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Remarks on time-space non-commutative field theories

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Abstract: We propose a physical interpretation of the perturbative breakdown of unitarity in time-like non-commutative field theories in terms of production of tachyonic particles. These particles may be viewed as a remnant of a continuous spectrum of undecoupled closed-string modes. In this way, we give a unified view of the string-theoretical and the field-theoretical no-go arguments against time-like non-commutative theories. We also perform a quantitative study of various locality and causality properties of non-commutative field theories at the quantum level.

Keywords: Non-Commutative Geometry, D-branes.

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1. Introduction

Quantum field theories in non-commutative spaces [1, 2] with non-commutative time coordinate are notoriously ill-defined. Heuristically, non-commutative particles of momentum p^μ can be regarded as rigid, extended dipoles oriented along the four-vector $L^\mu = \theta^{\mu\nu} p_\nu$, and interacting through the end-points [3]. It is already clear from this consideration that non-commutative time coordinates imply particles effectively "extended in time". Thus, the breakdown of naive criteria of local causality, in the form of "advanced effects" in tree-level scattering processes, should not come as a surprise (c.f. [4, 5]).

On a more technical level, these theories have no straightforward hamiltonian quantization (see however [6]) and are defined, in an operational sense, via the Feynman diagram expansion [7, 8]. It is thus necessary to check explicitly the unitarity of the theory. Indeed, in a theory with built-in nonlocality in time, it is hard to imagine an appropriate notion of causality which is at the same time useful and non trivial. It is more likely that these theories must be interpreted as S-matrix theories, in the same sense as critical string theory. Thus, from this very fundamental point of view, we may regard the existence of a consistent S-matrix as the weakest possible notion of causality.

More specifically, it was shown in [9, 10] that the stringy regularization of such systems is ill-defined. Following [11, 12], we can recover certain non-commutative field theories (NCFTs) as an appropriate low-energy limit of D-brane dynamics in the background of a constant electromagnetic field. While this limit is smooth in string perturbation theory for the case of a background magnetic field, it is problematic for a background electric field.[1] This is tied to the well-known instabilities of open-string dynamics in the presence of electric fields [14], i.e. beyond a maximal value of the electric field, the D-brane becomes effectively tachyonic. Since the time-noncommutativity is directly related to the electric field, and this must be large (in string units) in the low-energy limit, one finds that the low-energy limit always lies on the tachyonic regime of D-brane dynamics. Therefore, consistency of the string background requires the scale of time noncommutativity to be of the same order of magnitude as the string scale.

The breakdown of the stringy regularization for time-NCFT is a strong hint at the inconsistency of these models. Still, one can imagine some contrived analytic continuation of the background parameters, so that the open-string perturbation theory does converge in the formal low-energy limit to the series of Feynman diagrams of the time-NCFT. In particular, a formal continuation of the closed string coupling $g_s \to i g_s$ and an exchange of the roles of space and time in the plane of the electric field would do the job (c.f. [10]).

Therefore, it is desirable to find an internal inconsistency of the Feynman expansion in time-NCFTs. Such an inconsistency was found in [15], where a violation of the unitarity cutting rules was reported in a number of examples. The authors of [15] showed that scattering amplitudes have extra singularities that cannot be understood in terms of unitarity thresholds. In itself, this *does not* furnish a no-go argument for time-NCFT, since Feynman diagrams in ordinary theories are known to present the so-called "anomalous thresholds" whose interpretation in terms of unitarity rules is very problematic (see [16] for a review).

Thus, it is crucial to interpret physically the new singularities in order to evaluate the viability of time-NCFTs. One interesting possibility would be that NCFTs mimic open-string theory in the sense that unitarity of the S-matrix restricted to open-string initial and final states *requires* the introduction of closed strings as states in the asymptotic Hilbert space, since they contribute singularities in intermediate channels. In an analogous fashion, it is possible that the S-matrix of time-NCFT becomes unitary once we add appropriate new states to the asymptotic Hilbert space.

In fact, such a possibility is hinted at by the simplest example of a "non-commutative singularity", i.e. the case of the normal-ordering correction to the propagator of massless ϕ^4_* theory in four dimensions. In the commutative theory (or at the level of planar diagrams in the non-commutative theory) the normal-ordering diagram has no analytic structure, since it contributes a quadratically divergent constant renormalizing the mass. On the other hand, the non-planar diagram, viewed

[1] The marginal case of a null electromagnetic background $\mathbf{E}\cdot\mathbf{B} = \mathbf{E}^2 - \mathbf{B}^2 = 0$ is also smooth, c.f. [13].

as a one-to-one scattering amplitude, is given by

$$i\mathcal{A}(p) = -i\lambda \cdot \frac{1}{6} \int \frac{d^4q}{(2\pi)^4}\, e^{i\tilde{p}\,q}\, \frac{i}{q^2+i0} = -i\frac{\lambda}{24\pi^2}\frac{1}{p\circ p+i0}\,, \tag{1.1}$$

where $\tilde{p}^\mu \equiv p_\nu \theta^{\nu\mu}$ and $p\circ p \equiv -\tilde{p}^2$. Thus, we find a contribution to the imaginary part:

$$2\ \mathrm{Im}(\mathcal{A}) = \frac{\lambda}{12\pi}\,\delta(p\circ p)\,. \tag{1.2}$$

Let us now suppose that the initial energy positive, $p^0 > 0$, and that we have a single non-commutative plane $[x^0, x^1] = i\theta_e$. Then we can write the analog of the optical theorem for this quantity by introducing $1 = \int d^4k\, \delta(k-p)$ and obtain:

$$2\ \mathrm{Im}(\mathcal{A}) = \int \frac{d^3\mathbf{k}}{2(2\pi)^3|k_1|}\left(\frac{(2\pi)^2\lambda}{6\theta_e^2}\right)\delta(p-k)\,. \tag{1.3}$$

Thus, we see that the cutting rules can be formally recovered if we introduce new states $|\chi_k\rangle$ in the asymptotic Hilbert space with dispersion relation $k_0 = |k_1|$. They mix with the off-shell ϕ quanta with an effective coupling $\lambda_{\phi\chi} = \sqrt{4\pi^2\lambda/6\theta_e^2}$. All this is strongly reminiscent of the situation one finds in non-planar open-string scattering amplitudes. In that case, non-planar amplitudes show new poles, without a clear interpretation in terms of open-string intermediate states. It turns out that they just represent the amplitude for an open string to mix with closed strings. Therefore, in the particular case of the diagram considered here, the χ particles are analogous to closed strings. This analogy is sharpened by considering the "dipole" picture of non-commutative particles [3], since this is equivalent to *rigid* open strings. Then, the new states arise from "cutting the dipole" in the intermediate loop.

Notice that the effective coupling for producing the χ particles blows-up in the limit $\theta_e \to 0$, reflecting non-analiticity in the commutative limit — a typical UV/IR mixing. Another peculiar property of the χ particles is their lack of propagation in commutative spatial directions. The χ particles could be regarded as excitations of "χ-fields", a generalization of those proposed in [17, 18] in order to reconcile the Wilsonian interpretation of the renormalization group and the IR divergences found in spatially non-commutative theories. The main difference is that these fields are associated to propagating particles rather than being Lagrange multipliers. Thus, they cannot be simply considered as formal devices but *must* be included in any attemp to consistently construct the time-NCFT.

In the following section, we study the general structure of the "non-commutative singularities" at one-loop and attemp to give an interpretation along these lines. Using results from [19], we will show that the unitarity-violating singularities in time-NCFT can be manipulated into a form that is strongly reminiscent of undecoupled closed-string modes. Thus, the cutting rules can be formally satisfied by adding an appropriate set extra asymptotic degrees of freedom. On the other hand, unitarity is not restored in a strict sense, because the extra states are necessarily tachyonic. Thus, time-NCFT appears to be perturbatively inconsistent, even if we try to add new degrees of freedom into the problem. Our discussion closes the conceptual gap between the string-theoretical arguments of [9, 10] and the field-theoretical arguments of [15].

In the last section of the paper we come back to the issue of unitarity versus causality. We study in detail some criteria for microscopic causality at the quantum level. These criteria are relevant in situations where some Lorentz subgroup involving boosts remains unbroken. In particular, we will expose the impact of the quantum UV/IR mixing on the locality properties of the theory.

2. Non-commutative singularities at one-loop order

For a general one-loop diagram with N vertices in d-dimensional ϕ^n_* theory we have

$$i\mathcal{A}_{\{p_a\}} = \frac{(-i\lambda)^N}{|\Gamma|} \int \frac{d^d q}{(2\pi)^d} \,[\text{Moyal}] \prod_{a=1}^{N} \frac{i}{(q+Q_a)^2 - m^2 + i0}\,, \tag{2.1}$$

where $|\Gamma|$ is the symmetry factor of the diagram, and $Q_a + q$ is the momentum running through the a'th propagator. There is an overall momentum conservation delta function $\delta(\sum_a p_a)$ that we omit in the following. The Moyal phase can be factorized into the overall phase of the diagram $\mathcal{W}_{\rm NC}$, depending only on the external momenta p_a, and the non-trivial phase depending on the loop momentum q, given by

$$\exp(ip_\mu \theta^{\mu\nu} q_\nu) = \exp(i\tilde{p}\cdot q)\,,$$

where p is the total momentum flowing through the "non-planar channel".

We can evaluate the loop-momentum integral by introducing Feynman parameters x_a in the usual fashion:

$$\begin{aligned} i\mathcal{A} &= \frac{\lambda^N (N-1)!}{|\Gamma|}\,\mathcal{W}_{\rm NC} \int [dx] \int \frac{d^d q}{(2\pi)^d} \frac{e^{i\tilde{p}\,q}}{\left[\sum_a x_a (q+Q_a)^2 - m^2 + i0\right]^N} \\ &= \frac{\lambda^N (N-1)!}{|\Gamma|}\,\mathcal{W}_{\rm NC}\, e^{-i\tilde{p}\cdot \sum x_a Q_a} \int [dx] \int \frac{d^d q}{(2\pi)^d} \frac{e^{i\tilde{p}\,q}}{(q^2 - M_x^2 + i0)^N}\,, \end{aligned} \tag{2.2}$$

where

$$\int [dx] \equiv \int_0^1 \prod_a dx_a\, \delta\left(\sum_a x_a - 1\right),$$

and we have defined an x_a-dependent effective mass

$$M_x^2 = m^2 - \sum x_a Q_a^2 + \left(\sum x_a Q_a\right)^2. \tag{2.3}$$

The resulting momentum integral can be evaluated exactly in terms of appropriate Bessel functions (c.f. for example [20]):

$$\mathcal{A} = 2\left(-\frac{\lambda}{2}\right)^N (2\pi)^{-d/2}\,\frac{\mathcal{W}_{\rm NC}}{|\Gamma|}\times$$
$$\times\int [dx]\, e^{-i\tilde{p}\cdot\sum x_a Q_a}\,\left(M_x^2\right)^{\frac{d}{2}-N}\,\frac{K_{\frac{d}{2}-N}\left[\sqrt{M_x^2\,(p\circ p+i0)}\right]}{\left[\sqrt{M_x^2\,(p\circ p+i0)}\right]^{\frac{d}{2}-N}}\,, \tag{2.4}$$

with the convention that all branch cuts are drawn along the negative real axis.

The most significant property of this representation is the occurrence of a generic branch-point singularity at $p\circ p = 0$. The noncommutativity matrix $\theta^{\mu\nu}$ can be invariantly characterized as space-like or "magnetic", light-like or "null" and time-like or "electric" [13]. In the first two cases, we have $p\circ p \geq 0$ for real momenta, and we can only access the singularity at $p\circ p = 0$, i.e. the so-called UV/IR singularities of [17].

On the other hand, in the "electric" case we can access the full branch cuts along $p\circ p \leq 0$ with real momenta in the physical region, i.e. these singularities resemble particle-production cuts that are *characteristic* of time-NCFT. However, the examples studied in [15] show that the singularities at $p\circ p < 0$ do not satisfy the standard cutting rules, i.e. they do not have the standard interpretation in terms of production of ϕ-field quanta.

Unitarity thresholds in Feynman diagrams are associated to particles in a number of internal lines going on-shell. If a Schwinger parameter t_a is introduced for each propagator:

$$\frac{i}{p^2-m^2+i0} = \int_0^\infty dt_a\, e^{it_a\,(p^2-m^2+i0)}\,,$$

normal thresholds of one-loop diagrams correspond to exactly two of the Schwinger integrals being dominated by the region $t_a\to\infty$. Alternatively, we can replace the set of t_a parameters by Feynman parameters x_a, plus a global Schwinger parameter t, defined by $t = x_a t_a$, with $x_a\in[0,1]$ and $\sum_a x_a = 1$. Then $t=\sum_a t_a$ and *any* cutting of the diagram produces a singularity associated to the limit $t\to\infty$. Therefore, the parametric representation of (2.4) with respect to the (t,x_a) variables is useful in disentangling the new "non-commutative singularities" from the usual ones associated with unitarity cuts.

Taking advantage of the analysis in ref. [19] we compactify one spatial direction on a circle of length $L = 2\pi R$, which we assume to be *commutative*. We expect this to produce an effective mass in the non-planar channel, reminiscent of massess of closed-string winding modes. This will also allow us to make more precise the interpretation of $\tilde{p}^2 = -p\circ p$ as an invariant mass-squared.

The effect of the compactification at the level of the previous diagram is simply to discretize the momenta in that direction $q_R = n/R$, leading to a measure

$$\frac{1}{L}\sum_{n\in\mathbb{Z}}\int \frac{d^{d-1}q}{(2\pi)^{d-1}}\,. \tag{2.5}$$

A convenient way of writing this measure uses the identity $\sum_n \delta(x-n) = \sum_\ell e^{2\pi i \ell x}$ to rewrite the diagram as

$$i\mathcal{A} = \frac{(-i\lambda)^N}{|\Gamma|}\,\mathcal{W}_{\rm NC}\sum_{\ell\in\mathbb{Z}}\int \frac{d^d q}{(2\pi)^d} e^{iq\cdot\tilde{p}_\ell}\prod_{a=1}^{N}\frac{i}{(q+Q_a)^2 - m^2 + i0}\,, \tag{2.6}$$

where $\tilde{p}_\ell$ is a shifted momentum given by

$$(\tilde{p}_\ell)^\mu = p_\nu \theta^{\nu\mu} + L\,\ell\,\delta^\mu_R\,.$$

A more technical motivation for introducing the compactification is apparent in (2.6). Namely if we concentrate on the $\ell \neq 0$ sectors, the finite size of the circle L acts as an ultraviolet cutoff for the diagram. This is very convenient, since we would like to disentangle the occurrence of singularities related to particle production, from those at $p\circ p = 0$, inherent to the UV/IR effects of the theory. By selecting the $\ell \neq 0$ sectors we can effectively do so.

We can now introduce Feynman and Schwinger parameters as above:

$$\begin{aligned} i\mathcal{A} = \frac{(-i\lambda)^N}{|\Gamma|}\,\mathcal{W}_{\rm NC}\sum_{\ell\in\mathbb{Z}}\int [dx]\int \frac{d^d q}{(2\pi)^d}\,\times \\ \times\int_0^\infty dt\, t^{N-1} e^{iq\cdot\tilde{p}_\ell + it\sum_a x_a[(q+Q_a)^2 - m^2 + i0]}\,. \end{aligned} \tag{2.7}$$

Next, we evaluate the gaussian integral over q and perform a modular transformation of the Schwinger parameter $s = 1/4t$ to obtain:

$$\begin{aligned} i\mathcal{A} = \frac{(-i\lambda/4)^N}{|\Gamma|}\,\frac{\mathcal{W}_{\rm NC}}{(i\pi)^{d/2}}\sum_{\ell}\int [dx]\, e^{-i\sum x_a Q_a\cdot\tilde{p}_\ell}\,\times \\ \times\int_0^\infty ds\, s^{-N-1+\frac{d}{2}}\, e^{-\frac{i}{4s}(M_x^2 - i0)}\, e^{-is(\tilde{p}^2 - L^2\ell^2)}\,, \end{aligned} \tag{2.8}$$

where we have used $(\tilde{p}_\ell)^2 = \tilde{p}^2 - L^2\ell^2$, with $(L\ell)^2$ playing the expected role of an effective "winding mass", with dimension length-squared. This expression may also be obtained directly from (2.4) using the integral representation of the Bessel function.

The singularity structure of this expression is not standard. As stressed before, normal one-loop thresholds must be associated with on-shell intermediate quanta of the ϕ field. However, these contributions come from the region of integration of large proper times $t\to\infty$ or, equivalently, short dual proper times $s\to 0$. On the other hand, the integral (2.8) shows singularity structure in the opposite limit: $s\to\infty$, which is the ultraviolet domain in terms of the original ϕ-field quanta.

The analytic structure of the amplitudes can be inferred from the representation (2.8). For ordinary theories, one defines the amplitude by analytic continuation from the euclidean region for all momentum invariants (c.f. [21]). In this domain $M_x^2 > 0$ and the proper-time integral admits a Wick rotation $s \to is$ that makes it convergent at $s = 0$. The same Wick rotation renders the integral convergent in the ultraviolet, $s \to \infty$, provided a convenient cutoff is in place (in our case, $\ell \neq 0$). The only novelty in the non-commutative case is the requirement of excluding the real branch cut $\tilde{p}^2 \geq (L\ell)^2$ from the ordinary euclidean domain, in order to keep the integral convergent in the $s \to \infty$ limit.

This argument shows that any singularity at $-\tilde{p}^2 = p \circ p > 0$ is of "ordinary type", since it is in fact due to the $s \to 0$ limit. Thus, "non-commutative singularities" appear as real branch cuts for $p \circ p \leq -(L\ell)^2$. Notice that the Wick rotation of the proper-time parameter s is equivalent the regularization of the the large-s oscillatory phase by adding a small *positive* imaginary part to the winding mass, i.e. $(L\ell)^2 \to (L\ell)^2 + i0$. Then the behaviour in the vicinity of the branch points is given by a series of terms of the form:

$$\left(p \circ p + (L\ell)^2 + i0\right)^{k+N-\frac{d}{2}} \left[\log(p \circ p + (L\ell)^2 + i0)\right], \tag{2.9}$$

where the integer k comes from the Taylor expansion of the phase containing M_x^2, and the logarithm is present whenever the number $k + N - d/2$ is a positive integer or zero. In this formula, all branch cuts are conventionally drawn along the negative real axis.

What is the physical interpretation of these branch cuts? Guided by the example of the normal-ordering diagram in the introduction, together with the dipole picture, we would expect to find a series of pole singularities corresponding to the exchange of an infinite tower of "winding" χ_ℓ particles of "mass" $|L\ell|$. These states would naturally descend from closed-string winding modes in the low-energy limit.

The general expressions just derived imply that this picture is too naive. Namely, the leading $s \to \infty$ singularity is a pole at $p \circ p = -(L\ell)^2$ only for very special diagrams with $N = \frac{d-2}{2}$. In general, it is a softer branch-point singularity, which would suggest a multi-particle threshold, rather than the single-particle exchange implied by the dipole picture.

One possible interpretation of the cut uses a trick developed in [18], based on the simple observation that a branch cut can be viewed as a higher-dimensional pole, i.e. the negative half-integer powers of s in (2.8) may be traded by a gaussian integral over $d_\perp = 2N + 2 - d$ "transverse" momenta $\mathbf{z}_\perp$. In this way, we can approximate the amplitude in the vicinity of the $s \to \infty$ singularities as:

$$\mathcal{A} \approx \mathcal{W}_{\rm NC} \sum_{\ell \in \mathbb{Z}} \int \frac{d^{d_\perp}\mathbf{z}_\perp}{(2\pi)^{d_\perp}} \frac{f(p, \mathbf{z}_\perp, \ell)}{p \circ p + (L\ell)^2 + \mathbf{z}_\perp^2 + i0}, \tag{2.10}$$

for an appropriate "coupling" function $f(p, \mathbf{z}_\perp, \ell)$ that should be related to the couplings of χ particles to the *in* and *out* states in the non-planar channel. This function has a complicated momentum dependence, which makes this interpretation rather cumbersome. In addition, the number of extra "transverse" dimensions $d_\perp = 2N + 2 - d$ is completely *ad hoc* and depending on the particular diagram we consider, unlike the true number of transverse dimensions to a D-brane.

It is perhaps more appropriate to interpret the structure of the singularity in terms of a continuous spectrum of χ particles, so that the amplitude is approximated by

$$\mathcal{A} \approx \mathcal{W}_{\rm NC} \sum_{\ell \in \mathbb{Z}} \int d\mu^2 \, \frac{\rho(p, \ell, \mu^2)}{p \circ p + (L\ell)^2 + \mu^2 + i0} \,, \tag{2.11}$$

for an appropriate "spectral density" $\rho(p, \ell, \mu^2)$ that should be roughly proportional to the product $\langle {\rm out}, p, \ell \,|\chi(\mu^2)\rangle\langle\chi(\mu^2) \,|\, {\rm in}, p, \ell\rangle$, and will have a complicated structure due to the breakdown of Lorentz invariance. Its explicit form can be worked out in particular examples from the general expression (2.4).

However, even if the spectral density $\rho(\mu^2)$ had the right properties to be consistent with unitarity, the required on-shell condition of the χ particles, $p \circ p + (L\ell)^2 + \mu^2 = 0$, would be inconsistent with a positive-energy asymptotic Fock space. Solving it in the simplified situation of two orthogonal "electric and magnetic" noncommutative planes:

$$\omega_\theta(p) = \sqrt{\mathbf{p}_e^2 - \frac{1}{\theta_e^2} \left(\theta_m^2 \mathbf{p}_m^2 + (L\ell)^2 + \mu^2\right)} \,. \tag{2.12}$$

We see that, in general, the χ particles have tachyonic excitations for non-vanishing "electric" noncommutativity $\theta_e \neq 0$. The massless dispersion relation of the single χ particle in the example of the introduction generalizes to a continuum of tachyons. In addition, as soon as $\theta_m \neq 0$, the energy squared at fixed "mass" is unbounded from below.

In our view, this is the ultimate reason for the breakdown of unitarity in time-NCFT. Namely, the new singularities really represent production of tachyonic states, even if the cutting rules could eventually be satisfied in a formal sense.

2.1 Connection with string theory

The upshot of the discussion in the previous section is that, although the interpretation of the extra singularities, present for $\theta_e \neq 0$, in terms of new "closed-string" particles is very suggestive and even precise in some simple examples, the general structure is rather involved. In addition, it is found that unitarity is not preserved even if the cutting rules are formally restored, because the added part of the asymptotic Hilbert space contains tachyonic states.

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Given the intuitive "dipole" picture and the occurrence of "winding-like masses" for the effective χ particles in compact space, it would be desirable to establish a link between these *ad hoc* degrees of freedom and true closed strings in a model that would arise from a low-energy limit of string theory. The main challenge for such a discussion would be obtaining the required continuous spectrum of the χ particles for each value of the "winding number" ℓ, as well as the explanation of the sick dispersion relation (2.12).

Recall that NCFT amplitudes, when written in terms of Feynman parameters x_a and Schwinger parameter t, descend directly from the string counterparts in the Seiberg-Witten (SW) limit. For example, non-planar annulus amplitudes are integrals over the annulus modular parameter τ, defined so that the length of the annulus is $2\pi\tau$, and Koba-Nielsen parameters $\nu_a \in [0, 2\pi\tau)$. It was shown in a series of papers [22] that the string amplitude descends in the SW limit to the parametric representation (2.7) of the set of low-energy diagrams associated to the given string diagram. In this process, Koba-Nielsen moduli map to Feynman parameters according to $\nu_a = 2\pi\tau x_a$ and the annulus modular parameter maps to the Schwinger parameter via $t = 2\pi\alpha'\tau$.

Under a modular transformation, we can write the string amplitude in the closed-string channel as an overlap of the closed string propagator between D-brane boundary states.

$$\sum_\Psi \langle B|\Psi\rangle \left\langle \Psi \right| \frac{g_s^2}{\sqrt{-\det(g)}} \, \frac{-i}{\frac{\alpha'}{2}\left(g^{AB}p_A p_B - M_\Psi^2\right)} \left| \Psi \right\rangle \langle \Psi|B'\rangle \,, \tag{2.13}$$

where Ψ runs over all closed-string oscillator states, plus the momentum variables $\mathbf{p}_\perp$ transverse to the D-brane, and the winding quantum numbers ℓ contributing to M_Ψ:

$$M_\Psi^2 = \left(\frac{L\ell}{2\pi\alpha'}\right)^2 + \frac{N_{\rm osc}}{\alpha'} \,.$$

In (2.13), the dependence of the boundary states on Koba-Nielsen moduli has been obviated. The factor of g_s^2 comes from the two boundaries of the cylinder and the determinant factor comes from the canonical normalization of closed-string states propagating in a ten-dimensional bulk with metric g_{AB}. We choose this metric to be unity in the space transverse to the D-brane. On the other hand, both g_s and the world-volume components of the closed-string metric are determined by formulas in [12]:

$$g^{\mu\nu} = \eta^{\mu\nu} - \frac{1}{(2\pi\alpha')^2}\,\theta^{\mu\alpha}\,\eta_{\alpha\beta}\,\theta^{\beta\nu} \,, \qquad \frac{g_s^2}{\sqrt{-\det(g)}} = G_s^2 \,, \tag{2.14}$$

where G_s is the effective string coupling, which in turn determines the NCFT coupling in the SW limit.

In a proper-time representation, we write

$$\frac{-i}{\frac{\alpha'}{2}\left(g^{AB}p_A p_B - M_\Psi^2\right)} = \int_0^\infty d\tau_2\, e^{-i\tau_2\,\Delta_\Psi}\,,$$

where the modular parameter of the cylinder is related to that of the annulus by $\tau_2 = \pi/\tau$. On the other hand, the dual proper time for the χ particles is $s = 1/4t = \tau_2/8\pi^2\alpha'$. Therefore, the SW limit, involving $\alpha' \to 0$ at fixed s, takes $\tau_2 \to 0$, which *is not* the region where the amplitude can be approximated by a few low-lying closed-string states. Rather, in this scaling the whole tower of closed-string oscillator states contributes coherently. Using eq. (2.14), the proper-time kernel can be written as:

$$\tau_2\Delta_\Psi = s\left[-p\circ p - (L\ell)^2 + (2\pi\alpha')^2\, p^2 - (2\pi\alpha')^2\left(\mathbf{p}_\perp^2 + \frac{N_{\rm osc}}{\alpha'}\right)\right], \tag{2.15}$$

which is precisely the expression found in the proper-time integral (2.8): $s(\tilde{p}^2 - (L\ell)^2) = -s(p\circ p + (L\ell)^2)$, when we send $\alpha' \to 0$ at fixed p and $\theta^{\mu\nu}$. The effective gap induced by oscillator states of closed string on the spectrum of the χ particles is of order

$$\frac{\text{Oscillator gap}}{|p\circ p|} \sim \frac{\alpha'}{|p\circ p|} \longrightarrow 0\,,$$

so that, in the SW limit, the closed-string oscillator spectrum becomes effectively continuous on the scale of the χ particles. It follows that we cannot simply set $\alpha' = 0$ in (2.15) because an infinite set of modes, labelled by $N_{\rm osc}$ and the transverse momentum to the D-brane, contribute on the length scale relevant to the χ particles. This is the interpretation of the integral over the continuous "mass parameter"

$$\mu^2 \longrightarrow (2\pi\alpha')^2\left(\mathbf{p}_\perp^2 + \frac{N_{\rm osc}}{\alpha'}\right)$$

in eq. (2.11). In a proper treatment one would write a sum rule for the effective coupling of the χ particles, as in [19], involving a trace over all the closed-string spectrum, and using the detailed string amplitude and the known low-energy reduction to the expression (2.8). This trace generates the extra powers of s and the detailed structure found in (2.8) that define the properties of the χ particles, including the effective transverse dimension $d_\perp = 2N + 2 - d$, which *is not* directly related to the true number of transverse dimensions of the D-brane.

According to this picture, the χ fields are effective formal devices that represent the coherent coupling of an infinite number of closed-string states. The fact that they still resemble ordinary fields in some respects is a non-trivial property of NCFT, having a degenerate version of the string-theory's channel duality.

This construction also explains the tachyonic character of the χ particles for $\theta_e \neq 0$. From the formula (2.14) for the closed-string metric, we see that $g^{\mu\nu}$ only degenerates at the NCOS boundary $|\theta_e| = 2\pi\alpha'$, [9, 10], which is the only point where we can argue for the full decoupling of closed strings (see however [23]). For $|\theta_e| > 2\pi\alpha'$, including in particular the SW limit defining the time-NCFT, the role of space and time in the timelike non-commutative plane is exchanged [10]. This is exactly what is required to match the "empirical" dispersion relations (2.12) introduced above for the χ particles.

Therefore, we conclude that the SW limit of strings with timelike noncommutativity *does not* decouple closed-string states, even in the low-energy limit, since they show up as production of tachyonic states in non-planar amplitudes. This closes the circle of arguments in favour of the inconsistency of these theories. In this sense, our arguments bridge the gap between the criteria of [9, 10] and the purely field-theoretical one of [15].

3. Tests of locality and causality at the quantum level

In ordinary local quantum field theory, various technical concepts such as analiticity, microscopic causality and unitarity are roughly interchangable, due to some underlying "physical" requirements, such as Lorentz invariance and locality. NCFT violates these two physical conditions in a peculiar way, keeping still a controlable (and interesting) structure. Thus, NCFT is a nice laboratory to disentangle the relationships between the technical criteria cited above.

In particular, the breakdown of analiticity is to be expected generically in the case of non-commutative time, since the Moyal phases in amplitudes are not analytic functions of the energy in the upper half plane. This is related to the observed violations of classical causality criteria in [4, 5], and also renders invalid most derivations of dispersion relations by means of Cauchy's theorem (except for the two-point function, since the global Moyal phase is trivial in this case). Thus, we observe a close parallel between the violations of analiticity and those of unitarity.

On the other hand, the standard criterion of microcausality (local observables commute outside the relative light-cone) is more intuitive than the "technical" criterion based on analiticity, but it is clearly tied to the underlying Lorentz invariance of the theory. In fact, it loses much of its significance in situations where the light-cone itself has no dynamical meaning. Thus, we expect the microcausality criterion to break down "trivially" already for purely spatial noncommutativity. Still, since free propagation is Lorentz invariant in NCFT, the breakdown occurs necessarily as a result of the interactions and, in view of the UV/IR effects, it is an interesting question to determine the size of the "causality violation" in perturbation theory.[2]

A related interesting question is the following. One can turn on spatial and/or time noncommutativity while still preserving a certain Lorentz "little group". For example, in four dimensions, the frame x^μ satisfying $[x^0, x^1] = i\theta_e,\ [x^2, x^3] = i\theta_m$

[2]For other related discussions in different contexts see [24].

preserves a subgroup $\mathrm{SO}(1,1) \times \mathrm{SO}(2)$ of the four-dimensional Lorentz group. Thus, boosts along the x^1 axis are still a symmetry even for $\theta_e \neq 0$, and we can define a "two-dimensional" light-cone by the equation $x_e^2 \equiv (x^0)^2 - (x^1)^2 = 0$. In this situation, the microcausality criterion with respect to the four-dimensional light-cone $x^2 = x_e^2 - x_m^2 = 0$ has no particular meaning. However, the same criterion referred to the two-dimensional light-cone $x_e^2 = 0$ *is* still meaningful.

To analyze the issue we would like to compute the perturbative corrections to the commutator function

$$C(x) = \langle 0| \, [\phi(x), \phi(0)] \, |0\rangle \,. \tag{3.1}$$

In fact, it is technically more convenient to consider the related function given by the difference of retarded and advanced commutators

$$\overline{C}(x) \equiv C(x)_R - C(x)_A = \mathrm{sign}(x^0) \, C(x) \,, \tag{3.2}$$

which in turn is related to the imaginary part of the Feynman propagator in position space:

$$\overline{C}(x) = G_F(x) - G_F(x)^* = 2i \, \mathrm{Im} \, G_F(x) \,. \tag{3.3}$$

We can then *define* microscopic causality by the requirement that Im $G_F(x)$ be supported "inside" the light cone, i.e. Im $G_F(x) \propto \theta(x^2)$. This is certainly satisfied at the level of free fields, since the bare propagator is θ-independent. The obvious advantage of this definition for our purposes is that it extends naturally to the $\theta_e \neq 0$ case, where a hamiltonian construction of the commutator from its definition (3.1) is absent. In this case, we only have the Feynman rules as an operational definition of the theory, and one can readily compute perturbative corrections to Im $G_F(x)$.

The dressed propagator takes the form

$$G_F(x) = \int \frac{d^4p}{(2\pi)^4} \, e^{-ipx} \, \frac{i}{p^2 - m^2 - \Sigma + i0} \,, \tag{3.4}$$

where $\Sigma(p_e^2, p_m^2)$ is the 1PI self-energy, a function of the two $\mathrm{SO}(1,1) \times \mathrm{SO}(2)$ invariants of the problem:

$$p_e^2 = p_0^2 - p_1^2 \,, \qquad p_m^2 = p_2^2 + p_3^2 \,. \tag{3.5}$$

In terms of these quantities we have

$$p^2 = p_e^2 - p_m^2 \,, \qquad p \circ p = \theta_e^2 p_e^2 + \theta_m^2 p_m^2 \,. \tag{3.6}$$

In (3.4), the definition of Σ is such that m^2 gives the "physical" mass after renormalization by the planar diagrams. Thus, Σ includes the finite part of planar diagrams and the contribution from non-planar diagrams, which breaks Lorentz invariance.

On general grounds, we expect the analytic structure of the self-energy to present the normal thresholds for $p^2 > 0$, and the "non-commutative thresholds" for $p \circ p < 0$. Namely, poles at real positive values of p^2 if the theory develops bound states, the usual multi-particle cuts for $p^2 \geq 4m^2$, and the non-commutative cuts for $p \circ p < 0$ in the case $\theta_e \neq 0$.

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In what follows, we will *assume* that these singularities in the real axis exhaust all the singularities of the self-energy function in the physical sheet as a function of p_e^2. This assumption is motivated by various considerations. The planar contribution to the self-energy is exactly equal to that in the commutative theory, the overall Moyal phase being trivial due to momentum conservation. In addition, the non-planar contributions have a better high-energy behaviour than the planar ones, and all examples considered show the non-planar singularities accumulating in the real $p \circ p \leq 0$ line. Finally, the intuition from string theory points in the same direction, since $\tilde{p}^2 = -p \circ p$ is nothing but the "Mandelstam variable" in the closed-string channel.

In performing the Fourier transform to compute $G_F(x)$ it is convenient to use the corresponding "polar coordinates" with respect to the SO(2) and SO(1,1) groups. Thus, for the "magnetic" $(x^2, x^3) = x_m$ plane we change variables from $p_m = (p_2, p_3)$ to the invariant p_m^2 and SO(2) angle, and we can write:

$$\int \frac{d^2 p_m}{(2\pi)^2} \, e^{i p_m x_m} \, f(p_m^2) = \frac{1}{4\pi} \int_0^\infty dt \, J_0\left(\sqrt{t x_m^2}\right) \, f(t) \,, \tag{3.7}$$

for a general function of the magnetic invariant $f(p_m^2)$. In this expression, J_0 stands for the zeroth-order Bessel function.

On the other hand, the polar decomposition in the "electric" plane parametrizes the momenta $p_e = (p_0, p_1)$ in terms of the invariant p_e^2 and the SO(1,1) rapidity. In this case the contributions to the integral from the different signs of p_e^2 must be considered separately. The complete expression also depends on the sign of x_e^2. Thus, for $x_e^2 > 0$ one finds:

$$\begin{aligned}\theta(x_e^2) \int \frac{d^2 p_e}{(2\pi)^2} \, e^{-i p_e x_e} \, f(p_e^2) = \frac{\theta(x_e^2)}{4\pi} \int_0^\infty ds \, \Bigg[\frac{2}{\pi} \, f(-s) \, K_0\left(\sqrt{s x_e^2}\right) - \\ - f(s) \, N_0\left(\sqrt{s x_e^2}\right) \Bigg] \,,\end{aligned} \tag{3.8}$$

where N_0 denotes the zeroth-order Neumann function and K_0 is the zeroth-order McDonald function. We may simplify this expression under the assumption that the otherwise arbitrary function $f(-s)$ admits a "Wick rotation" $\sqrt{s} \to -i\sqrt{s}$ in the evaluation of the integral. Under this analytic continuation, the Bessel function transforms

$$K_0\left(\sqrt{s x_e^2}\right) \longrightarrow K_0\left(-i\sqrt{s x_e^2}\right) = -\frac{\pi}{2} \left[N_0\left(\sqrt{s x_e^2}\right) - i J_0\left(\sqrt{s x_e^2}\right) \right] ,$$

and the complete integral simplifies, since the N_0 functions cancel out. The final reduction formula is

$$\theta(x_e^2) \int \frac{d^2 p_e}{(2\pi)^2} \, e^{-i p_e x_e} \, f(p_e^2) = \frac{\theta(x_e^2)}{4\pi i} \int_0^\infty ds \, f(s) \, J_0\left(\sqrt{s x_e^2}\right) . \tag{3.9}$$

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Entirely analogous manipulations give a similar reduction formula for $x_e^2 < 0$. In this case, the elimination of the Neumann functions requires the opposite Wick rotation $\sqrt{s} \to i\sqrt{s}$, so that $f(s) \to f(-s)$. The result is

$$\theta(-x_e^2) \int \frac{d^2 p_e}{(2\pi)^2}\, e^{-ip_e x_e}\, f(p_e^2) = -\frac{\theta(-x_e^2)}{4\pi i} \int_0^\infty ds\, f(-s)\, J_0\left(\sqrt{-s x_e^2}\right). \tag{3.10}$$

With these preliminaries, we are ready to write down a general formula for the modified commutator function, using the general expression for the dressed propagator in momentum space.

The explicit assumptions that we need for the analytic structure of the self-energy can be summarized by demanding that

$$\frac{1}{z - t - m^2 - \Sigma(z,t)} \tag{3.11}$$

has only singularities in the real axis, as a function of the complex variable z. These singularities include the usual poles and cuts for $p^2 > 0$, $p \circ p > 0$, as well as the "non-commutative singularities" at $p \circ p \leq 0$. In particular, these conditions can be explicitly checked for all the one-loop examples considered in the literature.

There is a component respecting SO$(1,1)$ microcausality given by:

$$\begin{aligned}\overline{C}(x)_{x_e^2>0} = \frac{i}{8\pi^2}\,\theta(x_e^2) \int_0^\infty dt\, J_0\left(\sqrt{t x_m^2}\right) \times \\ \times \int_0^\infty ds\ \mathrm{Im}\, \frac{J_0\left(\sqrt{s x_e^2}\right)}{s - t - m^2 - \Sigma(s,t) + i0}\,.\end{aligned} \tag{3.12}$$

In addition, the component that violates the SO$(1,1)$ microcausality is given by

$$\begin{aligned}\overline{C}(x)_{x_e^2<0} = \frac{i}{8\pi^2}\,\theta(-x_e^2) \int_0^\infty dt\, J_0\left(\sqrt{t x_m^2}\right) \times \\ \times \int_0^\infty ds\ \mathrm{Im}\, \frac{J_0\left(\sqrt{-s x_e^2}\right)}{s + t + m^2 + \Sigma(-s,t) - i0}\,.\end{aligned} \tag{3.13}$$

From these general expressions we see that, as expected, four-dimensional microcausality is violated as a result of the breakdown of Lorentz invariance. There is no particular structure depending on $x^2 = x_e^2 - x_m^2$. On the other hand, there is room for violations of the SO$(1,1)$ microcausality criterion coming from the term proportional to $\theta(-x_e^2)$.

In general, a non-vanishing imaginary part of $\Sigma(-s,t)$ in the previous expression implies a violation of two-dimensional microcausality. Since $p \circ p = \theta_e^2 p_e^2 + \theta_m^2 p_m^2$ in terms of two-dimensional momenta (3.5), this corresponds, after the analytic continuation that leads to (3.13), to having a contribution to the imaginary part coming from the cut starting at $p_e^2 = -\theta_m^2 t/\theta_e^2$ and extending along the negative real axis. In (3.13), $s = -p_e^2$.

Another source of SO(1, 1)-causality violations is the possibility of having zeroes in the denominator of the integrand in (3.13) on the real axis: $s+t+m^2+\Sigma(-s,t) = 0$. Viewed as a two-dimensional dispersion relation this corresponds to the presence of tachyon poles.

The general structure can be understood by recalling that, under our analyticity assumptions, the function in (3.11) admits a dispersion relation of the form

$$\frac{1}{p_e^2 - p_m^2 - \Sigma + i0} = \int_{-\infty}^{\infty} du\, \frac{\sigma(u, p_m^2)}{p_e^2 - u + i0}\,, \tag{3.14}$$

where the "spectral function"

$$\sigma(u, p_m^2) = (\text{pole terms}) - \frac{1}{\pi} \frac{\text{Im } \Sigma(u, p_m^2)}{(u - p_m^2 - \text{Re } \Sigma)^2 + (\text{Im } \Sigma)^2} \tag{3.15}$$

splits naturally in two pieces, $\sigma_\pm(\mu^2) = \sigma(\pm\mu^2)$, respectively associated to the normal and "non-commutative" thresholds.

Incidentally, we notice that $\sigma_+(\mu^2)$ has the interpretation of a honest spectral function in a Källen-Lehmann representation, with positivity ensured by the unitarity relation $\text{Im } \Sigma \leq 0$ at the "normal thresholds". However, a glance at the explicit examples below shows that $\text{Im } \Sigma$ has no definite sign for the "non-commutative thresholds", so that the spectral interpretation of $\sigma_-(\mu^2)$ is problematic.[3]

Taking the appropriate Fourier transforms we can give a two-dimensional "spectral representation" of the commutator function $\overline{C}(x) = \overline{C}_+(x) + \overline{C}_-(x)$, where:

$$\overline{C}_\pm(x) = \int_{\mu_\pm^2}^{\infty} d\mu^2\, g_\pm(x_m, \mu^2)\, \mathcal{C}_e(x_e)_{\pm\mu^2}\,, \tag{3.16}$$

with

$$g_\pm(x_m, \mu^2) = \int \frac{d^2 p_m}{(2\pi)^2}\, e^{ip_m x_m}\, \sigma_\pm(p_m^2, \mu^2)$$

and $\mathcal{C}_e(x_e)_{\pm\mu^2}$ denoting the two-dimensional *free* commutator functions with mass squared $\pm\mu^2$:

$$\mathcal{C}_e(x_e)_{\pm\mu^2} = \pm\frac{\theta(\pm x_e^2)}{2i}\, J_0\left(\sqrt{\pm\mu^2 x_e^2}\right). \tag{3.17}$$

Thus, we see that violations of two-dimensional microcausality are associated to the contribution of the "spectral function" σ_- in the tachyonic branch, i.e. the non-commutative thresholds for emission of χ particles.

[3]This particular case of (2.11) makes it explicit that the spectral interpretation of the function $\rho(p, \ell, \mu^2)$ appearing in that equation will necessarily involve *indefinite norm* in the asymptotic Hilbert space of the χ particles.

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This shows that our SO$(1,1)$-invariant causality criterion is equivalent to unitarity. In particular, NCFTs with space-like noncommutativity are "causal" by the two-dimensional criterion.

3.1 Some examples

It is instructive to check the previous general statements with some simple examples. We consider the one-loop normal-ordering correction in massless ϕ^4_* theory, with renormalization conditions so that the "physical" mass vanishes after planar renormalization. Then, the non-planar graph is given by

$$\begin{aligned}\Sigma(p_e^2, p_m^2) &= i\frac{\lambda}{6}\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot\tilde{p}}}{k^2+i0} = \frac{\lambda}{24\pi^2}\frac{1}{p\circ p+i0} \\ &= \frac{\lambda}{24\pi^2}\frac{1}{\theta_e^2 p_e^2+\theta_m^2 p_m^2+i0}\,. \end{aligned} \tag{3.18}$$

There are no finite contributions to Im $\Sigma(\pm s, t)$, and the only source for $\overline{C}(x)$ comes from the pole terms. In the case of purely spatial noncommutativity one finds

$$\overline{C}(x)_{\theta_e=0} = \frac{1}{8\pi i}\theta(x_e^2)\int_0^\infty dt\, J_0\left(\sqrt{t x_m^2}\right)\, J_0\left(\sqrt{x_e^2}\,\sqrt{t+g^2/t}\right), \tag{3.19}$$

where $g^2 \equiv \lambda/24\pi^2\theta_m^2$.

It is interesting to notice the $t \to g^2/t$ "duality" in the argument of the second Bessel function; a rather transparent manifestation of the UV/IR effects. One can also check, using identities of Bessel functions, that $\overline{C}(x) \propto \delta(x^2)$ for $g = 0$, i.e. the modified commutator at $g^2 = 0$ is supported on the four-dimensional light-cone. This corresponds either to the free-field case, $\lambda = 0$, or to the case of infinite noncommutativity, $\theta_m = \infty$.

As expected, the violations of microscopic causality for purely spatial noncommutativity are tied to the breaking of Lorentz invariance. Thus, $\overline{C}(x)$ is non zero outside the four-dimensional light-cone $x^2 = x_e^2 - x_m^2 \leq 0$. On the other hand, the two-dimensional microcausality is not violated, since $\overline{C}(x)$ vanishes for $x_e^2 < 0$. It is also interesting to notice that $\overline{C}(x_m, x_e^2 = 0) = 0$ for any value of x_m^2.

An asymptotic expansion of (3.19) for large x_m^2 at constant ratio $a = x_e^2/x_m^2 < 1$ may be obtained by a saddle-point approximation:

$$\overline{C}(x)_{\theta_e=0} \propto \frac{1}{a}\frac{g^{1/4}}{|x_m|^{3/2}} \sim \frac{1}{a}\frac{\lambda^{1/8}}{\theta_m^{1/4}|x_m|^{3/2}}\,,$$

modulated by an oscillatory phase of frequency $\sqrt{2gax_m^2}$. We see that the non-locality is of long range, presumably as a consequece of the UV/IR mixing.

In the case $\theta_e \neq 0$, the special choice $\theta_e = \theta_m \neq 0$ is convenient to simplify the analysis, while maintaining all the qualitative features intact. The contributions to $\overline{C}(x)$ are again of pole type. However, now there is also a term proportional to $\theta(-x_e^2)$:

$$\overline{C}(x) = \frac{1}{8\pi i}\int_0^\infty dt J_0\left(\sqrt{t x_m^2}\right)\left[\theta(x_e^2)\frac{t+m_t^2}{2m_t^2}J_0\left(m_t\sqrt{x_e^2}\right) + \right.$$
$$\left. +\,\theta(-x_e^2)\frac{t-m_t^2}{2m_t^2}J_0\left(m_t\sqrt{-x_e^2}\right)\right], \qquad (3.20)$$

where $m_t^2 \equiv \sqrt{t^2+g^2}$, and g is still given as above.

If we interpret the pole in two-dimensional terms, the term proportional to $\theta(-x_e^2)$ comes from a "tachyonic" excitation. One can calculate exactly the integral for $x_m^2 = 0$ in terms of Bessel and Thomson functions. The piece that violates SO$(1,1)$ causality yields

$$\theta(-x_e^2)\,\overline{C}(x)_{\theta_e=\theta_m} = \frac{i}{8\pi}\,g\,\left[\frac{J_1(\xi\sqrt{2})}{\xi\sqrt{2}} - \mathrm{kei}_1(\xi)\,\mathrm{bei}_1(\xi) + \mathrm{ber}_1(\xi)\,\mathrm{ker}_1(\xi)\right], \qquad (3.21)$$

where $\xi \equiv \sqrt{-g x_e^2/2}$. The leading asymptotic behaviour for $x_m^2 = 0$ and large $\sqrt{-x_e^2}$ is

$$\overline{C}(x)_{\theta_e=\theta_m} \propto \sqrt{\frac{g}{-x_e^2}} \sim \frac{\lambda^{1/4}}{\sqrt{-\theta_e\, x_e^2}}\,,$$

modulated by an oscillating phase of frequency $\sqrt{-g x_e^2}$, so that the violation of two-dimensional microcausality is also of long range in this case.

Another instructive case is that of two-dimensional massive scalars. Here, Lorentz invariance is never broken, but time is always non commutative. The general formula for the commutator function is derived along similar lines and one obtains

$$\overline{C}(x) = \frac{i}{2\pi}\int_0^\infty ds\left[\mathrm{Im}\,\frac{\theta(x^2)J_0\left(\sqrt{s x^2}\right)}{s-m^2-\Sigma(s)+i0} + \mathrm{Im}\,\frac{\theta(-x^2)J_0\left(\sqrt{-s x^2}\right)}{s+m^2+\Sigma(-s)-i0}\right]. \qquad (3.22)$$

Taking for the self-energy the non-planar normal-ordering graph one finds

$$\Sigma(s) = \frac{\lambda}{12\pi}\,K_0\left(m\theta\sqrt{s+i0}\right). \qquad (3.23)$$

In fact, in this case one may evaluate the commutator in a power series in λ, since the corresponding integrals are convergent. The leading correction outside the light-cone is given by the function

$$\overline{C}(x)_{x^2<0} = \qquad (3.24)$$
$$= \frac{i\lambda}{24\pi}\begin{cases} \frac{\sqrt{-x^2}}{2m}\, I_1(m\sqrt{-x^2})\,K_0(m^2\theta) - \theta\, I_0(m\sqrt{-x^2})\,K_1(m^2\theta)\,, & \text{if } \sqrt{-x^2} < m\theta\,, \\ \theta\, K_0(m\sqrt{-x^2})\,I_1(m^2\theta) - \frac{\sqrt{-x^2}}{2m}\,K_1(m\sqrt{-x^2})\,I_0(m^2\theta)\,, & \text{if } \sqrt{-x^2} > m\theta\,. \end{cases}$$

This function increases away from the light cone to reach a maximum around $\sqrt{-x^2} \approx m\theta$. Then it decreases with exponential asymptotics $\exp(-m\sqrt{-x^2})$ controlled just by the mass of the field. Therefore, we have the expected behaviour, with breakdown of microcausality in spite of the preservation of Lorentz invariance. In this case however the light-cone fuzziness at $\mathcal{O}(\lambda)$ is of short range, with a width of order $m\theta$.

For completeness, we quote here the result of the same calculation for the light-like case [13], with $\theta^{02} = \theta^{12} \equiv \theta$, so that $p \circ p = \theta^2(p_0 - p_1)^2 = \theta^2 p_-^2$. Defining $\alpha \equiv g^2/(x^+)^2$ and $\beta \equiv -x^2/12$, we find the exact result

$$\begin{aligned} \overline{C}(x)_{\text{lightlike}} &= \frac{i}{8\pi^2}\int_0^\infty du\, \cos\left(\frac{x^2\, u}{4} + \frac{\alpha}{u^3}\right) \\ &= -\frac{8^{1/3}\sqrt{-g^2x^2}}{24|x^+|}\, G^{30}_{04}\left[\alpha^2\beta^3; \frac{1}{2}, \frac{1}{6}, -\frac{1}{6}; -\frac{1}{2}\right], \end{aligned} \tag{3.25}$$

where the second expression in terms of the Meijer G-function assumes $x^2 < 0$. Outside the four-dimensional light-cone, for large negative values of x^2, this function decays exponetially as $\exp\left(-(\sqrt{-x^2}/\ell)^{3/2}\right)$, with a characteristic length $\ell = (\theta x^+/\sqrt{\lambda})^{1/3}$. Notice, however, that the light-like configuration leaves no remnant of the four-dimensional light-cone. Therefore, this violation of four-dimensional microcausality is expected because of the breakdown of Lorentz invariance. It is interesting to note that the detailed form of the causality violation at fixed x^+ is of short range, i.e. exponentially supressed.

4. Conclusions

In this note we reexamine various issues of unitarity and causality in perturbative NCFT with time/space noncommutativity. We discuss general one-loop diagrams and we establish the locus of their "non-commutative singularities" to be $p \circ p \leq 0$. "Non-commutative singularities" refers to those singularities that have no simple interpretation in terms of standard cutting rules. The first examples of these singularities were found in [15].

More generally, if the theory is compactified on a circle of *commutative* length L, there are infinite sectors, labelled by an integer "winding number ℓ", with non-commutative singularities for $p \circ p \leq -(L\ell)^2$. These results are robust in the sense that they follow from analytic properties of parametric representations and do not rely on combined analytic continuations in momenta *and* the noncommutativity matrix $\theta^{\mu\nu}$, as in [15].

We can interpret these singularities in terms of new asymptotic states, the χ particles, analogous to closed strings. One considers an asymptotic Hilbert space of the form $\mathcal{H}_\infty = \mathcal{H}_\phi \otimes \mathcal{H}_\chi$, where $\mathcal{H}_\phi$ is the free Fock space of the ϕ-field quanta and $\mathcal{H}_\chi$ is the Fock space of χ particles. Then, a formal restoration of the cutting rules requires the χ particles to have a continuous spectrum and, in general, a tachyonic dispersion relation. Morever, a closer look at explicit examples for the two-point function reveals that $\mathcal{H}_\chi$ necessarily has *indefinite norm*.

Therefore, time-NCFT is perturbatively inconsistent because unphysical excitations, i.e. "ghostly tachyons", are produced in scattering. This is similar to the breakdown of string perturbation theory beyond the NCOS barrier [9, 10]. In models with a string regularization, we specifically show that the χ particles descend from closed strings in the low-energy limit. This provides a microscopic understanding of the peculiar properties of the χ particles, namely the continuous spectrum, the "winding masses" and the tachyonic dispersion relation. It also unifies the field-theoretical and stringy no-go arguments against time-NCFT.

We have also carried out a quantitative study of some causality criteria. Since NCFTs are theories of extendend "dipoles", local microcausality defined in terms of commutators of local observables, such as $[\phi(x), \phi(x')]$, loses much of its meaning, i.e. the "light-cone" has no significance due to the breaking of Lorentz invariance. However, if $\theta^{\mu\nu}$ is block-diagonal with a two-dimensional eigenspace including time, there is an unbroken subgroup of the Lorentz group including $SO(1,1)$ boosts, and a weaker notion of microcausality can be defined with respect to the corresponding two-dimensional light-cone, independently of whether the time coordinate is commutative or non commutative.

We compute quantum corrections to the commutator of scalar fields and find that $SO(1,1)$ microcausality is broken if and only if unitarity is broken. This result lends support to the intuitive idea that space-time non-local theories must be interpreted as S-matrix theories. Thus "causality" criteria must be referred to consistency conditions of the S-matrix.

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