Lambda$ is zero then we have a one-dimensional gauge group; otherwise it may be any subgroup generated by the two elements. Of course for abelian isometries $f^\Delta{}_{\Lambda\Sigma} = 0$ and the group is always a two-dimensional subgroup of the gauge group. It is interesting to observe that in five dimensions the corresponding section L^Λ is real so that the group generated by $k^u_\Lambda L^\Lambda$ is always a one-dimensional subgroup of the isometry group. Furthemore we note that the condition $k^u_\Lambda L^\Lambda = 0$, taking into account the equation (A.32),implies the following consistency condition on the quaternionic prepotentials:

$$-\lambda\, \varepsilon^{xyz}\, P^y_\Lambda\, P^z_\Sigma\, L^\Lambda \bar{L}^\Sigma = f^\Delta{}_{\Lambda\Sigma}\, P^x_\Delta\, L^\Lambda \bar{L}^\Sigma \tag{4.49}$$

which in the abelian case reduces to

$$\varepsilon^{xyz}\, P^y_\Lambda\, P^z_\Sigma\, L^\Lambda \bar{L}^\Sigma = 0\,. \tag{4.50}$$

Defining $P^x(L) = P^x_\Lambda L^\Lambda$ (and setting $\lambda = -1$), equation (4.49) can be rewritten in the suggestive form:

$$\overrightarrow{P}(L) \times \overrightarrow{P}(\bar{L}) = \overrightarrow{P}(L \times \bar{L})\,, \tag{4.51}$$

where

$$L \times \bar{L} \equiv f^\Delta{}_{\Lambda\Sigma} L^\Lambda \bar{L}^\Sigma\,. \tag{4.52}$$

5. Dual quaternionic manifolds and the gauging of their isometries

A particular interesting case where some "universal gauging" can be studied in a fairly general way is the special situation when the hypermultiplet manifold of quaternionic dimension $n+1$ is obtained by c-map [45] from a special Kähler manifold of complex dimensions n.

These manifolds, called dual quaternionic manifolds in reference [45], have a "solvable group of motion" whose Solvable Lie Algebra has dimension $2n+4$ [49]. This solvable Lie Algebra is associated to the rank one coset $\mathrm{SU}(1,n+2)/\,\mathrm{SU}(n+2)\otimes \mathrm{U}(1)$ and contains, as particular case, the "universal hypermultiplet" parametrizing $\mathrm{SU}(1,2)/\,\mathrm{U}(2)$ [45]. These symmetries are always present even if the special Kähler manifold has no isometries at all.

[9] The same is true for the special Kähler manifolds in the case of non abelian isometries.

In Calabi-Yau compactifications for type-IIB strings down to $D = 4$, $n = h_{1,1}$ and the solvable algebra of rank one is related to $2h_{1,1} + 2$ shift- symmetries of the RR-scalars, one shift symmetry of the NS-axion (dual to $b_{\mu\nu}$) making the $2h_{1,1} + 3$ nilpotent part of the group. The remaining Cartan generator is related to the scale symmetry of the dilaton. The maximal abelian ideal has dimension $h_{1,1}+2$ (of which $h_{1,1} + 1$ are RR abelian shifts).

In the case of the universal hypermultiplet the nilpotent subalgebra is three-dimensional, it is the Heisenberg algebra considerd in reference [46], where also the discrete remnant, after brane instanton corrections to Hypergeometry, was considered.

For a "dual quaternionic manifold" one can then always gauge the solvable group (non-abelian gauging) or restrict to the abelian gauging of its "maximal abelian ideal" of dimension $n + 2$. To achieve this gauging one must at least have $2n + 3$ vector multiplets ($n + 1$ in the abelian case).[10]

It is interesting that all the existing examples of gauging are particular cases of this general framework.

The gauging of the two shift-symmetries of the universal multiplet was considered in references [10, 6, 7] and correspond to the $n = 0$ case; it is obtained by turning on the H-fluxes of the two field-strengths of the NS and RR two-forms on a Calabi-Yau threefold. This case requires $h_{2,1} \geq 2$.

Another case considered in the literature is the case when the maximal compact subgroup of the isometry group is gauged. In order this to be the case we may consider dual quaternionic spaces which are coset spaces, in which case also the special Kähler manifold is a coset. The most general abelian compact gauging is obtained by gauging the Cartan subalgebra of the maximal compact subgroup. For the unitary series this has dimension $n + 2$ and for $n = 0$ reduces to the gauging of $\mathrm{U}(1)^2$ of the universal multiplet considered in reference [50].

For the $G_2/\,\mathrm{SO}(4)$ manifold dual to $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ special Kähler manifold, the gauging of the isometries was considered in reference [51].

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[10]For Calabi-Yau compactifications this would imply that $h_{2,1} - h_{1,1} = \frac{1}{2}\chi \geq 1$ in type IIB ($\chi \to -\chi$ in type IIA).

A. Glossary of quaternionic geometry

Both a quaternionic or a hyper-Kähler manifold $\mathcal{M}^{(hyper)}$ are $4n$-dimensional real manifolds endowed with a metric h:

$$ds^2 = h_{uv}(q) dq^u \otimes dq^v\,; \qquad u, v = 1, \ldots, 4n_H \tag{A.1}$$

and three complex structures

$$(J^x)\ :\ \ T(\mathcal{HM}) \longrightarrow T(\mathcal{HM}) \qquad (x = 1, 2, 3) \tag{A.2}$$

that satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy}\, \mathbb{1} + \epsilon^{xyz} J^z \tag{A.3}$$

and respect to which the metric is hermitian:

$$\forall \mathbf{X}, \mathbf{Y} \in T\mathcal{HM}\ :\quad h\left(J^x \mathbf{X}, J^x \mathbf{Y}\right) = h\left(\mathbf{X}, \mathbf{Y}\right) \qquad (x = 1, 2, 3)\,. \tag{A.4}$$

From eqs. (A.3), (A.4) it follows that one can introduce a triplet of 2-forms

$$K^x = K^x_{uv} dq^u \wedge dq^v; K^x_{uv} = h_{uw} (J^x)^w_v \tag{A.5}$$

that provides the generalization of the concept of Kähler form occurring in the complex case. The triplet K^x is named the *hyper-Kähler* form. It is an SU(2) Lie-algebra valued 2-form in the same way as the Kähler form is a U(1) Lie-algebra valued 2-form. In $N = 1$ supersymmetry there is a single complex structure and the scalar manifold has a Kähler structure implying that the Kähler 2-form is closed. If supersymmetry is local the Kähler 2-form can be identified with the curvature of the U(1) line-bundle and in this case the manifold is called a Hodge-Kähler manifold, while for rigid supersymmetry the line bundle is flat. Similar steps can be also taken here and lead to two possibilities: either hyper-Kähler or Quaternionic manifolds.

Let us introduce a principal SU(2)-bundle over $\mathcal{M}^{(hyper)}$. Let ω^x denote a connection on such a bundle. To obtain either a hyper-Kähler or a Quaternionic manifold we must impose the condition that the hyper-Kähler 2-form K^x is covariantly closed with respect to the connection ω^x:

$$\nabla K^x \equiv dK^x + \epsilon^{xyz} \omega^y \wedge K^z = 0\,. \tag{A.6}$$

The only difference between the two kinds of geometries resides in the structure of the SU(2)-bundle.

A hyper-Kähler manifold is a $4n$-dimensional manifold with the structure described above and such that the SU(2)-bundle is flat.

Defining the SU(2)-curvature by:

$$\Omega^x \ \equiv\ d\omega^x + \frac{1}{2} \epsilon^{xyz} \omega^y \wedge \omega^z \tag{A.7}$$

in the hyper-Kähler case we have:

$$\Omega^x = 0\,. \tag{A.8}$$

Viceversa a Quaternionic manifold is a $4n$-dimensional manifold with the structure described above and such that the curvature of the SU(2)-bundle is proportional to the hyper-Kähler 2-form. Hence, in the quaternionic case we can write:

$$\Omega^x = \lambda\, K^x \tag{A.9}$$

where λ is a non vanishing real number which, as we shall see, sets the scale of the manifold $\mathcal{M}^{(hyper)}$.

As a consequence of the above structure the Quaternionic manifold has a holonomy group of the following type:

$$\begin{aligned} \mathrm{Hol}(\mathcal{M}^{(\mathrm{hyper})}) &= \mathrm{SU}(2) \otimes \mathcal{H} \quad (\text{quaternionic}) \\ \mathcal{H} &\subset \mathrm{Sp}(2n_H, \mathbb{R}) \end{aligned} \tag{A.10}$$

Introducing flat indices $\{A, B, C = 1, 2\}$ $\{\alpha, \beta, \gamma = 1, \ldots, 2n\}$ that run, respectively, in the fundamental representations of SU(2) and $\mathrm{Sp}(2m, \mathbb{R})$, we can find a vielbein 1-form

$$\mathcal{U}^{A\alpha} = \mathcal{U}^{A\alpha}_u(q) dq^u \tag{A.11}$$

such that

$$h_{uv} = \mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v \mathbb{C}_{\alpha\beta} \epsilon_{AB}\,, \tag{A.12}$$

where $C_{\alpha\beta} = -C_{\beta\alpha}$ and $\epsilon_{AB} = -\epsilon_{BA}$ are, respectively, the flat $\mathrm{Sp}(2n)$ and $\mathrm{Sp}(2) \sim \mathrm{SU}(2)$ invariant metrics. The vielbein $\mathcal{U}^{A\alpha}$ is covariantly closed with respect to the SU(2)-connection ω^z and to some $\mathrm{Sp}(2m, \mathbb{R})$-Lie Algebra valued connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$:

$$\begin{aligned} \nabla \mathcal{U}^{A\alpha} \equiv d\mathcal{U}^{A\alpha} &+ \frac{i}{2}\omega^x \sigma_x^{AB} \wedge \mathcal{U}^{\alpha}_B + \\ &+ \Delta^{\alpha\beta} \wedge \mathcal{U}^{A\gamma} \mathbb{C}_{\beta\gamma} = 0\,, \end{aligned} \tag{A.13}$$

where $(\sigma^x)^{AB} = \epsilon^{AB} (\sigma^x)_A{}^C$ and $(\sigma^x)_A{}^B$ are the standard Pauli matrices. Furthermore $\mathcal{U}^{A\alpha}$ satisfies the reality condition:

$$\mathcal{U}_{A\alpha} \equiv (\mathcal{U}^{A\alpha})^* = \epsilon_{AB} \mathbb{C}_{\alpha\beta} \mathcal{U}^{B\beta}\,. \tag{A.14}$$

More specifically we can write a stronger version of eq. (A.12) [32]:

$$(\mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v + \mathcal{U}^{A\alpha}_v \mathcal{U}^{B\beta}_u) C_{\alpha\beta} = h_{uv} \epsilon^{AB}\,. \tag{A.15}$$

The inverse vielbein $\mathcal{U}^u_{A\alpha}$ is defined by

$$\mathcal{U}^u_{A\alpha} \mathcal{U}^{A\alpha}_v = \delta^u_v\,. \tag{A.16}$$

Flattening a pair of indices of the Riemann tensor $\mathcal{R}^{uv}{}_{ts}$ we obtain

$$\mathcal{R}^{uv}{}_{ts}\mathcal{U}^{\alpha A}_u\mathcal{U}^{\beta B}_v = -\frac{\mathrm{i}}{2}\Omega^x_{ts}(\sigma_x)^{AB}C^{\alpha\beta} + \mathbb{R}^{\alpha\beta}_{ts}\epsilon^{AB}\,, \tag{A.17}$$

where $\mathbb{R}^{\alpha\beta}_{ts}$ is the field strength of the $\mathrm{Sp}(2n_H)$ connection:

$$d\Delta^{\alpha\beta} + \Delta^{\alpha\gamma}\wedge\Delta^{\delta\beta}\mathbb{C}_{\gamma\delta} \equiv \mathbb{R}^{\alpha\beta} = \mathbb{R}^{\alpha\beta}_{ts}dq^t\wedge dq^s \tag{A.18}$$

and

$$\mathbb{R}^{\alpha\beta}_{uv} = \frac{\lambda}{2}\epsilon_{AB}(\mathcal{U}^{A\alpha}_u\mathcal{U}^{B\beta}_v - \mathcal{U}^{A\alpha}_v\mathcal{U}^{B\beta}_u) + \mathcal{U}^{A\gamma}_u\mathcal{U}^{B\delta}_v\epsilon_{AB}\mathbb{C}^{\alpha\rho}\mathbb{C}^{\beta\sigma}\Omega_{\gamma\delta\rho\sigma}\,, \tag{A.19}$$

where $\Omega_{\gamma\delta\rho\sigma}$ is a completely symmetric tensor. The previous equations imply that the Quaternionic manifold is an Einstein space with Ricci tensor given by[11]

$$\mathcal{R}_{uv} = \lambda(2+n_H)h_{uv}\,. \tag{A.20}$$

Note that if the manifold is hyper-Kähler, that is if equation (A.9) holds, then $\lambda = 0$ and the manifold is Ricci flat. Eq. (A.19) is the explicit statement that the Levi Civita connection associated with the metric h has a holonomy group contained in $\mathrm{SU}(2)\otimes\mathrm{Sp}(2n_H)$. Consider now eqs. (A.3), (A.5) and (A.9). We easily deduce the following relation:

$$h^{st}K^x_{us}K^y_{tw} = -\delta^{xy}h_{uw} + \epsilon^{xyz}K^z_{uw} \tag{A.21}$$

that holds true both in the hyper-Kähler and in the Quaternionic case. In the latter case, using eq. (A.9), equation (A.21) can be rewritten as follows:

$$h^{st}\Omega^x_{us}\Omega^y_{tw} = -\lambda^2\delta^{xy}h_{uw} + \lambda\epsilon^{xyz}\Omega^z_{uw}\,. \tag{A.22}$$

In the quaternionic case we can write:

$$\Omega^x_{A\alpha,B\beta} \equiv \Omega^x_{uv}\mathcal{U}^u_{A\alpha}\mathcal{U}^v_{B\beta} = -i\lambda C_{\alpha\beta}(\sigma_x)_{AB}\,. \tag{A.23}$$

Alternatively eq. (A.23) can be rewritten in an intrinsic form as

$$\Omega^x = \mathrm{i}\,\lambda C_{\alpha\beta}(\sigma_x)_{AB}\mathcal{U}^{\alpha A}\wedge\mathcal{U}^{\beta B} \tag{A.24}$$

whence we also get:

$$\frac{i}{2}\Omega^x(\sigma_x)^{AB} = \lambda\mathcal{U}^{A\alpha}\wedge\mathcal{U}^B_\alpha\,. \tag{A.25}$$

There exist quaternionic manifolds which are homogeneous symmetric manifolds (a list of homogeneous symmetric quaternionic spaces are given in [31]).

[11] Our convention for the Riemann tensor are as follows: $R^u_v \equiv d\Gamma^u_v + \Gamma^u_w\wedge\Gamma^w_v = R^u_{vrs}dq^r\wedge dq^s$ where Γ is the Levi-Civita connection 1-form. Therefore the Ricci tensor is $R_{vs} = R^u_{vrs}\delta^r_u$

In full analogy with the case of Kähler manifolds, to each Killing vector we can associate a triplet $\mathcal{P}^x_\Lambda(q)$ of 0-form prepotentials. Indeed we can set:

$$\mathbf{i}_{\mathbf{k}_\Lambda}\Omega^x = -\nabla\mathcal{P}^x_\Lambda \equiv -(d\mathcal{P}^x_\Lambda + \epsilon^{xyz}\omega^y\mathcal{P}^z_\Lambda)\,, \tag{A.26}$$

where ∇ denotes the SU(2)- covariant exterior derivative and $\mathbf{i}_{\mathbf{k}_\Lambda}$ denotes the contraction of a form with the vector $\mathbf{k}_\Lambda$. Using components the previous equation takes the form:

$$2k^v_\Lambda\Omega^x_{uv} \;=\; \nabla_u P^x_\Lambda\,. \tag{A.27}$$

Formula (A.27) can be inverted to give the Killing vector in terms of the prepotential:

$$h_{uw}k^w_\Lambda \;=\; -\frac{1}{6\lambda^2}\Omega^x_{uv}\nabla^v P^x_\Lambda\,. \tag{A.28}$$

The three-holomorphic Poisson bracket is defined as follows:

$$\{\mathcal{P}_\Lambda,\mathcal{P}_\Sigma\}^x \equiv \mathbf{i}_{\mathbf{k}_\Lambda}\mathbf{i}_{\mathbf{k}_\Sigma}K^x - \lambda\,\varepsilon^{xyz}\,\mathcal{P}^y_\Lambda\,\mathcal{P}^z_\Sigma\,, \tag{A.29}$$

where

$$\frac{1}{2}\mathbf{i}_{\mathbf{k}_\Lambda}\mathbf{i}_{\mathbf{k}_\Sigma}K^x \equiv \lambda\Omega^x_{uv}\,k^u_\Lambda\,k^v_\Sigma \tag{A.30}$$

and leads to the poissonian realization of the Lie algebra

$$\{\mathcal{P}_\Lambda,\mathcal{P}_\Sigma\}^x \;=\; f^\Delta{}_{\Lambda\Sigma}\,\mathcal{P}^x_\Delta \tag{A.31}$$

which in components reads:

$$\lambda\Omega^x_{uv}\,k^u_\Lambda\,k^v_\Sigma \;-\; \frac{\lambda}{2}\,\varepsilon^{xyz}\,\mathcal{P}^y_\Lambda\,\mathcal{P}^z_\Sigma \;=\; \frac{1}{2}\,f^\Delta{}_{\Lambda\Sigma}\,\mathcal{P}^x_\Delta\,. \tag{A.32}$$

From the Killing equation

$$\nabla_u k_v \,+\, \nabla_v k_u = 0 \tag{A.33}$$

using

$$[\nabla_u,\nabla_v]\,k_{w\Lambda} = -2\mathcal{R}_{uvw}{}^l k_{l\Lambda} \tag{A.34}$$

and the value of the Ricci tensor (A.20) one easily finds that k_u is an eigenfunction of the (covariant) laplacian:

$$\nabla_v\nabla^v k_u \;-\; 2\lambda(2+n_H)k_u = 0\,. \tag{A.35}$$

Furthermore by double differentiation of P^{AB}_Λ, using the identity:

$$\mathcal{U}^A_{\alpha u}\mathcal{U}^{B\alpha}_v \;=\; \frac{\mathrm{i}}{2\lambda}\Omega^{AB}_{uv} \;-\; \frac{1}{2}\epsilon^{AB}h_{uv} \tag{A.36}$$

we find that also the prepotential is an eigenfunction of the covariant laplacian:

$$\nabla_v\nabla^v P^x_\Lambda \;-\; 4n_H\lambda P^x_\Lambda = 0\,. \tag{A.37}$$

JHEP05(2001)034

As a check, inserting equation (A.28) into (A.35) and commuting the covariant derivative with the laplacian using the rule:

$$[\nabla_u, \nabla_v]\, P^x = 2\epsilon^{xyz}\Omega^z_{uv}P^y \tag{A.38}$$

we find that (A.35) and (A.37) are indeed consistent.

Finally we note that since $\nabla_u k_v \equiv \nabla_{[u}k_{v]}$ is a 2-form we can expand it on a basis of 2-forms given by Ω^{AB}_{uv} and $\mathcal{U}^{\alpha A}_{[u}\,\mathcal{U}^{\beta B}_{v]}\,\epsilon_{AB}$ which is part of the symplectic curvature 2-form given in equation (A.19).

Indeed one can write:

$$\nabla_u k_v \;=\; \frac{1}{4\lambda}\Omega^x_{uv}P^x \;-\; \frac{1}{2}\mathcal{U}^{\alpha A}_{[u}\,\mathcal{U}^{\beta B}_{v]}\,\epsilon_{AB}\mathcal{M}_{\alpha\beta}\,, \tag{A.39}$$

where $\mathcal{M}_{\alpha\beta}$ is the hyperino mass matrix defined as the complex conjugate of eq. (4.12). Equation (A.39) can be easily proved to hold by inverting the 2-form $\mathcal{U}^{\alpha A}_{[u}\,\mathcal{U}^{\beta B}_{v]}\,\epsilon_{AB}$ on the r.h.s. of (A.39) and antisymmetrizing either in the SU(2) indices or in the symplectic indices.

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JHEP05(2001)034