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The Algebraic Method

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Abstract

Combining the effect of an intermediate renormalization prescription (zero momentum subtraction) and the background field method (BFM), we show that the algebraic renormalization procedure needed for the computation of radiative corrections within non-invariant regularization schemes is drastically simplified. The present technique is suitable for gauge models and, here, is applied to the Standard Model. The use of the BFM allows a powerful organization of the counterterms and avoids complicated Slavnov-Taylor identities. Furthermore, the Becchi-Rouet-Stora-Tyutin (BRST) variation of background fields plays a special role in disentangling Ward-Takahashi identities (WTI) and Slavnov-Taylor identities (STI). Finally, the strategy to be applied to physical processes is exemplified for the process $b \to s\gamma$.

1 Introduction

The method of algebraic renormalization has been intensively used as a tool for proving renormalizability of various models (see, e.g., [1]). However, its full value has not yet been widely appreciated by the practitioners. Indeed, the theoretical understanding of algebraic renormalization does not lead automatically to a practical advice for higher-loop calculations. We have to recall that the quantization of gauge models in covariant gauges is necessarily characterized by complicate STIs among the Green functions. Already for simple models, the algebraic renormalization — which essentially provides a technique to restore the symmetries broken by non-invariant regularization schemes — appears particularly difficult [2]. In the case of the Standard Model (SM) or its supersymmetric extensions, the number of fields, of couplings and of independent renormalization constants forbid the naive application of the method [3, 4, 5, 6].

In a recent paper [7], the algebraic renormalization has been considered with regard to practical applications. A procedure has been suggested and worked out which allows an efficient determination of the breaking terms and of the corresponding counterterms. Actually the computation can be reduced to the evaluation of universal, i.e. regularization-scheme-independent, counterterms.

In this paper, we present an improved strategy to simplify the renormalization program within a non-invariant regularization scheme by using the advantages of the BFM [8, 9, 10]. The latter simplifies consistently the analysis and provides an effective procedure for computing all necessary counterterms.

The difference between the conventional approach without the BFM and the new method presented here is essentially due to the fact that the number of independent breaking terms is highly reduced. This is achieved by implementing the WTIs also for the background fields. In a first step the corresponding counterterms are computed, which is relatively simple due to the linearity of the WTIs. The counterterms are in turn incorporated into the STIs where, as a consequence, only a few background gauge invariant counterterms are needed to restore the non-linear equations. Thus we completely avoid the use of the conventional STIs, obtained by differentiating the functional STI with respect to the quantum fields and at least one ghost field. However, we cannot totally skip the STIs obtained with at least one anti-field. Next to the presence of the BRST variation of the background fields the other main new aspect of our method is the effective solution of the remaining STIs by exploiting the background gauge invariance.

For many models, it is quite easy to find a regularization technique which preserves background gauge invariance at one- or two-loop order due to the linearity of the WTIs. In such cases, the only missing counterterms are those needed to restore the STIs. Owing the background gauge invariance, the number of independent non-invariant counterterms are reduced and, according to our procedure, the program can be further simplified. In addition, exploiting completely the BFM, only one-loop non-invariant counterterms are really necessary to perform the computation of physical amplitudes.

The practical use of a such an optimized algebraic scheme is at least two-fold: the impressive experimental precision mainly reached at the electron-positron colliders LEP and

SLC and at the proton anti-proton collider TEVATRON has made it mandatory to evaluate higher-order quantum correction where the algebraic inconsistencies within the naive dimensional regularization scheme are unavoidable. It also seems desirable to have a powerful alternative for cross-checks. Moreover, using the dimensional scheme in supersymmetric theories one needs a practical procedure to restore the Ward identities of supersymmetry in the final step of the renormalization procedure.

In [11], we applied the method of Ref. [7] to the three-gauge-boson vertices involving two W bosons and a photon or Z boson. These contributions constitute a building block to the important W pair production process $e^+e^- \to W^+W^-$, which plays a crucial role at LEP2. Also the two-loop electroweak muon decay amplitude was discussed within this approach [12]. The application of these techniques to supersymmetric examples will be presented in a forthcoming paper.

The paper is organized as follows: In Section 2 we recall the quantization within the BFM and introduce our conventions. In Section 3.1 we discuss the conventional algebraic method. In Section 3.2 our method for the computation of the breaking terms $\Delta^{(n)}$ to all orders is presented. This technique, used in conjunction with the BFM, simplifies the task of the complete analysis. It is based on intermediate renormalization prescriptions. In Section 3.3, we describe the structure of the complete set of identities necessary to renormalize the SM and how they should be organized in order to provide the most efficient procedure in practical computations. Finally, in Section 4, we apply the previous analysis to the physical process $b \to s\gamma$ and present our conclusions in Section 5. In Appendix A we discuss the linearized Slavnov-Taylor operator and the couplings not presented in [7]. Besides the WTIs and STIs there exist also other functional identities which are discussed in Appendix B. There, also the reduced functional is briefly introduced. All possible background gauge invariant counterterms, which in our method could occur to fix the STIs, are listed in Appendix C.

2 Quantization within the Background Field Gauge

For the reader's convenience and in order to establish our conventions, we briefly illustrate the quantization procedure for the SM in the background field gauge. We recall the BRST symmetry (extended to background fields) [13, 8, 10, 4] and the corresponding STIs, the gauge fixing and, finally, the WTIs for the background gauge invariance. Auxiliary material and supplementary constraints such as the Abelian Anti-ghost equation [10] are discussed in Appendix B. For more details we refer to [7].

A generic field is denoted by ϕ, while Φ_i stands for scalar matter fields, i.e. Goldstone ($G^\pm$, G^0) and Higgs (H) bosons. A generic gauge boson field is denoted by V_μ^B and the ghost and the anti-ghost fields by c^B and $\bar{c}^B$, respectively. The index B collectively denotes the adjoint representation for the group $SU(3) \times SU(2) \times U(1)$. The symbols V_μ^a and c^a are used to denote gluon fields and the corresponding ghosts in the adjoint representation of the Lie algebra $su(3)$. The background fields are marked with a hat in order to distinguish them from their quantum counterparts. Q_i denotes the electric charge of a quark q_i. The Greek indices belong to the adjoint representation for $su(2)$ and Latin indices of the beginning of the alphabet $a, b, c, \dots$ run over the adjoint representation of the Lie algebra $su(3)$. Latin indices from the middle of the alphabet $i, j, k, \dots$ denote the representation of scalars which is taken to be real ($t^A_{ij} = -t^A_{ji}$). The generators t^A satisfy the commutation relations $[t^A, t^B] = f^{ABC} t^C$ where f^{ABC} denotes the structure constant of the gauge group. We use the symbol ψ_I for fermions and the generators in the fermion representation are identified by T^A_{IJ} where the indices $I, J, K, \dots$ collect the spin, isospin, colour and chirality degrees of freedom.

Let us also introduce three different types of effective actions, which will be used in the following. The Green functions Γ are regularized and subtracted by means of any chosen scheme. The Green functions $\hat{\Gamma}$ are subtracted using Taylor expansion (see Section 3.2). This implies that all sub-divergences are already renormalized properly. Finally, $\mathrm{I\!\Gamma}$ denotes the renormalized symmetric Green functions, which satisfy the relevant WTIs and STIs. A complete explanation of the conventions, quantum numbers and symmetry transformations of the fields is provided in Ref. [7].

The quantization of the theory can only be achieved by introducing a suitable gauge fixing $\mathcal{L}_{GF}$ and the corresponding Faddeev-Popov terms $\mathcal{L}_{\Phi\Pi}$. Then the classical action reads

$$\mathrm{I\!\Gamma}_0 \;=\; \Gamma_0 \;=\; \int \mathrm{d}^4x \left(\mathcal{L}_{INV} + \sum_i \phi^*_i \, s\phi^i + \mathcal{L}_{GF} + \mathcal{L}_{\Phi\Pi} \right) . \tag{2.1}$$

Both $\mathcal{L}_{GF}$ and $\mathcal{L}_{\Phi\Pi}$ break the local gauge invariance, leaving the theory invariant under the BRST [13] transformations. The BRST symmetry is crucial for proving the unitarity of the S-matrix and the gauge independence[1] of physical observables. Therefore it must be implemented to all orders. Owing to the non-linearity of the BRST transformations [13], the renormalization of some composite operators (namely $s\phi_i$ where ϕ_i is a generic field of the SM and s is the BRST generator) is necessary. This is usually done by adding the composite operators $s\phi_i$ coupled to BRST-invariant external sources ϕ^*_i (anti-fields) to the classical action, namely we add the sum $\sum_i \phi^*_i \, s\phi^i$, which is restricted to those fields undergoing a non-linear transformation [15].

To renormalize properly the SM in the background gauge, one needs to implement the equations of motion for the background fields at the quantum level. The most efficient way to this end is to extend the BRST symmetry to the background fields

$$\begin{array}{llll} s\hat{W}^3_\mu = \Omega^3_\mu \,, & s\Omega^3_\mu = 0 \,, & s\hat{G}^0 = \Omega^0 \,, & s\Omega^0 = 0 \,, \\ s\hat{W}^\pm_\mu = \Omega^\pm_\mu \,, & s\Omega^\pm_\mu = 0 \,, & s\hat{G}^\pm = \Omega^\pm \,, & s\Omega^\pm = 0 \,, \\ s\hat{G}^a_\mu = \Omega^a_\mu \,, & s\Omega^a_\mu = 0 \,, & s\hat{H} = \Omega^H \,, & s\Omega^H = 0 \,, \end{array} \tag{2.2}$$

where $\Omega^\pm_\mu, \Omega^3_\mu$ and Ω^a_μ are (classical) vector fields with the same quantum numbers as the gauge bosons W, Z and G^a_μ, but ghost charge $+1$ (like a ghost field). $\Omega^\pm, \Omega^0$ and Ω^H are scalar fields with ghost number $+1$.

[1]The BRST symmetry extended to background fields is essential for quantizing a gauge model in the background gauge. Within that framework the unitarity of the model, the gauge parameter independence and the independence of the S-matrix from the background fields can be proven [14].

Finally, the BRST symmetry extended to the background fields (cf., e.g., [7]) is implemented at the quantum level by establishing the corresponding STI in the functional form

$$
\begin{aligned}
\mathcal{S}(\mathrm{I\!\Gamma})[\phi] \;=\; & \int \mathrm{d}^4x \Bigg\{ \Bigg[(s_W\partial_\mu c_Z + c_W\partial_\mu c_A)\left(s_W\frac{\delta \mathrm{I\!\Gamma}}{\delta Z_\mu} + c_W\frac{\delta \mathrm{I\!\Gamma}}{\delta A_\mu}\right) \\
& + \frac{\delta \mathrm{I\!\Gamma}}{\delta W^{*,3}_\mu}\left(c_W\frac{\delta \mathrm{I\!\Gamma}}{\delta Z_\mu} - s_W\frac{\delta \mathrm{I\!\Gamma}}{\delta A_\mu}\right) + \frac{\delta \mathrm{I\!\Gamma}}{\delta W^{*,\pm}_\mu}\frac{\delta \mathrm{I\!\Gamma}}{\delta W^{\mp}_\mu} + \frac{\delta \mathrm{I\!\Gamma}}{\delta V^{*,a}_\mu}\frac{\delta \mathrm{I\!\Gamma}}{\delta V^{a}_\mu} + \frac{\delta \mathrm{I\!\Gamma}}{\delta c^{*,\pm}}\frac{\delta \mathrm{I\!\Gamma}}{\delta c^{\mp}} \\
& + \frac{\delta \mathrm{I\!\Gamma}}{\delta c^{*,3}}\left(c_W\frac{\delta \mathrm{I\!\Gamma}}{\delta c_Z} - s_W\frac{\delta \mathrm{I\!\Gamma}}{\delta c_A}\right) + \frac{\delta \mathrm{I\!\Gamma}}{\delta c^{*,a}}\frac{\delta \mathrm{I\!\Gamma}}{\delta c^{a}} + \frac{\delta \mathrm{I\!\Gamma}}{\delta G^{*,\pm}}\frac{\delta \mathrm{I\!\Gamma}}{\delta G^{\mp}} + \frac{\delta \mathrm{I\!\Gamma}}{\delta G^{*,0}}\frac{\delta \mathrm{I\!\Gamma}}{\delta G^{0}} \\
& + \frac{\delta \mathrm{I\!\Gamma}}{\delta H^{*}}\frac{\delta \mathrm{I\!\Gamma}}{\delta H} + \sum_{I=L,Q,u,d,e}\left(\frac{\delta \mathrm{I\!\Gamma}}{\delta \bar\psi^{*I}}\frac{\delta \mathrm{I\!\Gamma}}{\delta \psi^{I}} + \frac{\delta \mathrm{I\!\Gamma}}{\delta \psi^{*I}}\frac{\delta \mathrm{I\!\Gamma}}{\delta \bar\psi^{I}}\right) + \sum_{\alpha=A,Z,\pm,a} b_\alpha\frac{\delta \mathrm{I\!\Gamma}}{\delta \bar c^{\alpha}}\Bigg] \\
& + \Omega^3_\mu\left[c_W\left(\frac{\delta \mathrm{I\!\Gamma}}{\delta \hat Z_\mu} - \frac{\delta \mathrm{I\!\Gamma}}{\delta Z_\mu}\right) - s_W\left(\frac{\delta \mathrm{I\!\Gamma}}{\delta \hat A_\mu} - \frac{\delta \mathrm{I\!\Gamma}}{\delta A_\mu}\right)\right] \\
& + \Omega^\pm_\mu\left(\frac{\delta \mathrm{I\!\Gamma}}{\delta \hat W^{\pm}_\mu} - \frac{\delta \mathrm{I\!\Gamma}}{\delta W^{\pm}_\mu}\right) + \Omega^a_\mu\left(\frac{\delta \mathrm{I\!\Gamma}}{\delta \hat V^{a}_\mu} - \frac{\delta \mathrm{I\!\Gamma}}{\delta V^{a}_\mu}\right) \\
& + \Omega^\pm\left(\frac{\delta \mathrm{I\!\Gamma}}{\delta \hat G^{\pm}} - \frac{\delta \mathrm{I\!\Gamma}}{\delta G^{\pm}}\right) + \Omega^0\left(\frac{\delta \mathrm{I\!\Gamma}}{\delta \hat G^{0}} - \frac{\delta \mathrm{I\!\Gamma}}{\delta G^{0}}\right) + \Omega^H\left(\frac{\delta \mathrm{I\!\Gamma}}{\delta \hat H} - \frac{\delta \mathrm{I\!\Gamma}}{\delta H}\right)\Bigg\} \\
\;=\; & 0\,, \qquad (2.3)
\end{aligned}
$$

where the notation $A^\pm B^\mp = A^+B^- + A^-B^+$ has been used. Here s_W and c_W denote the sine and cosine of the Weinberg angle θ_W, and b_α are the so-called Nakanishi-Lautrup multipliers.

The first sum in the fourth line of Eq. (2.3) includes the left-handed doublets and the right-handed singlets. For the BRST source fields no Weinberg rotation has been introduced. We stress that this formula represents the complete non-linear STI to all orders. The first two and the last term in the first square brackets correspond to the linear BRST variation of the $U(1)$ abelian gauge field and the BRST transformations of the anti-ghost fields.

In order to specify the gauge-fixing, we introduce the equation of motion for the b fields corresponding to the various gauge fields in the SM

$$
\frac{\delta \mathrm{I\!\Gamma}}{\delta b_C} \;=\; \mathcal{F}_C(V,\Phi_i) + \xi_C b_C\,, \qquad (2.4)
$$

where $\mathcal{F}_C$ are the gauge fixing functions. ξ_C ($C = A, Z, \pm, a$) are the corresponding gauge parameters. In the case of the background gauge fixing the functions $\mathcal{F}_C$ are explicitly given in formula (A.2) of [7].

For each generator of the gauge group $SU(3)\times SU(2)\times U(1)$, we consider the corresponding local infinitesimal parameters. They are denoted by $\lambda_A(x), \lambda_Z(x)$ and $\lambda_\pm(x)$ for the electroweak part and by $\lambda_a(x)$ for the QCD sector. The functional WTI for the effective action $\mathrm{I\!\Gamma}$ thus reads

$$
\mathcal{W}_{(\lambda)}(\mathrm{I\!\Gamma}) \;=\; \sum_\phi \int \mathrm{d}^4x \left(\delta_{\lambda(x)}\phi\right)\frac{\delta \mathrm{I\!\Gamma}}{\delta\phi(x)} = \sum_{B=A,Z,\pm,a}\int \mathrm{d}^4x\,\lambda_B(x)\mathcal{W}_B(x)(\mathrm{I\!\Gamma}) \;=\; 0\,, \qquad (2.5)
$$

where the variations $\delta_\lambda \phi(x)$ are explicitly given in the Appendix of [7]. The sum runs over all possible fields and anti-fields. $\mathcal{W}_\alpha(x)$ is called Ward-Takahashi operator. It acts on the functional $\mathbb{\Gamma}[\phi]$.

The principal difference between the STIs (2.3) and the WTIs (2.5) for the background gauge invariance is due to the linearity of the latter. This means that the WTIs are linear in the functional $\mathbb{\Gamma}$ and they therefore relate Green functions of the same orders while for the STIs there is an interplay between higher- and lower-order radiative corrections.

In Appendix B, we consider the supplementary equations needed in order to complete the algebraic structure of the functional equations. Although these constraints are not independent from (2.3), (2.4) and (2.5) they turn out to be useful in order to reduce the complexity of the algebraic method.

As a final remark of this section we want to mention that in the forthcoming part of this paper instead of Eq. (2.1) we will use the reduced functional which differs only slightly from Γ_0 in the gauge-fixing part as is explained explicitly in Appendix B (see Eqs. (B.2) and (B.3)). This has the consequence that some of the formulae are simpler. In particular, in Eq. (2.3) the last term in the fourth line is absent. As this is a minor technical point and confusion is excluded, we denote also the reduced functional by Γ_0.

3 Renormalization

3.1 The algebraic method

The Quantum Action Principle (QAP) [16] is the fundamental theorem of renormalization theory. It guarantees the locality of the counterterms and the polynomial character of the renormalization procedure. The QAP also implies that all breaking terms of the STIs and WTIs are local and that they can be fully characterized in terms of classical fields, their quantum numbers and symmetry properties.

In the case of STIs, the QAP implies that, if the Green functions $\mathbb{\Gamma}^{(n-1)}$ satisfy all the symmetry constraints at lower orders, the (subtracted) n^{th}-order Green functions $\Gamma^{(n)}$ fulfill them up to local insertions $\Delta_S^{(n)}$ in the one-particle-irreducible (1PI) Green functions:

$$[\mathcal{S}(\Gamma)]^{(n)} \quad = \quad \Delta_S^{(n)} \,. \tag{3.1}$$

Here $\Delta_S^{(n)}$ is an integrated, Lorentz-invariant polynomial of the fields, of the anti-fields and of their derivatives, $\Delta_S^{(n)} = \sum_i \gamma_{S,i}^{(n)} \int \mathrm{d}^4x \, \Delta_{S,i}(x)$, with ultra-violet (UV) degree ≤ 4 and infra-red (IR) degree ≥ 3 (assuming four space-time dimensions). In the same way, the WTIs are spoiled by breaking terms of the form

$$\mathcal{W}_{(\lambda)} \left(\Gamma^{(n)}\right) \quad = \quad \Delta_W^{(n)}(\lambda) \,, \tag{3.2}$$

where $\Delta_W^{(n)}(\lambda) = \sum_i \gamma_{W,i}^{(n)} \int \mathrm{d}^4x \, \Delta_{W,i}(x,\lambda)$ is again an integrated, Lorentz-invariant polynomial, depending linearly on the infinitesimal parameter λ, with UV degree ≤ 4 and IR degree ≥ 3.

Although Eqs. (3.1) and (3.2) apply to any renormalization scheme, the coefficients $\gamma_{S,i}^{(n)}$ and $\gamma_{W,i}^{(n)}$ of the various $\Delta^{(n)}$'s depend on the particular scheme adopted and on the order of the computation. In fact, the definitions of $\Delta_S^{(n)}$ and $\Delta_W^{(n)}(\lambda)$ rely on specific conventions for composite operators. Thus a renormalization description for the composite operators is necessary. Usually one uses the concept of Normal Product Operators introduced by Zimmermann [17] or the conventional counterterm technique, which is preferable from the practical point of view.

Once the breaking terms $\Delta_S^{(n)}$ and $\Delta_W^{(n)}(\lambda)$ are given, the main objective of the algebraic method [13, 1] can be discussed. This essentially entails a prescription to restore the identities by suitable local counterterms[2], $\Gamma^{CT,(n)} = \sum_i \xi_i^{(n)} \int d^4x\, \mathcal{L}_i^{CT}(x)$, such that one has at $n^{\rm th}$ order:

$$\begin{aligned}
\Big[\mathcal{S}(\mathrm{I\!\Gamma})\Big]^{(n)} \;&\equiv\; \mathcal{S}_0(\Gamma^{(n)}) + \sum_{j=1}^{n-1}\Big(\mathrm{I\!\Gamma}^{(j)}, \mathrm{I\!\Gamma}^{(n-j)}\Big) - \;\mathcal{S}_0(\Gamma^{CT,(n)}) \;=\; 0\,,\\
\mathcal{W}_{(\lambda)}\left(\mathrm{I\!\Gamma}^{(n)}\right) \;&\equiv\; \mathcal{W}_{(\lambda)}\left(\Gamma^{(n)}\right) - \mathcal{W}_{(\lambda)}\left(\Gamma^{CT,(n)}\right) \;=\; 0\,,
\end{aligned} \tag{3.3}$$

where the decomposition given in Eqs. (A.1) and (A.2) has been used. Notice that, since the Green functions $\mathrm{I\!\Gamma}^{(j)}$ with $j < n$ are already fixed, only $\Gamma^{(n)}$ has to be adjusted by the local counterterms $\Gamma^{CT,(n)}$.

In practice the problem amounts to solving the algebraic equations

$$\mathcal{S}_0\left(\Gamma^{CT,(n)}\right) \;=\; \Delta_S^{(n)}\,, \qquad\qquad \mathcal{W}_{(\lambda)}\left(\Gamma^{CT,(n)}\right) \;=\; \Delta_W^{(n)}(\lambda)\,, \tag{3.4}$$

where $\mathcal{S}_0$ is defined below Eq. (A.2). The solution fixes a subset of the coefficients $\xi_i^{(n)}$ of the counterterms in terms of $\gamma_{S,i}^{(n)}$ and $\gamma_{W,i}^{(n)}$. These equations turn out to be solvable in the absence of anomalies, where only the consistency conditions have to be used. Moreover, because of a non-trivial kernel of the operators $\mathcal{S}_0$ and $\mathcal{W}_{(\lambda)}$ (i.e. the space of invariant counterterms), one is allowed to impose renormalization conditions by tuning the free parameters of the model (namely the remaining coefficients $\xi_i^{(n)}$ of $\Gamma^{CT,(n)}$)

$$\mathcal{N}_{\mathcal{I}}\left(\mathrm{I\!\Gamma}^{(n)}\right) \;=\; \mathcal{N}_{\mathcal{I}}\left(\Gamma^{(n)} + \Gamma^{CT,(n)}\right) \;=\; 0\,, \tag{3.5}$$

where the index $\mathcal{I}$ runs over all independent parameters of the SM and $\mathcal{N}_{\mathcal{I}}$ denotes the normalization condition operators.

The existence and uniqueness of $\Gamma^{CT,(n)}$ as a solution of the system (3.3)–(3.5) has been proven in [13, 18, 19] for gauge theories, in [20, 6] for supersymmetric models, and in [3, 4, 10] for non semi-simple models coupled to fermion and scalar fields. Indeed, the main purpose of the algebraic renormalization is to demonstrate the existence of a unique solution (up to normalization conditions) to the algebraic problem (3.3)–(3.4) in the absence of anomalies. Unfortunately, this does not necessarily provide a practical technique to compute the breaking terms $\Delta^{(n)}$ and to determine the corresponding counterterms $\Gamma^{CT,(n)}$.

[2]See Appendix C for the notation.

3.2 Simplifying the breaking terms

As already mentioned above, regardless of which regularization scheme is used, the calculation of the $\Delta^{(n)}$'s in (3.4) is quite tedious and gets even more complicated at higher orders (see, e.g., [2] for a complete one-loop computation). In general, one has to calculate all Green functions that occur in the complete set of STIs or WTIs. Inserting them in the chosen identities then determines $\Delta^{(n)}$. The additional computations necessary in the conventional algebraic method can be slightly reduced: instead of calculating all Green functions that occur in the full set of STIs, one can compute only those at special points, namely for zero momentum, for on-shell momentum or for large external momenta. As a consequence, the breaking terms, $\Delta^{(n)}$, are simply related to those Green functions that are evaluated at these special points. Clearly, if on-shell renormalization conditions are used in the calculation, the on-shell method could be superior to the zero-momentum subtraction. At large momentum, one takes advantage of Weinberg's theorem [21]. In [22], based on the zero-momentum subtraction, a procedure has been formulated to discuss the complete renormalization of the abelian Higgs model. Here, we adopt the same procedure in the more general context of the SM.

Owing to locality and to bounded mass dimension (as fixed by power counting and the QAP) of each single monomial $\Delta_{S,i}(x)$ and $\Delta_{W,j}(x,\lambda)$ of $\Delta_S^{(n)}$ and $\Delta_{(\lambda)}^{(n)}$, respectively, there exist two non-negative integers δ_S and δ_W such that

$$\left(1 - T^{\delta_S}_{p_1,\ldots,p_n}\right) \Delta_S(p_1,\ldots,p_n) \;=\; 0\,, \qquad \left(1 - T^{\delta_W}_{p_1,\ldots,p_n}\right) \Delta_W(p_1,\ldots,p_n,\lambda) \;=\; 0\,, \tag{3.6}$$

where T^δ is the Taylor subtraction operator; $\Delta_{S,i}(p_1,\ldots,p_n)$ and $\Delta_{W,j}(p_1,\ldots,p_n,\lambda)$ are the Fourier transformed polynomials of $\Delta_{S,i}(x)$ and of $\Delta_{W,j}(x,\lambda)$. In power counting renormalizable theories the degrees δ_S and δ_W are independent of the loop order n.

Acting with the zero-momentum subtraction operator $(1 - T^\delta_{S/W})$ on both sides of Eqs. (3.1) and (3.2) leads to

$$\begin{aligned} (1 - T^{\delta_S})\Big[\mathcal{S}(\Gamma)\Big]^{(n)} \;&=\; (1 - T^{\delta_S})\Delta_S^{(n)} \;=\; 0\,, \\ (1 - T^{\delta_W})\mathcal{W}_{(\lambda)}\,(\Gamma)^{(n)} \;&=\; (1 - T^{\delta_W})\Delta_W^{(n)}(\lambda) \;=\; 0\,. \end{aligned} \tag{3.7}$$

Thus the $\Delta^{(n)}$'s are subtracted away. The *functional* Taylor subtraction operator T^δ is defined in (A.4) of Appendix A. At the moment we assume that the zero-momentum subtraction is possible. This means that the vertex functions have to be sufficiently regular at zero momenta. In the presence of massless particles, zero-momentum subtractions of the regularized function $\Gamma^{(n)}$ might lead to IR divergences. A practical solution of this problem is discussed in [11]. In particular, the decomposition of $\mathcal{S}_0$ is made in order to disentangle the IR divergent Green functions from the IR finite ones. Consequently one has to analyze the new breaking terms arising from the commutator between this new operator and the STI and the WTI operator. Moreover, one can take advantage of the fact that the breaking terms are IR safe for general reasons provided there are no IR anomalies in the model.

The l.h.s. of Eqs. (3.7) has not yet the correct form. Actually, our aim is to obtain new STIs and WTIs for the subtracted Green functions, i.e. for $\hat\Gamma^{(n)} = (1 - T^{\delta_{pc}})\Gamma^{(n)}$, where δ_{pc}

is the naive power counting degree. Generically, we have $\delta_W,\delta_S \geq \delta_{pc}$. For that purpose we commute the Taylor operation with the Slavnov-Taylor operator $\mathcal{S}_0$ and with the Ward-Takahashi operator $\mathcal{W}_{(\lambda)}$ where it is convenient to adopt the decomposition (3.3) into a linearized operator plus bilinear terms for the STIs case. The part involving the linearized operator leads to

$$(1-T^{\delta_S})\mathcal{S}_0(\Gamma^{(n)}) \;=\; \mathcal{S}_0\left(\hat{\Gamma}^{(n)}\right)+\mathcal{S}_0\left(T^{\delta_{pc}}\Gamma^{(n)}\right)-T^{\delta_S}\left(\mathcal{S}_0(\Gamma^{(n)})\right), \tag{3.8}$$

and, correspondingly, for the WTIs case, we have

$$(1-T^{\delta_W})\mathcal{W}_{(\lambda)}(\Gamma^{(n)}) \;=\; \mathcal{W}_{(\lambda)}\left(\hat{\Gamma}^{(n)}\right)+\mathcal{W}_{(\lambda)}\left(T^{\delta_{pc}}\Gamma^{(n)}\right)-T^{\delta_W}\left(\mathcal{W}_{(\lambda)}(\Gamma^{(n)})\right). \tag{3.9}$$

These equations express the fact that $\mathcal{S}_0$ and $\mathcal{W}_{(\lambda)}$ are in general not homogeneous in the fields. In particular, this is the case for theories with spontaneous symmetry breaking. Notice that, although the Taylor operator is scale-invariant, it does not commute with $\mathcal{S}_0$ and $\mathcal{W}_{(\lambda)}$ of the SM. The difference between δ_W, δ_S and δ_{pc} leads to over-subtractions in $\Gamma^{(n)}$. Therefore, breaking terms generated by the last two terms on the r.h.s. of Eqs. (3.8) and (3.9) have to be introduced. Furthermore, the action of the Taylor operator can be split into the naive contribution $\sum_{j=1}^{n-1}\left(\mathrm{I\!\Gamma}^{(j)}, \mathrm{I\!\Gamma}^{(n-j)}\right)$ plus the local terms obtained by Taylor expansion. These local terms also contribute to the new breaking terms.

Finally, by applying the Taylor operator on (3.1) and using (3.7) and (3.8) we obtain

$$\begin{aligned}(1-T^{\delta_S})\Big[\mathcal{S}(\Gamma)\Big]^{(n)} \;&=\; \mathcal{S}_0\left[(1-T^{\delta_{pc}})\Gamma^{(n)}\right]+\sum_{j=1}^{n-1}\left(\mathrm{I\!\Gamma}^{(j)}, \mathrm{I\!\Gamma}^{(n-j)}\right) \\ &\quad -\Big[T^{\delta_S}\mathcal{S}_0-\mathcal{S}_0T^{\delta_{pc}}\Big](\Gamma^{(n)})-T^{\delta_S}\sum_{j=1}^{n-1}\left(\mathrm{I\!\Gamma}^{(j)}, \mathrm{I\!\Gamma}^{(n-j)}\right) \;=\; 0\,.\end{aligned} \tag{3.10}$$

The terms in the second line of (3.10) represent the new local breaking terms which correspond to the subtracted function at the n^{th} order, $\hat{\Gamma}^{(n)}=(1-T^{\delta_{pc}})\Gamma^{(n)}$. Thus, it is convenient to define the new breaking terms as

$$\Psi_S^{(n)} \;=\; \Big[T^{\delta_S}\mathcal{S}_0-\mathcal{S}_0T^{\delta_{\mathrm{pc}}}\Big](\Gamma^{(n)})+T^{\delta_S}\sum_{j=1}^{n-1}\left(\mathrm{I\!\Gamma}^{(j)}, \mathrm{I\!\Gamma}^{(n-j)}\right). \tag{3.11}$$

We emphasize that they are universal in the sense that they do not depend on the specific regularization used in the calculation — in contrast to $\Delta_S^{(n)}$ in Eq. (3.1) because they are UV-finite.

Applying the same steps to the WTIs, we end up with

$$\Psi_W^{(n)}(\lambda) \;=\; \Big[T^{\delta_W}\mathcal{W}_{(\lambda)}-\mathcal{W}_{(\lambda)}T^{\delta_{\mathrm{pc}}}\Big](\Gamma^{(n)})\,. \tag{3.12}$$

As a consequence of the linearity of the WTIs, Eq. (3.12) is simpler than Eq. (3.11). The form of the new breaking terms $\Psi_W^{(n)}(\lambda)$ does not depend on the loop-order. In addition, the only source for $\Psi_W^{(n)}(\lambda)$ essentially consists in the different UV behaviour of each term in the functional operator $\mathcal{W}_{(\lambda)}$, which lead to over-subtractions. The sole source for $\Psi_W^{(n)}(\lambda)$ is the spontaneous symmetry-breaking mechanism.

In the described procedure the use of the Taylor expansion is motivated by practical considerations. In momentum space the technique for evaluating Green functions with zero external momenta is quite elaborated and, in some cases, the corresponding integrals can even be solved analytically at the three-loop order [23].

The algebraic problem is now reduced to finding the proper counterterms $\Xi^{(n)}$ which solve the equations

$$\mathcal{S}_0\left(\Xi^{(n)}\right) = -\Psi_S^{(n)}\,, \qquad \mathcal{W}_{(\lambda)}\left(\Xi^{(n)}\right) = -\Psi_W^{(n)}\,. \tag{3.13}$$

Finally, in terms of $\Xi^{(n)}$ the final correct vertex function at $n^{\rm th}$ order reads

$$\mathrm{I\!\Gamma}^{(n)} \;=\; \hat{\Gamma}^{(n)} + \Xi^{(n)} \;=\; \left(1 - T^{\delta_{\rm pc}}\right)\Gamma^{(n)} + \Xi^{(n)}\,. \tag{3.14}$$

In order to illustrate the discussion of this subsection, we want to present a simple example. We consider the renormalization of the three-point function $\Gamma^{(n)}_{\hat{W}^+_\nu \bar{q} b}$. This amplitude is necessary in the discussion of the process $b \to s\gamma$ which is presented in Section 4. Assuming that at order $n-1$ the renormalization has already been worked out, the amplitude $\Gamma^{(n)}_{\hat{W}^+_\nu \bar{q} b}$ satisfies the WTI

$$\begin{aligned} & i\,(p_q+p_b)_\nu\,\Gamma^{(n)}_{\hat{W}^+_\nu \bar{q} b}(p_q,p_b) + i\,M_W\,\Gamma^{(n)}_{\hat{G}^+ \bar{q} b}(p_q,p_b) \\ & \qquad + \frac{ie}{s_W\sqrt{2}}\left[\Gamma^{(n)}_{\bar{q}q'}(-p_q)V_{q'b}P_L - V_{qq'}P_R\Gamma^{(n)}_{\bar{q}'b}(p_b)\right] \;=\; \Delta^{(n)}_{W,\lambda^+\bar{q}b}(p_q,p_b)\,, \end{aligned} \tag{3.15}$$

where $q = u, c, t$ and the sum over q' is understood. After adopting a renormalization scheme the n-loop Green functions and thus the breaking term $\Delta^{(n)}_{W,\lambda^+\bar{q}b}$ are computed. According to Eq. (3.12) we get after application of the zero momentum subtraction $(1 - T^1_{p_q,p_b})$

$$\begin{aligned} & i\,(p_q+p_b)_\nu\left[\left(1 - T^0_{p_q,p_b}\right)\Gamma^{(n)}_{\hat{W}^+_\nu \bar{q} b}(p_q,p_b)\right] \\ & \quad + i\,M_W\left[\left(1 - T^0_{p_q,p_b}\right)\Gamma^{(n)}_{\hat{G}^+ \bar{q} b}(p_q,p_b)\right] \\ & \quad + \frac{i\,e}{s_W\sqrt{2}}\left\{\left[(1-T^1_{p_q})\Gamma^{(n)}_{\bar{q}q'}(-p_q)\right]V_{q'b}P_L - V_{qq'}P_R\left[(1-T^1_{p_b})\Gamma^{(n)}_{\bar{q}'b}(p_b)\right]\right\} \;=\; \\ & \qquad\qquad iM_W\left(p_q\partial_{p_q} + p_b\partial_{p_b}\right)\Gamma^{(n)}_{\hat{G}^+\bar{q}b}(p_q,p_b)\Big|_{p_b=p_q=0} \;\equiv\; \Psi^{W,(n)}_{\lambda^+\bar{q}b}(p_q,p_b)\,, \end{aligned} \tag{3.16}$$

where we kept all quark masses different from zero. In the square brackets, the Taylor-subtracted Green function are collected. On the r.h.s. only one finite Green function[3] evaluated for zero external momenta appears. It has been generated by the over-subtraction

[3] The case where some masses are set to zero and IR divergences occur are discussed in detail in [11].

and constitutes the new breaking term $\Psi^{W,(n)}_{\lambda^+\bar{q}b}$. Notice that, in contrast to $\Delta^{(n)}_{W,\lambda^+\bar{q}b}$, the expression of $\Psi^{W,(n)}_{\lambda^+\bar{q}b}$ is UV finite quantity and only depends on one Green function.

In our example the breaking term $\Psi^{W,(n)}_{\lambda^+\bar{q}b}$ is absorbed by the counterterm (cf. Eq. (C.9) of Appendix C)

$$\Xi^{W,(n)}_2 \;=\; \int \mathrm{d}^4x \left(\xi^{W,(n)}_{\bar{q}bW,L}\, \bar{q}\; \hat{\not{W}}^+ P_L b + \xi^{W,(n)}_{\bar{q}bW,R}\, \bar{q}\; \hat{\not{W}}^+ P_R\, b\right) , \tag{3.17}$$

where the parameters $\xi^{W,(n)}_{\bar{q}bW,L}$ and $\xi^{W,(n)}_{\bar{q}bW,R}$ have to be tuned and the wave function renormalization for the quarks have to be adjusted. In Section 4 the above equations will be needed at one-loop order, i.e. for $n = 1$.

Besides the obvious advantages of the Taylor expansion around zero momenta — which essentially reduces the number of non-vanishing contributions to the $\Delta^{(n)}$'s (for instance, mass counterterms are not needed to restore the STIs or WTIs) — there remain complicate expressions for the "universal" breaking terms $\Psi^{(n)}_S$ as given in Eqs. (3.11). Mainly, there are terms that depend on lower-order Green functions, recalling the non-linear nature of the BRST symmetry. Therefore, the WTIs of the background gauge invariance have to be exploited completely in order to further increase the practical gain of our method with respect to the conventional approach. In fact, as we will show below, because of algebraic relations between the functional operators $\mathcal{S}_0$ and $\mathcal{W}_{(\lambda)}$, restoring the WTIs implies a partial restoration also of the STIs. This will significantly simplify the evaluation of $\Psi^{(n)}_S$. A further simplification in the calculation of the universal breaking terms, $\Psi^{(n)}_S$ and $\Psi^{(n)}_W$, can be achieved by a suitable choice of the renormalization conditions.

3.3 Computation of counterterms

The main new aspect in the present analysis is the BRST variation of the background fields. In combination with the advantages of the intermediate renormalization, we are able to simplify the procedure of the algebraic renormalization. However, an essential ingredient for an efficient evaluation of counterterms $\Xi^{(n)}$ is the hierarchical structure among functional identities. This implies that we can restore the identities one after the other without spoiling those that are already recovered. Organizing the counterterms in such a way that in a first step the WTIs and in a second one the STIs can be restored can only be achieved if the fixing of the STIs does not destroy the already restored WTIs. This means that the counterterms needed to restore the STIs must be invariant under the action of the WTIs, i.e. they must be background gauge invariant. Clearly, this is only possible if the breaking terms of the STIs are background gauge invariant. The latter is a consequence of consistency conditions between the breaking terms of STIs and of WTIs.

In fact, we have to recall that the operators $\mathcal{S}_{\mathrm{I\!\Gamma}}$ and $\mathcal{W}_{(\lambda)}$ form an algebra[4] [13, 3, 10]

$$\mathcal{S}^2_{\mathrm{I\!\Gamma}} = 0 \quad \text{if} \quad \mathcal{S}(\mathrm{I\!\Gamma}) \;=\; 0\,,$$

[4]Here we used the notation $(\lambda \wedge \beta)^a = f^{abc}\lambda^b\beta^c, (\lambda \wedge \beta)^A = f^{A+-}\lambda^+\beta^-, \ldots$. Notice in addition, that in the present section we are only dealing with the WTIs and STIs. However, our considerations are easily extended to the complete set of functional identities of the SM, as will be recalled in Appendix B.

$$
\begin{aligned}
\mathcal{S}_{\mathbb{I}\Gamma}\Big(\mathcal{W}_{(\lambda)}(\mathbb{I}\Gamma)\Big)-\mathcal{W}_{(\lambda)}\Big(\mathcal{S}(\mathbb{I}\Gamma)\Big) &= 0\,,\\
\mathcal{W}_{(\lambda)}\Big(\mathcal{W}_{(\beta)}(\mathbb{I}\Gamma)\Big)-\mathcal{W}_{(\beta)}\Big(\mathcal{W}_{(\lambda)}(\mathbb{I}\Gamma)\Big) &= \mathcal{W}_{(\lambda\wedge\beta)}(\mathbb{I}\Gamma)\,,
\end{aligned}
\tag{3.18}
$$

which, applied to the breaking terms $\Psi_S^{(n)}$ and $\Psi_W^{(n)}(\lambda)$ in Eqs. (3.11) and (3.12), leads to the so-called Wess-Zumino consistency conditions

$$
\mathcal{W}_{(\lambda)}\Big(\Psi_W^{(n)}(\beta)\Big)-\mathcal{W}_{(\beta)}\Big(\Psi_W^{(n)}(\lambda)\Big) = \Psi_W^{(n)}(\lambda\wedge\beta)\,, \tag{3.19}
$$

$$
\mathcal{S}_0\Big(\Psi_W^{(n)}(\lambda)\Big)-\mathcal{W}_{(\lambda)}\Big(\Psi_S^{(n)}\Big) = 0\,, \tag{3.20}
$$

$$
\mathcal{S}_0(\Psi_S^{(n)}) = 0\,. \tag{3.21}
$$

From well-established results of the cohomology analysis of the Wess-Zumino consistency conditions (3.19) for the functional operators $\mathcal{W}_{(\lambda)}$ [13, 3, 18], we know that, in the absence of anomalies, the counterterms $\Xi^{W,(n)}$ can be constructed such that

$$
\mathcal{W}_{(\lambda)}\Big(\Xi^{W,(n)}\Big) = -\Psi_W^{(n)}(\lambda)\,.
$$

Clearly, since $\mathcal{W}_{(\lambda)}$ has a non-trivial kernel, this equation cannot fix background gauge invariant counterterms. A suitable choice of the latter and the organization of the complete set of breaking terms according to the quantum numbers of the fields and anti-fields significantly simplify the computation of $\Xi^{W,(n)}$, as is shown below.

By adding the counterterms $\Xi^{W,(n)}$ to the Green functions $\hat{\Gamma}^{(n)}$, we see that the WTIs are restored and the STIs are consequently modified

$$
\begin{aligned}
\mathcal{W}_{(\lambda)}\Big(\hat{\Gamma}^{(n)}+\Xi^{W,(n)}\Big) &= \Psi_W^{(n)}(\lambda)+\mathcal{W}_{(\lambda)}\Big(\Xi^{W,(n)}\Big) = 0\,,\\
\Big[\mathcal{S}(\hat{\Gamma}+\Xi^{W})\Big]^{(n)} &= \Psi_S^{(n)}+\mathcal{S}_0\Big(\Xi^{W,(n)}\Big) \equiv \hat{\Psi}_S^{(n)}\,.
\end{aligned}
$$

Moreover, as a consequence of Eq. (3.20), the new breaking terms $\hat{\Psi}_S^{(n)}$ are explicitly background gauge invariant as can be seen as follows:

$$
\mathcal{W}_{(\lambda)}\Big(\hat{\Psi}_S^{(n)}\Big)=\mathcal{W}_{(\lambda)}\Big(\Psi_S^{(n)}\Big)+\mathcal{W}_{(\lambda)}\mathcal{S}_0\Big(\Xi^{W,(n)}\Big)=\mathcal{W}_{(\lambda)}\Big(\Psi_S^{(n)}\Big)-\mathcal{S}_0\Big(\Psi_W^{(n)}(\lambda)\Big)=0\,.
$$

The difference between a conventional approach without the BFM and our method is essentially that the number of independent breaking terms $\hat{\Psi}_S^{(n)}$ is significantly reduced. Indeed, only background gauge invariant counterterms[5] $\Xi^{S,(n)}$ are needed to restore non-linear STIs (3.12)

$$
\mathcal{W}_{(\lambda)}\Big(\Xi^{S,(n)}\Big)=0\,,\qquad \mathcal{S}_0\Big(\Xi^{S,(n)}\Big)=-\hat{\Psi}_S^{(n)}\,. \tag{3.22}
$$

However, this is not the total benefit we obtain from the inclusion of the BFM. It is also possible to simplify the calculation of the second equation in (3.22). To this end we

[5] In Appendix C the complete structure of the counterterms is given.

decompose the counterterms $\Xi^{S,(n)}$ according to the dependence on the anti-fields ϕ^* and on the fields Ω

$$
\begin{aligned}
\Xi^{S,(n)} &= \Xi^{S,(n)}_O[\phi,\hat\phi] + \Xi^{S,(n)}_{\#}[\phi,\hat\phi,\phi^*,\Omega]\,, \\
\Xi^{S,(n)}_{\#}[\phi,\hat\phi,\phi^*,\Omega] &= \Xi^{S,(n)}_{\#,O}[\phi,\hat\phi,\phi^*] + \Xi^{S,(n)}_{\Omega}[\phi^*,\Omega]\,,
\end{aligned}
\tag{3.23}
$$

where $\Xi^{S,(n)}_O[\phi,\hat\phi]$ depends only on quantum and background fields. Note that due to power counting the dependence of $\Xi^{S,(n)}_{\#}[\phi,\hat\phi,\phi^*,\Omega]$ (and thus also of $\Xi^{S,(n)}_{\Omega}[\phi^*,\Omega]$) on ϕ^* and Ω is only linear. For the same reason $\Xi^{S,(n)}_{\Omega}[\phi^*,\Omega]$ does not depend on ϕ and $\hat\phi$. The separation of Ω-dependent and Ω-independent terms in the second line of Eq. (3.23) turns out to be useful as we will see below. In addition, the background gauge invariance of $\Xi^{S,(n)}$ implies that each contribution to the r.h.s. of Eq. (3.23) is a linear combination of invariant polynomials.

On the other hand, the background gauge invariant breaking terms $\hat\Psi^{(n)}_S$ can also be decomposed according to the occurrence of ϕ^* and Ω,

$$
\hat\Psi^{(n)}_S = \hat\Psi^{(n)}_{S,O}[\phi,\hat\phi] + \hat\Psi^{(n)}_{S,\#}[\phi,\hat\phi,\phi^*] + \hat\Psi^{(n)}_{S,\Omega O}[\phi,\hat\phi,\Omega] + \hat\Psi^{(n)}_{S,\Omega\#}[\phi,\hat\phi,\phi^*,\Omega]\,, \tag{3.24}
$$

where the corresponding dependence is again linear.

Our final aim is to re-write the second equation of (3.22) using the decompositions introduced in (3.23) and (3.24). In this respect the following definitions turn out to be useful

$$
X^0_\phi(x) \equiv \left.\frac{\delta\Gamma_0}{\delta\phi(x)}\right|_{\phi^*=0}\,, \qquad X^{\#}_\phi(x) \equiv \frac{\delta\Gamma_0}{\delta\phi(x)} - X^0_\phi(x)\,,
$$

where Γ_0 is the classical reduced functional. X^0_ϕ coincides with the classical equations of motion of the field ϕ and $X^{\#}_\phi$ is the anti-field part of the equations of motion. It is also useful to introduce the compact notation

$$
\begin{aligned}
\sum_\phi \int \mathrm{d}^4x\, \Omega\left(\frac{\delta}{\delta\hat\phi} - \frac{\delta}{\delta\phi}\right) \equiv{}& \int \mathrm{d}^4x \sum_{A=3,\pm,a} \Omega^{\mu A}\left(\frac{\delta}{\delta\hat V^{A\mu}} - \frac{\delta}{\delta V^{A\mu}}\right) \\
&+ \int \mathrm{d}^4x \sum_{i=H,0,\pm} \Omega^i\left(\frac{\delta}{\delta\hat\Phi_i} - \frac{\delta}{\delta\Phi_i}\right).
\end{aligned}
$$

We can then write Eq. (3.22) using Eqs. (2.3), (A.1) and (A.2) in the following schematic way

$$
\sum_\phi \int \mathrm{d}^4x \left[\frac{\delta\Gamma_0}{\delta\phi}\frac{\delta\Xi^{S,(n)}}{\delta\phi^*} + \frac{\delta\Xi^{S,(n)}}{\delta\phi}\frac{\delta\Gamma_0}{\delta\phi^*} + \Omega\left(\frac{\delta}{\delta\hat\phi} - \frac{\delta}{\delta\phi}\right)\Xi^{S,(n)}\right] = \hat\Psi^{(n)}_S\,.
$$

In this equation we insert Eqs. (3.23) and (3.24) and sort the terms according to their dependence on ϕ^* and Ω. This leads to the system of equations

$$
\sum_\phi \int \mathrm{d}^4x \left(s\,\phi\,\frac{\delta}{\delta\phi} + X^{\#}_\phi \frac{\delta}{\delta\phi^*}\right)\Xi^{S,(n)}_{\#,O} = \hat\Psi^{(n)}_{S,\#}\,,
$$

$$\sum_{\phi}\int \mathrm{d}^4x \left[X^{\#}_{\phi}\frac{\delta}{\delta\phi^*}\Xi^{S,(n)}_{\Omega} + \Omega\left(\frac{\delta}{\delta\hat{\phi}} - \frac{\delta}{\delta\phi}\right)\Xi^{S,(n)}_{\#,O}\right] = \hat{\Psi}^{(n)}_{S,\Omega\#}\,,$$

$$\sum_{\phi}\int \mathrm{d}^4x \left[X^{0}_{\phi}\frac{\delta}{\delta\phi^*}\Xi^{S,(n)}_{\Omega} + \Omega\left(\frac{\delta}{\delta\hat{\phi}} - \frac{\delta}{\delta\phi}\right)\Xi^{S,(n)}_{O}\right] = \hat{\Psi}^{(n)}_{S,\Omega O}\,, \qquad (3.25)$$

which can be solved for the counterterms $\Xi^{S,(n)}_{O}$, $\Xi^{S,(n)}_{\#,O}$ and $\Xi^{S,(n)}_{\Omega}$. For completeness we also list the equation for $\hat{\Psi}^{(n)}_{S,O}$ which reads

$$\sum_{\phi}\int \mathrm{d}^4x \left[s\,\phi\,\frac{\delta}{\delta\phi}\Xi^{S,(n)}_{O} + X^{0}_{\phi}\frac{\delta}{\delta\phi^*}\Xi^{S,(n)}_{\#,O}\right] = \hat{\Psi}^{(n)}_{S,O}\,. \qquad (3.26)$$

In principle Eq. (3.26) could be combined with the first equation of (3.25) in order to obtain $\Xi^{S,(n)}_{O}$. However, the resulting expressions become more complicated and it is more advantageous to use only the system (3.25).

In the derivation of Eqs. (3.25) and (3.26) we have used

$$\frac{\delta\Gamma_0}{\delta\Omega} = 0\,, \qquad \frac{\delta\Gamma_0}{\delta\phi^*} = s\,\phi\,, \qquad (3.27)$$

where $s\,\phi$ is the classical BRST transformation of ϕ. The first equation is a consequence of the fact that we used the reduced functional and the second equation in (3.27) immediately follows from (2.1).

At this point some comments on (3.25) are in order. The big advantage of our approach is that conventional STIs obtained by differentiating Eq. (2.3) with respect to one ghost field c^B and some quantum fields are completely avoided. More precisely, it is not necessary to evaluate the breaking terms $\hat{\Psi}^{(n)}_{S,O}[\phi,\hat{\phi}]$ which enormously reduces the calculational effort. The complete set of counterterms $\Xi^{S,(n)}_{O}$ are fixed in terms of background Green functions by the extension of the BRST symmetry obtained by including the background fields. A further simplification is due to the power counting degrees of Ω and their Lorentz properties. The needed STIs are simpler than those obtained by differentiating with respect to a ghost field c^B.

Notice that, although we can completely avoid the STIs for quantum fields, we cannot totally skip the STIs obtained with at least one anti-field, namely the first equation of system (3.25). However, this requires some remarks. The reason for this has to be ascribed to the difference between BRST symmetry and gauge symmetry. Indeed, the anti-field-dependent terms are BRST-invariant and not gauge invariant. This means that in the gauge invariant part, the fields ϕ and $\hat{\phi}$ appear only in the combination $Z_{\phi}\phi + \hat{\phi}$, while in the anti-field-dependent terms this is not valid and, in principle, all the possible combinations could appear (cf. Appendix C). Nevertheless, the background gauge invariance already removes some of the breaking terms $\hat{\Psi}^{(n)}_{S,\#}$, leaving only few free parameters. This will be explicitly shown in the forthcoming section.

At this point we should spend some words on the computation of the breaking terms $\hat{\Psi}^{(n)}_{S,\#}$, $\hat{\Psi}^{(n)}_{S,\Omega O}$ and $\hat{\Psi}^{(n)}_{S,\Omega\#}$ which appear on the r.h.s. of Eq. (3.25). Guided by the structure of the second and third equations of (3.25) it appears useful to consider relations between the Green functions $\Gamma_{\phi_1\phi_2\dots}$ and $\Gamma_{\hat{\phi}_1\phi_2\dots}$ where one quantum field is replaced by the corresponding background field. Such relations, that contain $\hat{\Psi}^{(n)}_{S,\Omega O}$ and $\hat{\Psi}^{(n)}_{S,\Omega\#}$ as breaking term, can be derived as follows:

- Substitute $\hat{\phi}_1$ by Ω_1 (the BRST variation of $\hat{\phi}_1$) and take the derivative of the STI (2.3) with respect to Ω_1 and $\phi_2 \dots$

$$\left.\frac{\delta\mathcal{S}(\Gamma)}{\delta\Omega_1(p_1)\delta\phi_2(p_2)\dots}\right|_{\phi=0} = \Gamma_{\hat{\phi}_1\phi_2\dots} - \Gamma_{\phi_1\phi_2\dots} + \dots = \Delta_{S,\Omega_1\phi_2\dots}\,, \tag{3.28}$$

  where the ellipses contain those terms which are quadratic in Γ.

- Recall that Ω carries a Faddeev-Popov charge +1. Thus on the r.h.s. of Eq. (3.28) only the Green functions with vanishing ghost charge are different from zero. This, together with the Lorentz invariance, selects non-vanishing contributions from the ellipses in (3.28).

Concerning the extraction of identities for $\hat{\Psi}^{(n)}_{S,\#}$ (see the first equation of (3.25)) we refer to the detailed discussion presented in Sections 2.3 and 3.2 of Ref. [7].

The existence of a solution to the system (3.25) is guaranteed by the Wess-Zumino consistency conditions (3.19) and (3.20). The solution is not unique since the Slavnov-Taylor operator $\mathcal{S}_0$ has a non-trivial kernel. This means that there exist background gauge invariant polynomials that satisfy $\mathcal{S}_0\left(\Xi^{N,(n)}\right) = 0$. In addition, also the invariant counterterms $\Xi^{N,(n)}$ can be split into an anti-field-independent part $\Xi^{N,(n)}_O[\phi,\hat{\phi}]$ and an anti-field-dependent one $\Xi^{N,(n)}_\#[\phi,\hat{\phi},\phi^*,\Omega]$. From this decomposition, it is straightforward to see that the anti-field-dependent counterterms $\Xi^{N,(n)}_\#[\phi,\hat{\phi},\phi^*,\Omega]$ parameterize the unphysical normalization conditions (such as wave function and gauge parameter renormalizations). The invariant counterterms $\Xi^{N,(n)}_O[\phi,\hat{\phi}]$ turn out to depend upon linear combinations $\tilde{\phi} = Z_\phi\phi + \hat{\phi}$ (this can be seen by solving the homogeneous system (3.25)). They parameterize the physical normalization conditions that can be employed to match the physical parameters.

Having solved Eqs. (3.25), the result of the algebraic problem (3.4) can be summarized into the following equations:

$$\begin{aligned} \mathrm{I\!\Gamma}^{(n)} &= \Gamma^{(n)} + \Gamma^{CT,(n)}\,, \\ \Gamma^{CT,(n)} &= -T^{\delta_{pc}}\Gamma^{(n)} + \Xi^{W,(n)} + \Xi^{S,(n)} + \Xi^{N,(n)}\,. \end{aligned} \tag{3.29}$$

The last term, $\Xi^{N,(n)}$, is an invariant counterterm and has to be added in order to tune the normalization conditions on physical data.

Before discussing the more complex example of $b \to s\gamma$ in Section 4, we would like to consider as a simple case the gluon two-point function. In this example it is assumed that the breaking terms are already known and background gauge invariant and Eqs. (3.25) are solved using the decomposition introduced in Eqs. (3.23) and (3.24).

The final aim is the computation of the counterterms for the quantum, the background and the corresponding mixed two-point functions denoted by $\Xi^{S,(n)}_{V^a_\mu V^b_\nu}(p)$, $\Xi^{S,(n)}_{\hat{V}^a_\mu \hat{V}^b_\nu}(p)$ and $\Xi^{S,(n)}_{\hat{V}^a_\mu V^b_\nu}(p)$, respectively. In the notation of Eq. (3.23) they correspond to $\Xi^{S,(n)}_O$ which occurs in the last equation in (3.25). Differentiation with respect to Ω^b_ν and $\hat{V}^a_\mu$ or Ω^b_ν and V^a_μ leads to the two equations

$$
\begin{aligned}
-\Gamma_{0,\hat{V}^a_\mu V^c_\rho}(p)\, \Xi^{S,(n)}_{V^{*,c}_\rho \Omega^b_\nu}(p) + \Xi^{S,(n)}_{\hat{V}^a_\mu \hat{V}^b_\nu}(p) - \Xi^{S,(n)}_{\hat{V}^a_\mu V^b_\nu}(p) &= \hat{\Psi}^{(n)}_{S,\Omega^b_\nu \hat{V}^a_\mu}(-p)\,, \\
-\Gamma_{0,V^a_\mu V^c_\rho}(p)\, \Xi^{S,(n)}_{V^{*,c}_\rho \Omega^b_\nu}(p) + \Xi^{S,(n)}_{V^a_\mu \hat{V}^b_\nu}(p) - \Xi^{S,(n)}_{V^a_\mu V^b_\nu}(p) &= \hat{\Psi}^{(n)}_{S,\Omega^b_\nu V^a_\mu}(-p)\,,
\end{aligned} \tag{3.30}
$$

which contain four unknown counterterms. These equations can be used to determine $\Xi^{S,(n)}_{V^a_\mu V^b_\nu}(p)$ and $\Xi^{S,(n)}_{\hat{V}^a_\mu V^b_\nu}(p)$, as $\Xi^{S,(n)}_{\hat{V}^a_\mu \hat{V}^b_\nu}(p)$ is fixed by normalization conditions for physical parameters [9]. In addition they are transverse as a consequence of the WTI. Furthermore, the counterterm $\Xi^{S,(n)}_{V^{*,c}_\rho \Omega^b_\nu}(p)$ is constrained by the second equation of (3.25). In particular, differentiating with respect to c^c, Ω^a_μ and $V^{*,b}_\nu$ gives

$$
-\Gamma_{0,V^d_\rho V^{*,b}_\nu c^c}(p,q)\, \Xi^{S,(n)}_{V^{*,d}_\rho \Omega^a_\mu}(-p-q) - \Xi^{S,(n)}_{V^a_\mu V^{*,b}_\nu c^c}(p,q) + \Xi^{S,(n)}_{\hat{V}^a_\mu V^{*,b}_\nu c^c}(p,q) = \hat{\Psi}^{(n)}_{S,\,\Omega^a_\mu V^{*,b}_\nu c^c}(p,q)\,. \tag{3.31}
$$

This equation contains the new unknown terms $\Xi^{S,(n)}_{\hat{V}^a_\mu V^{*,b}_\nu c^c}$ and $\Xi^{S,(n)}_{V^a_\mu V^{*,b}_\nu c^c}$. The latter also occurs in the first equation of (3.25) after taking the derivatives with respect to c^a, c^b and $V^{*,c}_\nu$

$$
\begin{aligned}
-\Gamma_{0,c^a V^{*,d}_\rho}(p+q)\Xi^{S,(n)}_{V^d_\rho V^{*,c}_\nu c^b}(q,p) + \Gamma_{0,c^b V^{*,d}_\rho}(-p)\Xi^{S,(n)}_{V^d_\rho V^{*,c}_\nu c^a}(q,-p-q) \\
- \Gamma_{0,c^a c^b c^{*,d}}(p,q)\Xi^{S,(n)}_{c^d V^{*,c}_\nu}(q) - \Gamma_{0,V^d_\rho V^{*,c}_\nu c^b}(q,p)\Xi^{S,(n)}_{c^a V^{*,d}_\rho}(p+q) \\
+ \Gamma_{0,V^d_\rho V^{*,c}_\nu c^a}(q,-p-q)\Xi^{S,(n)}_{c^b V^{*,d}_\rho}(-p) - \Gamma_{0,c^d V^{*,c}_\nu}(q)\Xi^{S,(n)}_{c^a c^b c^{*,d}}(p,q) &= \hat{\Psi}^{(n)}_{S,c^a c^b V^{*,c}_\nu}(p,q)\,.
\end{aligned} \tag{3.32}
$$

Thus, after exploiting (3.25) we are left with the counterterms $\Xi^{S,(n)}_{\hat{V}^a_\mu V^{*,b}_\nu c^c}$, $\Xi^{S,(n)}_{c^a V^{*,d}_\rho}$ and $\Xi^{S,(n)}_{c^a c^b c^{*,d}}$ which are not fixed by the Eqs. (3.30), (3.31) and (3.32). However, due to background gauge invariance the counterterms $\Xi^{S,(n)}_{\hat{V}^a_\mu V^{*,b}_\nu c^c}$ and $\Xi^{S,(n)}_{c^a V^{*,d}_\rho}$ are not independent, rather one has $ip_\nu \Xi^{S,(n)}_{\hat{V}^a_\mu V^{*,b}_\nu c^c} = f^{dbc}\Xi^{S,(n)}_{c^a V^{*,d}_\mu}$. Furthermore, $\Xi^{S,(n)}_{c^a V^{*,d}_\rho}$ and $\Xi^{S,(n)}_{c^a c^b c^{*,d}}$ are fixed via normalization conditions, namely the wave function renormalization for the quantum gauge field V^a_μ and the one for the ghost field c^a, respectively[6]. Thus all counterterms appearing in the computation of vector boson two-point functions are determined.

Notice that Eqs. (3.30), (3.31) and (3.32) are quite simple and they can easily be solved in terms of $\Xi^{(n)}$. Furthermore, once the breaking terms, which are expressed in terms of vacuum

[6] See also Section 4 below.

integrals, are computed also the individual coefficients of the counterterms (see Appendix C, Eq. (C.6)) are available. In this context we want to stress again that the counterterms $\Xi^{S,(n)}_{V^a_\mu V^b_\nu}(p)$, $\Xi^{S,(n)}_{\hat{V}^a_\mu \hat{V}^b_\nu}(p)$ and $\Xi^{S,(n)}_{\hat{V}^a_\mu V^b_\nu}(p)$ have to be background gauge invariant.

4 Application

Based on the theoretical analysis of the previous section, we consider in this section the important process $b \to s\gamma$ (see, e.g., Ref. [24] where the algebraic method has been applied to $b \to s\gamma$ or Refs. [25, 26] for recent summaries on QCD and electroweak corrections, respectively.) and formulate a strategy for computing the corresponding amplitudes at two-loop order independent of any regularization. This amplitude is used as an illustrating example in order to show how the complete procedure works in detail. We assume that all symmetry identities which are involved are broken and have to be restored. However, we stress again that our procedure is independent of the regularization and a suitable choice allows for many additional simplifications within our approach. Indeed, it is easy to see that many breaking terms discussed in the following do not occur in a reasonable non-invariant regularization.

Since at the classical level there are no flavour changing neutral currents (FCNC) in the SM, the amplitude $\Gamma_{\hat{A}_\mu \bar{s} b}$ vanishes at tree level. To compute it up to two loops, one has to take into account only one-loop counterterms as the two-loop amplitude — even if superficially divergent — is directly related by the linear WTIs to a superficially convergent amplitude, which is completely identified once the one-loop counterterms are known. If one had an invariant regularization, the two-loop amplitude would appear superficially convergent. However, in the absence of a non-invariant regularization, the amplitude might turn out to be divergent. Therefore, there remains the problem of computing the one-loop non-invariant counterterms in an efficient way.

For the physical decay rate $b \to s\gamma$, in principle, only the form factor of the magnetic moment has to be computed. The discussion in this section, however, is kept more general and the complete $\hat{A}\bar{b}s$ vertex is considered.

From the topological structure of the two-loop diagrams contributing to the amplitude $\Gamma^{(2)}_{\hat{A}_\mu b \bar{s}}$, it is evident that the 1PI three- and four-point functions with external gauge or scalar quantum fields (e.g. $\Gamma^{(1)}_{V^\alpha_\mu V^\beta_\nu V^\gamma_\rho}$ or $\Gamma^{(1)}_{V^\alpha_\mu V^\beta_\nu V^\gamma_\rho V^\delta_\sigma}$) or with ghost fields (e.g. $\Gamma^{(1)}_{V^\alpha_\mu \bar{c}^\beta c^\gamma}$) do not appear as one-loop sub-graphs. Actually, the renormalization of sub-divergences with more than two (gauge or scalar) quantum fields enters the BFM calculations only starting from the three-loop order. Thus, we only have to consider the three-point functions

$$\begin{aligned}
&\Gamma^{(1)}_{\hat{A}_\mu \bar{s} b}(p_s, p_b)\,, \quad \Gamma^{(1)}_{\hat{A}_\mu \bar{q}_2 q_1}(p_2, p_1)\,, \quad \Gamma^{(1)}_{\hat{A}_\mu W^+_\nu W^-_\rho}(p_+, p_-)\,, \quad \Gamma^{(1)}_{\hat{A}_\mu G^+ G^-}(p_+, p_-)\,, \\
&\Gamma^{(1)}_{\hat{A}_\mu G^\pm W^\mp_\nu}(p_\pm, p_\mp)\,, \quad \Gamma^{(1)}_{\hat{A}_\mu \bar{c}^\pm c^\mp}(p_\pm, p_\mp)\,, \quad \Gamma^{(1)}_{W^\pm_\mu \bar{q}_2 q_1}(p_2, p_1)\,, \quad \Gamma^{(1)}_{G^\pm \bar{q}_2 q_1}(p_2, p_1)\,,
\end{aligned} \tag{4.1}$$

and the two-point functions

$$\begin{aligned} &\mathrm{I\!\Gamma}^{(1)}_{\bar{s}b}(p_b)\,, \qquad \mathrm{I\!\Gamma}^{(1)}_{\bar{q}_2 q_1}(p_1)\,, \qquad \mathrm{I\!\Gamma}^{(1)}_{W^+_\nu W^-_\rho}(p_-)\,, \\ &\mathrm{I\!\Gamma}^{(1)}_{G^+G^-}(p_-)\,, \;\; \mathrm{I\!\Gamma}^{(1)}_{G^\pm W^\mp_\mu}(p_\mp)\,, \;\; \mathrm{I\!\Gamma}^{(1)}_{\bar{c}^\pm c^\mp}(p_\mp)\,. \end{aligned} \tag{4.2}$$

In Eqs. (4.1) and (4.2) q_1 and q_2 are two generic quark fields. Notice that the Green functions $\mathrm{I\!\Gamma}_{\hat{A}_\mu G^+ W^-_\nu}$ and $\mathrm{I\!\Gamma}_{\hat{A}_\mu W^+_\nu G^-}$ have no tree-level contribution (see, e.g., [9]). At the one-loop level, however, contributions may appear as soon as a non-invariant regularization scheme is used.

For the computation we follow the strategy outlined in Section 3.3. In a first step we exploit the WTIs for the background fields and fix all possible counterterms. In a second step the remaining counterterms are determined by STIs and, finally, we tune the free parameters of the theory to match the normalization conditions. The complete set of counterterms, be they used to restore the WTIs (Ξ^W) or the STIs (Ξ^S), or to implement the normalization conditions (Ξ^N), can be separated into anti-field-dependent counterterms $\Xi_\#[\phi, \hat{\phi}, \phi^*, \Omega]$ and anti-field-independent ones $\Xi_O[\phi, \hat{\phi}]$ (cf. Eq. (3.23)) according to the ghost number. In principle the former should be determined first as can be deduced from the triangular structure of Eq. (3.25). However, for pedagogical purposes, we discuss the renormalization starting directly from the analysis of the amplitudes (4.1) and (4.2), i.e. from the last equation in (3.25). The anti-field-dependent counterterms turn out to be necessary along the discussion. Moreover, as mentioned in Section 3.2, the intermediate zero-momentum subtraction is applied to simplify the computation.

In order to provide a guidance for the reader we briefly outline the rest of this section. We start with the WTIs containing those Green functions of our list (see Eqs. (4.1) and (4.2)) involving background fields and determine the corresponding counterterms. In our case this means that we have to consider the identities involving Green functions with an external background photon. In the corresponding equations two-point Green functions with external quantum fields appear. According to our procedure this means that in a first step the WTIs for the background two-point functions have to be considered (cf. point 2 below) and afterwards the STIs for the quantum counterparts (cf. point 3) are investigated. This completes the determination of the counterterms for the background three-point functions in Eq. (4.1). In a next step (cf. point 4) the quantum three-point functions of (4.1) are discussed. To keep the discussion simple we will postpone the treatment of those Green functions that do not contribute to the counterterms of the quantum fields in points 1 to 4. The corresponding identities are discussed in point 5.

1. Background three-point functions[7]. The main goal of our analysis is to obtain the counterterm for the amplitude $\Gamma_{\hat{A}_\mu \bar{s}b}$ that constitutes the central object in the process $b \to s\gamma$. In this part we discuss the one-loop amplitudes involving the background photon $\hat{A}$ and two fermions or gauge bosons, respectively.

[7]In the following the reduced functional discussed briefly at the end of Section 2 and in Appendix B. This slightly simplifies the forthcoming equations.

The WTI which contains the amplitude $\Gamma^{(1)}_{\hat{A}_\mu \bar{q}_2 q_1}$ reads (cf. [7]):

$$\left.\frac{\delta^3 \mathcal{W}_{(\lambda)}(\Gamma^{(1)})}{\delta\lambda_A(-p_1-p_2)\delta\bar{q}_2(p_2)\delta q_1(p_1)}\right|_{\phi=0} =$$
$$i\,(p_2+p_1)^\mu\,\Gamma^{(1)}_{\hat{A}_\mu \bar{q}_2 q_1}(p_2,p_1) + ieQ_q\left(\Gamma^{(1)}_{\bar{q}_2 q_1}(p_1) - \Gamma^{(1)}_{\bar{q}_2 q_1}(-p_2)\right) \;=\; \Delta_{W,\lambda_A \bar{q}_2 q_1}\,, \qquad (4.3)$$

where Q_q is the common charge of the quarks q_1 and q_2. The equation for $\Gamma^{(1)}_{\hat{A}_\mu \bar{s} b}$ is obtained by the replacements $q_2 \to s$ and $q_1 \to b$. As the breaking term $\Delta^W_{\lambda_A \bar{q}_2 q_1}$ has mass dimension one we remove them by acting with $(1-T^1_{p_1,p_2})$. The resulting Green functions

$$\hat{\Gamma}^{(1)}_{\hat{A}_\mu \bar{q}_2 q_1}(p_2,p_1) \;=\; (1-T^0_{p_1,p_2})\Gamma^{(1)}_{\hat{A}_\mu \bar{q}_2 q_1}(p_2,p_1)\,, \qquad \hat{\Gamma}^{(1)}_{\bar{q}_2 q_1}(p) \;=\; (1-T^1_p)\Gamma^{(1)}_{\bar{q}_2 q_1}(p)\,, \quad (4.4)$$

automatically satisfy the WTI, which means that the counterterm $\Xi^{W,(1)}[\phi,\hat{\phi}]$ is zero. However, we still have the freedom to impose normalization conditions for the quark self-energies. In particular, we can add the (BRST and background gauge invariant) counterterms

$$\Xi^{N,(1)}_1 \;=\; \sum_{\psi\psi'} \int \mathrm{d}^4x \Big(\xi^{N,(1)}_{\bar{\psi}\nabla\psi'} \bar{\psi}^I \;\not{\nabla}_{IJ}\psi'^{,J} + \xi^{N,(1)}_{\bar{\psi}\psi'\Phi,m} Y^{i,IJ}_m (\Phi+\hat{\Phi}+v)_i \bar{\psi}_I \psi'_J + \text{h.c.}\Big)\,, \quad (4.5)$$

where $\xi^{N,(1)}_{\bar{\psi}\nabla\psi'}$ and $\xi^{N,(1)}_{\bar{\psi}\psi'\Phi,m}$ are the coefficients of the counterterms and $Y^{i,IJ}_m$ are invariant tensors of the fermion and scalar representations (cf. Appendix C, Eq. (C.9)). v^i is the vacuum expectation value. The free parameters are fixed by normalization conditions for the CKM matrix elements[8] and by quark mass renormalizations [28, 29, 27] (see also [10] for a complete discussion of normalization conditions within the BFM framework). In particular, the diagonal part in Eq. (4.5) can be written as

$$\Xi^{N,(1)}_{\bar{q}q}(p) \;=\; \xi^{(1)}_{2,q}\,(\not{p} - m_q) + \xi^{(1)}_q m_q\,.$$

According to Eq. (3.29) the symmetrical three-point function reads

$$\begin{aligned}\mathrm{I\!\Gamma}^{(1)}_{\hat{A}_\mu \bar{q}_2 q_1}(p_2,p_1) \;&=\; \Gamma^{(1)}_{\hat{A}_\mu \bar{q}_2 q_1}(p_2,p_1) + \Gamma^{(1),CT}_{\hat{A}_\mu \bar{q}_2 q_1}(p_2,p_1)\\ &=\; \Gamma^{(1)}_{\hat{A}_\mu \bar{q}_2 q_1}(p_2,p_1) - T^0_{p_1,p_2}\Gamma^{(1)}_{\hat{A}_\mu \bar{q}_2 q_1}(p_2,p_1) - \frac{eQ_q}{2}\left(\xi^{(1)}_{2,q_1} + \xi^{(1)}_{2,q_2}\right)\gamma^\mu\,, \quad (4.6)\end{aligned}$$

and analogously for $\mathrm{I\!\Gamma}^{(1)}_{\bar{q}_2 q_1}$.

Next to the fermionic three-point functions also the bosonic ones of (4.1) have to be considered. The corresponding WTIs read (see, e.g., [7])

$$\left.\frac{\delta^3 \mathcal{W}_{(\lambda)}(\Gamma^{(1)})}{\delta\lambda_A(-p_+-p_-)\delta W^+_\rho(p_+)\delta W^-_\sigma(p_-)}\right|_{\phi=0} =$$

[8]Notice that the WTI also implies special normalization conditions for the CKM elements as described in [27].

$$
\begin{aligned}
& i\,(p_+ + p_-)^\mu\, \Gamma^{(1)}_{\hat{A}_\mu W^+_\rho W^-_\sigma}(p_+,p_-) - ie\left(\Gamma^{(1)}_{W^+_\rho W^-_\sigma}(p_-) - \Gamma^{(1)}_{W^+_\rho W^-_\sigma}(-p_+)\right) = \Delta^{(1)}_{W,\hat{A}_\mu W^+_\rho W^-_\sigma}\,, \\
& \left.\frac{\delta^3 \mathcal{W}_{(\lambda)}(\Gamma^{(1)})}{\delta\lambda_A(-p_+-p_-)\delta G^+(p_+)\delta G^-(p_-)}\right|_{\phi=0} = \\
& i\,(p_+ + p_-)^\mu\, \Gamma^{(1)}_{\hat{A}_\mu G^+ G^-}(p_+,p_-) - ie\left(\Gamma^{(1)}_{G^+G^-}(p_-) - \Gamma^{(1)}_{G^+G^-}(-p_+)\right) = \Delta^{(1)}_{W,\hat{A}_\mu G^+ G^-}\,, \\
& \left.\frac{\delta^3 \mathcal{W}_{(\lambda)}(\Gamma^{(1)})}{\delta\lambda_A(-p_+-p_-)\delta W^+_\rho(p_+)\delta G^-(p_-)}\right|_{\phi=0} = \\
& i\,(p_+ + p_-)^\mu\, \Gamma^{(1)}_{\hat{A}_\mu W^+_\rho G^-}(p_+,p_-) - ie\left(\Gamma^{(1)}_{W^+_\rho G^-}(p_-) - \Gamma^{(1)}_{W^+_\rho G^-}(-p_+)\right) = \Delta^{(1)}_{W,\hat{A}_\mu W^+_\rho G^-}\,, \qquad (4.7)
\end{aligned}
$$

where we have omitted their hermitian counterparts. Acting with $(1 - T^2_{p_+,p_-})$ removes all the breaking terms on the r.h.s.. However, the two-point (quantum) Green functions in (4.7) are not yet completely fixed as there is still freedom to add background gauge invariant counterterms arising from STIs (Ξ^S) and normalization conditions (Ξ^N). In order to determine the counterterms Ξ^S we first have to consider the WTIs for the two-point Green functions with external background fields (see point 2) as described in Section 3.3. In a next step the Eqs. (3.25) are solved step-by-step and, finally, the normalization conditions are implemented.

Note that due to the linearity of the WTIs the equations of this section are not restricted to the one-loop order but have the same form at any order.

2. Background two-point functions. Before treating the quantum two-point functions of (4.2) we have to deal with the corresponding background counterparts. They are constrained by the WTIs

$$
\begin{aligned}
ip_\mu \Gamma^{(1)}_{\hat{W}^+_\mu \hat{W}^-_\nu}(p) + iM_W \Gamma^{(1)}_{\hat{G}^+ \hat{W}^-_\nu}(p) &= \Delta^{(1)}_{W,\lambda^+ \hat{W}^-_\mu}\,, \\
ip_\mu \Gamma^{(1)}_{\hat{W}^+_\mu \hat{G}^-}(p) + iM_W \Gamma^{(1)}_{\hat{G}^+ \hat{G}^-}(p) &= \Delta^{(1)}_{W,\lambda^+ \hat{G}^-}\,. \qquad (4.8)
\end{aligned}
$$

The r.h.s. of Eqs.(4.8) have mass dimension three as can be seen by power-counting arguments. Therefore we have to act from the left with the Taylor operator $(1 - T^3_p)$ which leads to

$$
\begin{aligned}
\Psi^{W,(1)}_{\lambda_+ \hat{W}^-_\mu} &= ip_\mu \hat{\Gamma}^{(1)}_{\hat{W}^+_\mu \hat{W}^-_\nu}(p) + iM_W \hat{\Gamma}^{(1)}_{\hat{G}^+ \hat{W}^-_\nu}(p) = iM_W \frac{1}{3!} p^\mu p^\rho p^\sigma \partial_{p^\mu}\partial_{p^\rho}\partial_{p^\sigma} \Gamma^{(1)}_{\hat{G}^+ \hat{W}^-_\nu}(p)\Big|_{p=0}\,, \\
\Psi^{W,(1)}_{\lambda_+ \hat{G}^-} &= ip_\mu \hat{\Gamma}^{(1)}_{\hat{W}^+_\mu \hat{G}^-}(p) + iM_W \hat{\Gamma}^{(1)}_{\hat{G}^+ \hat{G}^-}(p) = 0\,. \qquad (4.9)
\end{aligned}
$$

As a consequence, only the counterterm

$$
\Xi^{W,(1)}_1 = \int d^4x \left[\xi^{W,(1)}_{\partial W^2,1}\, \partial^\mu \hat{W}^+_\mu \partial^\nu \hat{W}^-_\nu + \xi^{W,(1)}_{\partial W^2,2}\, \partial^\mu \hat{W}^+_\nu \partial^\mu \hat{W}^-_\nu\right]\,, \qquad (4.10)
$$

is needed to restore the WTIs (4.8). Notice that, in the case where the used renormalization scheme is invariant under background gauge symmetry, the breaking terms (4.9) vanish and, therefore, only the STIs (which are discussed in point 3) have to be restored.

As in the previous section, the same WTIs hold also here to all loop orders, which is due to the linearity of the equations.

3. Quantum two-point functions. One of the main new features of the method presented in this paper is the use of a particular set of STIs as derived in Eqs. (3.25). In the example presented at the end of Section 3.3 the last equation of (3.25) applied to the two-point functions is shown in (3.30). In order to get the breaking terms on the r.h.s. one has to proceed as indicated after (3.25). In our case the differentiation of (2.3) with respect to $\Omega^\pm$ and $W^\mp$ or $G^\mp$ has to be performed. Altogether this leads to eight equations, which, however, all have the same form. For demonstration we only present one of them:

$$\frac{\delta^2\mathcal{S}(\Gamma)}{\delta\Omega^+_\nu(-p)\delta W^-_\mu(p)}\Big|_{\phi=0} = \\ \Gamma_{\hat{W}^+_\nu W^-_\mu} - \left(g_{\nu\rho} + \Gamma_{W^{*,-}_\rho \Omega^+_\nu}\right)\Gamma_{W^+_\rho W^-_\mu} + \Gamma_{\Omega^+_\nu G^{*,-}}\Gamma_{G^+ W^-_\mu} = \Delta^S_{\Omega^+_\nu W^-_\mu}\,, \tag{4.11}$$

where the dependence on the external momenta is suppressed. For our purposes only the one-loop approximation of these equations are needed[9]. Furthermore, it is useful to combine two equations in such a way that the two-point functions of a background and a quantum field drop out. After zero-momentum subtraction of the form $(1-T^2_p)$ one obtains

$$\begin{aligned}
&\hat{\Gamma}^{(1)}_{\hat{W}^+_\nu \hat{W}^-_\mu} - \hat{\Gamma}^{(1)}_{W^+_\nu W^-_\mu} \\
&\quad - \hat{\Gamma}^{(1)}_{W^{*,+}_\rho \Omega^-_\mu}\Gamma^{(0)}_{W^+_\nu W^-_\rho} - \hat{\Gamma}^{(1)}_{W^{*,-}_\rho \Omega^+_\nu}\Gamma^{(0)}_{W^-_\mu W^+_\rho} - \hat{\Gamma}^{(1)}_{G^{*,+}\Omega^-_\mu}\Gamma^{(0)}_{G^- W^+_\nu} - \hat{\Gamma}^{(1)}_{G^{*,-}\Omega^+_\nu}\Gamma^{(0)}_{G^+ W^-_\mu} = \hat{\Psi}^{S,(1)}_{1,\nu\mu}\,, \\
&\hat{\Gamma}^{(1)}_{\hat{W}^+_\nu \hat{G}^-} - \hat{\Gamma}^{(1)}_{W^+_\nu G^-} \\
&\quad - \hat{\Gamma}^{(1)}_{W^{*,+}_\rho \Omega^-}\Gamma^{(0)}_{W^+_\nu W^-_\rho} - \hat{\Gamma}^{(1)}_{G^{*,-}\Omega^+_\nu}\Gamma^{(0)}_{G^+ G^-} - \hat{\Gamma}^{(1)}_{W^{*,-}_\rho \Omega^+_\nu}\Gamma^{(0)}_{G^- W^+_\rho} - \hat{\Gamma}^{(1)}_{G^{*,+}\Omega^-}\Gamma^{(0)}_{G^- W^+_\nu} = \hat{\Psi}^{S,(1)}_{2,\nu}\,, \\
&\hat{\Gamma}^{(1)}_{\hat{G}^+ \hat{G}^-} - \hat{\Gamma}^{(1)}_{G^+ G^-} \\
&\quad - \hat{\Gamma}^{(1)}_{W^{*,+}_\rho \Omega^-}\Gamma^{(0)}_{G^+ W^-_\rho} - \hat{\Gamma}^{(1)}_{W^{*,-}_\rho \Omega^+}\Gamma^{(0)}_{G^- W^+_\rho} - \hat{\Gamma}^{(1)}_{G^{*,+}\Omega^-}\Gamma^{(0)}_{G^- G^+} - \hat{\Gamma}^{(1)}_{G^{*,-}\Omega^+}\Gamma^{(0)}_{G^+ G^-} = \hat{\Psi}^{S,(1)}_{3}\,,
\end{aligned} \tag{4.12}$$

where the equation containing $\hat{\Gamma}^{(1)}_{W^-_\nu G^+}$ is not shown. The breaking terms are given by

$$\begin{aligned}
\Psi^{S,(1)}_{1,\nu\mu} &= -i\left(M^2_W p^\rho p^\sigma \partial_{p^\rho}\partial_{p^\sigma}\Gamma^{(1)}_{W^{*,-}_\mu \Omega^+_\nu}(p) + 2M_W p_\mu p^\sigma \partial_{p^\sigma}\Gamma^{(1)}_{G^{*,-}\Omega^+_\nu}(p)\right)\Big|_{p=0}\,, \\
\Psi^{S,(1)}_{2,\nu} &= -iM^2_W p^\sigma \partial_{p^\sigma}\Gamma^{(1)}_{W^{*,+}_\nu \Omega^-}(p)\Big|_{p=0}\,, \\
\Psi^{S,(1)}_{3} &= 2iM_W p^\nu p^\sigma \partial_{p^\sigma}\Gamma^{(1)}_{W^{*,+}_\nu \Omega^-}(p)\Big|_{p=0}\,,
\end{aligned} \tag{4.13}$$

where the properties of the Green functions under Hermitian conjugation have been used. Note again that the breaking terms in Eq. (4.13) are finite and thus do not depend on the regularization.

[9]Note that, one can take into account that $\Gamma^{(0)}_{\hat{W}^+_\nu W^-_\mu} = \Gamma^{(0)}_{\hat{W}^+_\nu \hat{W}^-_\mu} = \Gamma^{(0)}_{W^+_\nu W^-_\mu}$ and analogously for WG and GG Green functions and that the Green functions involving Ω and an anti-field vanish at tree level.

The breaking terms can be absorbed by counterterms for $\hat{\Gamma}^{(1)}_{W^+_\nu W^-_\mu}$ and $\hat{\Gamma}^{(1)}_{G^+G^-}$ of the form

$$\begin{aligned}\Xi^{S,(1)}_1 &= \int d^4x \Big[\xi^{S,(1)}_{\nabla V^2,1}\,\hat\nabla^\mu W^+_\mu \hat\nabla^\nu W^-_\nu + \xi^{S,(1)}_{\nabla V^2,2}\,\hat\nabla^\mu W^+_\nu \hat\nabla^\mu W^-_\nu \\ &+ \xi^{S,(1)}_{\nabla\Phi^2}\,\hat\nabla_\mu G^+ \hat\nabla^\mu G^- + \Big(\xi_{\Phi^2,1} + \xi_{\Phi^2\hat\Phi^2,1} v^2\Big) G^+G^- \\ &+ \xi^{S,(1)}_{\nabla\Phi V\Phi}\Big(\hat\nabla^\mu W^+_\mu G^- + \hat\nabla^\mu W^-_\mu G^+\Big)\Big]\,, \end{aligned} \tag{4.14}$$

which can be extracted from background gauge invariant counterterms of Eqs. (C.6), (C.7) and (C.8) in Appendix C.

For the practical computation of the counterterm of Eq. (4.14) we now use (3.25). Applied to two-point functions the third equation can already be found in (3.30). Adding the two equations leads in our case to

$$\begin{aligned}\Xi^{S,(1)}_{1,\hat W^+_\nu \hat W^-_\mu} - \Xi^{S,(1)}_{1,W^+_\nu W^-_\mu} - \Big(\Xi^{S,(1)}_{W^{*,+}_\rho\Omega^-_\mu}\Gamma_{0,W^+_\mu W^-_\rho} + \Xi^{S,(1)}_{W^{*,-}_\rho\Omega^+_\nu}\Gamma_{0,W^-_\mu W^+_\rho}\Big) \\ - \Big(\Xi^{S,(1)}_{G^{*,+}\Omega^-_\mu}\Gamma_{0,G^-W^+_\nu} + \Xi^{S,(1)}_{G^{*,-}\Omega^+_\nu}\Gamma_{0,G^+W^-_\mu}\Big) \;=\; \Psi^{S,(1)}_{1,\nu\mu}\,. \end{aligned} \tag{4.15}$$

Similar equations are obtained for $\Xi^{S,(1)}_{1,W^+_\nu G^-}$ and $\Xi^{S,(1)}_{1,G^+G^-}$ where $\Psi^{S,(1)}_{2,\nu}$ and $\Psi^{S,(1)}_3$ appear on the r.h.s., respectively.

Before explicitly computing the coefficients $\xi^{S,(n)}_i$ in Eq. (4.14), we have to determine the counterterms $\Xi^{S,(1)}_{W^{*,+}_\rho\Omega^-_\mu}$ and $\Xi^{S,(1)}_{G^{*,+}\Omega^-_\mu}$ and perform the renormalization of the Green functions $\hat\Gamma^{(1)}_{W^{*,+}_\rho\Omega^-_\mu}$ and $\hat\Gamma^{(1)}_{G^{*,+}\Omega^-_\mu}$. This will be discussed in point 5 below. The renormalization of background Green functions in Eqs. (4.12) is given by the WTIs (4.8).

Concerning the normalization conditions we are allowed to add the counterterm

$$\begin{aligned}\Xi^{N,(1)}_2 &= \int d^4x \Bigg[-\frac{\xi^{N,(1)}_{F^2}}{4}\Big(F^\alpha_{\mu\nu}(V+\hat V)F^{\mu\nu}_\alpha(V+\hat V)\Big) \\ &+ \frac{\xi^{N,(1)}_{\nabla\Phi^2}}{2}\nabla_\mu(\Phi+\hat\Phi+v)^i\nabla^\mu(\Phi+\hat\Phi+v)_i\Bigg]\,. \end{aligned}$$

The coefficients $\xi^{N,(1)}_{F^2}$ and $\xi^{N,(1)}_{\nabla\Phi^2}$ are tuned in order to fix the mass of the W boson, M_W, and the weak mixing angle c_W [9]. Notice that the $\xi^{N,(1)}_{F^2}$ and $\xi^{N,(1)}_{\nabla\Phi^2}$ can be expressed as a combination of coefficients in Eqs. (C.6), (C.7), and (C.8), by requiring the BRST symmetry.

This discussion completes the renormalization of the background three-point functions. In the list of contributing Green functions, cf. Eqs. (4.1) and (4.2), only the quantum three-point functions and the ones involving ghosts are missing. They will be treated in points 4 and 5.

4. Quantum three-point functions. In this paragraph we consider the Green function of (4.1) involving two fermions and a W or a Goldstone boson. According to our procedure we again have to consider the corresponding background Green functions first.

The background amplitude belonging to $\Gamma^{(1)}_{W^+_\nu \bar{q}b}$ satisfies the one-loop identity of Eq. (3.15). The breaking and the counterterms are given by Eqs. (3.16) and (3.17), respectively. Analogous equations hold for the Green function $\Gamma^{(1)}_{W^+_\nu \bar{s}q}$ where (b,q) is replaced by (q,s). Note that the counterterm $\Xi_2^{W,(1)}$ has no terms involving a Goldstone-fermion vertex. Such contributions vanish through the zero momentum subtraction. The one-loop coefficients $\xi^{W,(1)}_{\bar{q}bW,L/R}$ have been explicitly computed in [11] in the case of QCD corrections. Note that the quark two-point functions have already been fixed by normalization conditions (cf. Eq. (4.5)).

The STIs for the amplitudes $\Gamma^{(1)}_{W^+_\nu \bar{q}b}$ and $\Gamma^{(1)}_{G^+\bar{q}b}$ which correspond to the last equation of (3.25) are given by

$$\begin{aligned}
&\hat{\Gamma}^{(1)}_{\hat{W}^+_\nu \bar{q}b}(p_q,p_b) - \hat{\Gamma}^{(1)}_{W^+_\nu \bar{q}b}(p_q,p_b) + \hat{\Gamma}^{(1)}_{\Omega^+_\nu W^{*,-}_\rho}(p_q+p_b)\Gamma^{(0)}_{W^+_\rho \bar{q}b} \\
&\quad + \hat{\Gamma}^{(1)}_{\Omega^+_\nu G^{*,-}}(p_q+p_b)\Gamma^{(0)}_{\hat{G}^+\bar{q}b} - \Gamma^{(0)}_{\bar{q}q'}(-p_q)\Gamma^{(1)}_{\Omega^+_\nu \bar{q}'^* b}(p_q,p_b) - \Gamma^{(1)}_{\Omega^+_\nu \bar{q}q'^*}(p_q,p_b)\Gamma^{(0)}_{\bar{q}'b}(p_b) \;=\; \Psi^{S,(1)}_{\Omega^+_\nu \bar{q}b}\,, \\
&\hat{\Gamma}^{(1)}_{\hat{G}^+\bar{q}b}(p_q,p_b) - \hat{\Gamma}^{(1)}_{G^+\bar{q}b}(p_q,p_b) + \hat{\Gamma}^{(1)}_{\Omega^+ W^{*,-}_\rho}(p_q+p_b)\Gamma^{(0)}_{W^+_\rho \bar{q}b} \\
&\quad + \hat{\Gamma}^{(1)}_{\Omega^+ G^{*,-}}(p_q+p_b)\Gamma^{(0)}_{\hat{G}^+\bar{q}b} - \Gamma^{(0)}_{\bar{q}q'}(-p_q)\Gamma^{(1)}_{\Omega^+ \bar{q}'^* b}(p_q,p_b) - \Gamma^{(1)}_{\Omega^+ \bar{q}q'^*}(p_q,p_b)\Gamma^{(0)}_{\bar{q}'b}(p_b) \;=\; \Psi^{S,(1)}_{\Omega^+ \bar{q}b}\,,
\end{aligned} \tag{4.16}$$

where zero momentum subtraction has already been applied. They are obtained by considering the derivatives of Eq. (2.3) with respect to $\bar{q}$, b and Ω^+_ν or Ω^+. The breaking terms are given by

$$\begin{aligned}
\Psi^{S,(1)}_{\Omega^+_\nu \bar{q}b} &= i\left(m_q \Gamma^{(1)}_{\Omega^+_\nu \bar{q}^* b}(0,0) + m_b \Gamma^{(1)}_{\Omega^+_\nu \bar{q}b^*}(0,0)\right)\,, \\
\Psi^{S,(1)}_{\Omega^+ \bar{q}b} &= i\left(m_q \Gamma^{(1)}_{\Omega^+ \bar{q}^* b}(0,0) + m_b \Gamma^{(1)}_{\Omega^+ \bar{q}b^*}(0,0)\right)\,,
\end{aligned} \tag{4.17}$$

where the Green functions on the r.h.s. are finite. They are removed by introducing a counterterm for the quantum fields (cf. Eq. (C.9))

$$\begin{aligned}
\Xi_2^{S,(1)} &= \int \mathrm{d}^4x \left[\xi^{S,(1)}_{\bar{q}bW,L}\,\bar{q}\,\not{W}^+ P_L b + \xi^{S,(n)}_{\bar{q}bW,R}\,\bar{q}\,\not{W}^+ P_R\, b \right. \\
&\qquad \left. + \xi^{S,(1)}_{\bar{q}bG,L}\, G^+\,\bar{q} P_L b + \xi^{S,(n)}_{\bar{q}bG,R}\, G^+\,\bar{q} P_R\, b + \text{h.c.}\right]\,,
\end{aligned}$$

where the values of the coefficients $\xi_i^{S,(1)}$ depend on the normalization of the Green functions involving an Ω field.

We refrain from listing the equations that determine the counterterms $\Xi^{S,(1)}_{2,W^+_\mu \bar{q}b}$ and $\Xi^{S,(1)}_{2,G^+\bar{q}b}$ (i.e. the ones corresponding to (4.15)) as the structure is similar to Eq. (4.16). The one-loop Green functions have to be replaced by the corresponding counterterms.

Note that the equations for the vertices involving the quarks s and q are in complete analogy to the ones presented above.

At this point of our analysis all the counterterms for the quantum fields are expressed in terms of the counterterms $\Xi^{S,(1)}_{W^*\Omega}$ and $\Xi^{S,(1)}_{G^*\Omega}$. They will be discussed below.

5. Ghost Green functions. In the following we discuss the Green functions involving Faddeev-Popov ghosts or the fields Ω.

In the list of Green functions contributing to $b \to s\gamma$ at two loops there are the amplitudes $\Gamma_{\hat{A}\bar{c}^\pm c^\mp}$ and $\Gamma_{\bar{c}^\pm c^\mp}$ which are related through the WTI

$$\left.\frac{\delta^3 \mathcal{W}_{(\lambda)}(\Gamma^{(1)})}{\delta\lambda_A(-p_+ - p_-)\delta c^+(p_+)\delta\bar{c}^-(p_-)}\right|_{\phi=0} =$$
$$i\,(p_+ + p_-)^\mu\,\Gamma^{(1)}_{\hat{A}_\mu c^+\bar{c}^-}(p_+,p_-) - ie\left(\Gamma^{(1)}_{c^+\bar{c}^-}(p_-) - \Gamma^{(1)}_{c^+\bar{c}^-}(-p_+)\right) \;=\; \Delta^W_{\lambda_A c^+\bar{c}^-}\,,$$

and its Hermitian conjugate. Acting with the Taylor operator $(1 - T^1_{p_+,p_-})$ removes the breaking term $\Delta^W_{\lambda_A c^+\bar{c}^-}$ and no counterterm is needed to restore the identity. However, there is still freedom to add background gauge invariant counterterms to the amplitudes $\hat{\Gamma}^{(1)}_{c^\pm\bar{c}^\mp}$. The latter are related to those with external anti-fields by means of the Faddeev-Popov equations (cf. Appendix B and [10, 4]) which read

$$\Gamma^{(1)}_{c_\pm\bar{c}_\mp}(p) \pm ip^\mu\Gamma^{(1)}_{c_\pm W^{*,\mp}_\mu}(p) - \xi_W M_{\pm,G^\mp}\Gamma^{(1)}_{c_\pm G^{*,\mp}}(p) \;=\; \Delta^{(1)}_{F,c^\pm\bar{c}^\mp}\,. \tag{4.18}$$

Thus the counterterms for $\Gamma^{(1)}_{c_\pm\bar{c}_\mp}$ have to be chosen in such a way that $\Delta^{(1)}_{F,c^\pm\bar{c}^\mp}$ is removed. In general they read

$$\Xi^{F,(1)}_1 \;=\; \int \mathrm{d}^4x\Big(\xi^{F,(1)}_1\hat{\nabla}_\mu\bar{c}^+\hat{\nabla}_\mu c^- + \xi^{F,(1)}_2\bar{c}^+c^- + \mathrm{h.c.}\Big)\,. \tag{4.19}$$

However, this does not completely fix the ghost two-point functions as also the anti-field-dependent Green functions in (4.18) have to be fixed.

In the remaining part of this subsection we discuss the renormalization of the missing two-point functions, namely those involving Ω fields, like $\hat{\Gamma}^{(1)}_{\Omega^\pm G^{*,\mp}}$ and $\hat{\Gamma}^{(1)}_{\Omega^\pm_\mu W^{*,\mp}_\nu}$, and the ones with Faddeev-Popov ghosts and anti-fields, $\Gamma^{(1)}_{c^\pm W^{*,\mp}_\mu}$ and $\Gamma^{(1)}_{c^\pm G^{*,\mp}}$. Actually, the following considerations are significantly simplified in the framework of dimensional regularization [30, 31] (see [2] for a practical calculation), since only those identities involving fermions will produce breaking terms and since there is no tree-level coupling of fermions with ghost fields. However, we present the general analysis as outlined in Section 3.3. Also at higher orders this part of the discussion will be useful even in dimensional regularization.

In a first step we want to mention that the Green functions $\Gamma^{(1)}_{c^\mp W^{*,\pm}_\mu}$, $\Gamma^{(1)}_{c^A W^{*,3}_\mu}$, $\Gamma^{(1)}_{c^Z W^{*,3}_\mu}$ and $\Gamma^{(1)}_{c^{*,3}c^+c^-}$ are fixed by normalization conditions for the wave function of the ghost fields. A convenient choice corresponds to

$$-i\,\partial_{p^\mu}\mathrm{I\!\Gamma}_{c^A W^{*,3}_\mu}\Big|_{p=0} \;=\; s_W\,, \qquad\qquad \mathrm{I\!\Gamma}_{c^{*,3}c^+c^-}\big|_{p^+=p^-=0} = -ie\frac{c_W}{s_W}\,. \tag{4.20}$$

Then the WTI

$$i(p+q)_\nu\Gamma^{(1)}_{\hat{W}^+_\nu W^{*,3}_\mu c^-}(p,q) + iM_W\Gamma^{(1)}_{\hat{G}^+ W^{*,3}_\mu c^-}(p,q) \;=$$
$$+i\frac{e}{s_W}\left(\Gamma^{(1)}_{W^{*,+}_\mu c^-}(q) + s_W\Gamma^{(1)}_{W^{3,*}_\mu c^A}(-p) - c_W\Gamma^{(1)}_{W^{3,*}_\mu c^Z}(-p)\right) + \Delta^{W,(1)}_{\lambda_+ W^{*,3}_\mu c^-}\,, \tag{4.21}$$

can be used to obtain the counterterm for $\Gamma^{(1)}_{\hat{W}^+_\nu W^{*,3}_\mu c^-}$ and for $\Gamma^{(1)}_{W^{*,\mp}_\mu c^\pm}$ which removes the breaking term $\Delta^{W,(1)}_{\lambda_+ W^{*,3}_\mu c^-}$. On the other hand, from the STI

$$\begin{aligned}
&i(p+q)_\nu \Gamma^{(1)}_{W^+_\nu W^{*,3}_\mu c^-}(p,q) + iq_\nu \Gamma^{(1)}_{W^-_\nu W^{*,3}_\mu c^+}(p,-p-q) \\
&+ iM_W \Gamma^{(1)}_{G^+ W^{*,3}_\mu c^-}(p,q) + iM_W \Gamma^{(1)}_{G^- W^{*,3}_\mu c^+}(p,-p-q) \;= \\
&\qquad + i\frac{e}{s_W}\Big(\Gamma^{(1)}_{W^{*,+}_\mu c^-}(q) + \Gamma^{(1)}_{W^{*,-}_\mu c^+}(-p-q) + s_W \Gamma^{(1)}_{W^{3,*}_\mu c^A}(-p) - c_W \Gamma^{(1)}_{W^{3,*}_\mu c^Z}(-p)\Big) \\
&\qquad - ip_\mu \Gamma^{(1)}_{c^{*,3} c^+ c^-}(-p-q,q) + \Delta^{S,(1)}_{c^+ W^{*,3}_\mu c^-}\,, \qquad (4.22)
\end{aligned}$$

the Green function $\Gamma^{(1)}_{W^+_\nu W^{*,3}_\mu c^-}$ is determined and the counterterms given in Eqs. (C.3) remove the breaking term $\Delta^{S,(1)}_{W^{*,3}_\mu c^+ c^-}$. Finally, in the STI

$$\begin{aligned}
&\Gamma^{(1)}_{\hat{W}^+_\nu W^{*,3}_\mu c^-}(p,q) - \Gamma^{(1)}_{W^+_\nu W^{*,3}_\mu c^-}(p,q) + i\frac{e}{s_W}\Gamma^{(1)}_{\Omega^+_\nu W^{*,-}_\mu}(p) \;= \\
&\qquad - iq^\rho \Gamma^{(1)}_{W^-_\rho W^{*,3}_\mu \Omega^+_\nu}(p,-p-q) - iM_W \Gamma^{(1)}_{G^- W^{*,3}_\mu \Omega^+_\nu}(p,-p-q) \\
&\qquad - ip_\mu \Gamma^{(1)}_{c^{*,3}\Omega^+_\nu c^-}(-p-q,q) + \Delta^{S,(1)}_{\Omega^+_\nu W^{*,3}_\mu c^-}\,, \qquad (4.23)
\end{aligned}$$

we only have finite Green functions on the r.h.s. and thus from this equation it is possible to determine the renormalization of $\Gamma^{(1)}_{\Omega^+_\nu W^{*,-}_\mu}$.

In principle, also for $\Gamma^{(1)}_{\Omega^+_\nu W^{*,-}_\mu}$ a normalization condition can be chosen. However, the corresponding parameters in Eq. (C.5) are automatically fixed by the symmetries of the theory.

A complete equivalent system can be derived to treat the two-point functions with a scalar Ω field and with $W^{*,-}_\mu$ replaced by $G^{*,-}$. In the latter case the renormalization of $\Gamma^{(1)}_{c^- G^{*,+}}$ and $\Gamma^{(1)}_{c^i G^{*,0}}$ $(i = A, Z)$ is needed in order to fix the ghost mass parameters. For a detailed discussion we refer to [10, 4].

The equations presented in this paragraph follow the method outlined in Section 3.3. In particular, Eq. (4.21) fixes the Green function $\Gamma^{(1)}_{\hat{W}^+_\nu W^{*,3}_\mu c^-}(p,q)$ which involves the background field $\hat{W}^+_\nu$. Furthermore, Eqs. (4.22) and (4.23), which correspond to the first and second equations of (3.25), respectively, contain the corresponding quantum Green functions and Eqs. (4.20) fix the normalization conditions.

Notice that the renormalization of the amplitudes with external anti-fields and ghost fields is fairly arbitrary for the present computation. These normalization conditions do not influence the physical observables in the process $b \to s\gamma$.

6. QED gauge coupling. Besides the normalization conditions discussed above, we have to treat the QED coupling constant as remaining free parameter. Its renormalization is achieved by introducing the counterterm

$$\Xi^{N,(1)}_3 \;=\; \int \mathrm{d}^4x \left[-\frac{\xi^{N,(1)}_{F^2,3}}{4}\, F_{\mu\nu}(V+\hat{V})F^{\mu\nu}(V+\hat{V})\right],$$

where $F_{\mu\nu}(V + \hat{V})$ is the abelian field strength.

As already mentioned above the analysis presented in this section is quite general. Thus, in order to conclude we briefly mention the simplifications due to the use of Dimensional Regularization accompanied with the 't Hooft-Veltman definition of γ_5 [30, 31].

First, all identities which do not involve fermion lines are preserved. In particular the breaking terms of Eqs. (4.12) are zero as only diagrams involving virtual ghost particles contribute. For the same reason the ghost sector is highly simplified as there is no fermionic contribution to Eqs. (4.21), (4.22) and (4.23). Furthermore, those breaking terms to be computed from diagrams which don't involve a chiral vertex are also zero. In the above analysis this would correspond to Eq. (4.3).

As a final remark, we would like to emphasize that owing a background gauge invariant regularization, Eqs. (4.3), (4.7), (4.8), (3.15), (4.18) and (4.21) are automatically preserved and only the Eqs. (4.12), (4.16), (4.22) and (4.23) have to be studied in detail.

From the analysis performed in the example of this section, it is clear that the strategy outlined in Section 3.3 can effectively be applied to each process of the SM. One can also see, that the zero-momentum subtraction significantly simplifies the practical computation of the non-invariant counterterms.

5 Conclusions

In this paper a general procedure to perform the renormalization using the algebraic renormalization and the background field method is discussed. It is shown that the computational problems to evaluate the non-invariant counterterms within a generic subtraction scheme can be drastically reduced in the framework of the BFM and by means of an intermediate subtraction.

Recently, several progress have been done in constructing new regularization schemes for chiral gauge theories at perturbative [32, 33, 34] and at non-perturbative level [35]. However, at the practical level, the algebraic renormalization with a non-invariant regularization scheme in our formulation turns out to be still superiour. In addition, owing a scheme which is explicitly background gauge invariant only few counterterms (cf. Appendix C) are indeed necessary to restore the STIs. Finally, even in case that a scheme would exist which is invariant under all the symmetries of the SM, the present paper provides a complete analysis of the relations between Green function with external background fields and those with quantum ones.

To summarize, we want to give a brief outline for possible applications of the method. In general, the method can be divided into the following two main steps:

1. *Anti-field-dependent counterterms* $\Xi^{(n)}_{\#}[\phi, \hat{\phi}, \phi^*, \Omega]$.

   In a first step one considers the WTIs and fixes all possible counterterms using the gauge symmetry constraints. This amounts to determining the coefficients of the counterterms $\Xi^{W,(n)}_{\#}[\phi, \hat{\phi}, \phi^*, \Omega]$ (see, for instance, Eq. (4.21)).

The second step concerns the computation of the counterterms which are left after fixing the WTIs, namely $\Xi^{S,(n)}_{\#}[\phi, \hat{\phi}, \phi^*, \Omega]$ (see, e.g., Eqs. (4.22) and (4.23)). In particular, to restore the relation between the background and the quantum fields (cf. the second and the third equations of (3.25) and in the example Eq. (4.23)) one needs to study STIs obtained by differentiating Eq. (2.3) with respect to one anti-field and two ghost fields and with respect to one anti-field, Ω and one ghost field (cf. the first and the second equations of (3.25) and Eq. (4.23)). Finally, the remaining free parameters $\Xi^{N,(n)}_{\#}[\phi, \hat{\phi}, \phi^*, \Omega]$ are fixed by normalization conditions (cf. Eqs. (4.20)).

2. *Anti-field-independent counterterms* $\Xi^{(n)}_{O}[\phi, \hat{\phi}]$.

   The anti-field-independent counterterms constitute the main part of the diagram computations. At first, one fixes all the possible counterterms using the background gauge symmetry. This is done by exploiting the corresponding WTI and computing $\Xi^{W,(n)}_{O}[\phi, \hat{\phi}]$ (compare Eqs. (3.15), (4.3), (4.7) and (4.8)). Then, one restores the STIs by $\Xi^{S}_{O}[\phi, \hat{\phi}]$ (see Eqs. (4.11), (4.12), and (4.16)), and finally the free parameters $\Xi^{N}_{O}[\phi, \hat{\phi}]$ can be tuned on the physical data by means of normalization conditions.

   Note that, as is shown in the example, in a first step the two-point functions are considered. Afterwards the three-point functions are fixed, then the four-point functions etc.. The success of this procedure is guaranteed by the consistency conditions between $\mathcal{S}_0$ and $\mathcal{W}_{(\lambda)}$ as outlined in Appendix C of [7].

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A Linearized Slavnov-Taylor operator, coupling of Ω fields and functional Taylor operator

The linearized Slavnov-Taylor operator for a generic functional $\mathcal{F}$ is given by

$$
\begin{aligned}
\mathcal{S}_{\rm I\!\Gamma}(\mathcal{F}) \equiv & \int \mathrm{d}^4x \Bigg\{ (s_W \partial_\mu c_Z + c_W \partial_\mu c_A) \left(s_W \frac{\delta \mathcal{F}}{\delta Z_\mu} + c_W \frac{\delta \mathcal{F}}{\delta A_\mu} \right) \\
& + \sum_{\alpha = A, Z, \pm, a} b_\alpha \frac{\delta \mathcal{F}}{\delta \bar{c}^\alpha} + ({\rm I\!\Gamma}, \mathcal{F}) + (\mathcal{F}, {\rm I\!\Gamma}) \\
& + \Omega^3_\mu \left[c_W \left(\frac{\delta \mathcal{F}}{\delta \hat{Z}_\mu} - \frac{\delta \mathcal{F}}{\delta Z_\mu} \right) - s_W \left(\frac{\delta \mathcal{F}}{\delta \hat{A}_\mu} - \frac{\delta \mathcal{F}}{\delta A_\mu} \right) \right] \\
& + \Omega^\pm_\mu \left(\frac{\delta \mathcal{F}}{\delta \hat{W}^\pm_\mu} - \frac{\delta \mathcal{F}}{\delta W^\pm_\mu} \right) + \Omega^a_\mu \left(\frac{\delta \mathcal{F}}{\delta \hat{G}^a_\mu} - \frac{\delta \mathcal{F}}{\delta G^a_\mu} \right)
\end{aligned}
$$

$$+\,\Omega^{\pm}\left(\frac{\delta\mathcal{F}}{\delta\hat{G}^{\pm}}-\frac{\delta\mathcal{F}}{\delta G^{\pm}}\right)+\Omega^{0}\left(\frac{\delta\mathcal{F}}{\delta\hat{G}^{0}}-\frac{\delta\mathcal{F}}{\delta G^{0}}\right)+\Omega^{H}\left(\frac{\delta\mathcal{F}}{\delta\hat{H}}-\frac{\delta\mathcal{F}}{\delta H}\right)\Bigg\}\,, \qquad \text{(A.1)}$$

where

$$\begin{aligned}(X,Y) \;=\; & \int d^4x\,\Bigg[\frac{\delta X}{\delta W^{*,3}_\mu}\frac{\delta Y}{\delta W^3_\mu}+\frac{\delta X}{\delta W^{*,\pm}_\mu}\frac{\delta Y}{\delta W^{\mp}_\mu}+\frac{\delta X}{\delta G^{*,a}_\mu}\frac{\delta Y}{\delta G^a_\mu}+\frac{\delta X}{\delta c^{*,\pm}}\frac{\delta Y}{\delta c^{\mp}}+\frac{\delta X}{\delta c^{*,3}}\frac{\delta Y}{\delta c^3}\\
&+\frac{\delta X}{\delta c^{*,a}}\frac{\delta Y}{\delta c^a}+\frac{\delta X}{\delta c^{*,\pm}}\frac{\delta Y}{\delta c^{\mp}}+\frac{\delta X}{\delta G^{*,\pm}}\frac{\delta Y}{\delta G^{\mp}}+\frac{\delta X}{\delta G^{*,0}}\frac{\delta Y}{\delta G^{0}}+\frac{\delta X}{\delta H^{*}}\frac{\delta Y}{\delta H}\\
&+\sum_{I=L,Q,u,d,e}\left(\frac{\delta X}{\delta\bar{\psi}^{*I}}\frac{\delta Y}{\delta\psi^{I}}+\frac{\delta X}{\delta\psi^{*I}}\frac{\delta Y}{\delta\bar{\psi}^{I}}\right)\Bigg]\,. \end{aligned}\qquad \text{(A.2)}$$

Since $\mathcal{S}(\mathrm{I\!\Gamma})=0$, the operator $\mathcal{S}_{\mathrm{I\!\Gamma}}$ is nilpotent. We also introduce the tree-level linearized operator $\mathcal{S}_0\equiv\mathcal{S}_{\Gamma_0}$ where Γ_0 is the tree-level action.

In the method discussed in this paper the couplings of the Ω-fields ($\mathcal{L}_{\Phi\Pi,\Omega}$) represent an important piece of information. In fact they are essential in the derivation the STIs for the counterterms involving background fields. In Ref. [7], the general building blocks of the Lagrangian with anti-fields has been given and the couplings with Ω-fields are contained in Eqs. (A.2) and (A.3) of Appendix A of [7]. However, for the convenience of the reader, we present the couplings $\mathcal{L}_{\Phi\Pi,\Omega}$ in the explicit form:

$$\begin{aligned}\mathcal{L}_{\Phi\Pi,\Omega} \;=\; & \Omega^3_\mu\left\{\left(c_W\partial_\mu\bar{c}^Z-s_W\partial_\mu\bar{c}^A\right)-\frac{i\,e}{s_W}\left[\left(W^+_\mu+\hat{W}^+_\mu\right)\bar{c}^- -\left(W^-_\mu+\hat{W}^-_\mu\right)\bar{c}^+\right]\right\}\\
&+\Omega^{\mp}_\mu\Big\{\partial_\mu\bar{c}^{\pm}\mp ie\left(W^{\pm}_\mu+\hat{W}^{\pm}_\mu\right)\left(\bar{c}^A-\frac{c_W}{s_W}\bar{c}^Z\right)\\
&\qquad\pm ie\bar{c}^{\pm}\left[\left(A_\mu+\hat{A}_\mu\right)-\frac{c_W}{s_W}\left(Z_\mu+\hat{Z}_\mu\right)\right]\Big\}\\
&+\Omega^a_\mu\left\{\partial_\mu\bar{c}^a-g_sf^{abc}\left(G^b_\mu+\hat{G}^b_\mu\right)\bar{c}^c\right\}\\
&+\Omega^H\left\{\frac{ie\,\xi_W}{2s_W}\left[\left(G^++\hat{G}^+\right)\bar{c}^- -\left(G^-+\hat{G}^-\right)\bar{c}^+\right]+\frac{e\,\xi_Z}{2s_Wc_W}\left(G^0+\hat{G}^0\right)\bar{c}^Z\right\}\\
&+\Omega^{\mp}\Big\{\pm\frac{ie\,\xi_W}{2s_W}\left[H+\hat{H}+v\pm i\left(G^0+\hat{G}^0\right)\right]\bar{c}^{\pm}\\
&\qquad\mp ie\left(G^{\pm}+\hat{G}^{\pm}\right)\left(\xi_A\bar{c}^A-\xi_Z\,\frac{c_W^2-s_W^2}{2c_Ws_W}\bar{c}^Z\right)\Big\}\\
&+\Omega^0\left\{\frac{e\,\xi_W}{2s_W}\left[\left(G^++\hat{G}^+\right)\bar{c}^- +\left(G^-+\hat{G}^-\right)\bar{c}^+\right]-\frac{e\,\xi_Z}{2s_Wc_W}\left(H+\hat{H}+v\right)\bar{c}^Z\right\}\,,\end{aligned}\qquad \text{(A.3)}$$

where $\bar{c}^Z,\bar{c}^A,\bar{c}^{\pm}$ and $\bar{c}^a$ are the anti-ghost fields. Notice that the Feynman rules for these new vertices are related to the couplings of the anti-fields with quantum fields (cf. Eq. (A.1) of [7]) simply by exchanging the ghost fields with the anti-ghost fields.

The Taylor operator T^δ of the functional Γ is defined as follows. One first considers the relevant amplitude which results from functional derivatives with respect to fields denoted by subscripts $\Gamma_{\phi_1(p_1)\phi_2(p_2)\dots\phi_m(p_m)}$ with $\sum_{j=1}^m p_j = 0$. Then the Taylor expansion T^δ in the independent momenta up to degree δ acts formally as

$$T^\delta \Gamma = \sum_{m=1}^{\infty} \int \prod_{i=1}^{m} \mathrm{d}^4 p_i \phi_i(p_i) \delta^4(\textstyle\sum_{j=1}^m p_j) T^\delta_{p_1,\dots,p_m} \Gamma_{\phi_1(p_1)\phi_2(p_2)\dots\phi_m(p_m)}\Big|_{\sum_{j=1}^m p_j = 0} . \quad \text{(A.4)}$$

A remarkable property of T^δ is that $T^{\delta_1} T^{\delta_2} = T^\delta$ with $\delta = \min\{\delta_1, \delta_2\}$. Note that the Taylor operator is scale-invariant, but it does not commute with spontaneous symmetry breaking.

B Auxiliary functional constraints

We recall that the SM in the BFM [10] is completely defined in terms of the following functional identities (up to normalization conditions)

1. The Nakanishi-Lautrup identities (2.4), which implement the gauge fixing conditions to all orders,
2. the Abelian Anti-ghost Equation (in the case of BFM see second reference in [10], Eq. (4.28)),
3. the non-abelian WTI given in Eq. (2.5) for the background gauge invariance,
4. the STI given by Eq. (2.3) for the BRST symmetry,
5. the Faddeev-Popov equations of motion (see [10, 4]), and
6. the abelian WTI given by Eq. (2.5) for the background gauge invariance restricted to the $U(1)$ factor.

The sets 1 to 4 of functional identities are imposed on the theory by requiring invariance under the corresponding symmetries. The sets 5 and 6 are derived constraints of the commutation relations of previous functional identities. However, for practical purposes, they turn out to be relevant. In the following, we briefly discuss this issue. For more details we refer to the literature.

For a generic non-invariant scheme all possible functional identities can be spoiled by local breaking terms and the method presented in this paper can be applied to all of them. In addition, since all the identities are linear (except the STI), the intermediate subtraction at zero momentum drastically reduces the number of breaking terms and the latter can easily be removed by breaking terms.

In this paper we emphasized the role of the WTIs and the STIs, however, also the supplementary constraints should be taken into account. As a consequence, it is possible to define a reduced functional generator and to renormalize certain anti-field-dependent amplitude in terms of Green functions with external ghost fields.

The Nakanishi-Lautrup identities (2.4) are discussed in detail in [10, 4]. In [4], the gauge fixing depends only on the scalar background fields $\hat{\Phi}$. There a complete discussion for the renormalization of (2.4) is presented. In [10], the same analysis is performed in the background 't Hooft gauge fixing. In particular, we notice that the zero momentum subtraction already removes all the possible breaking terms and no additional non-invariant counterterms are indeed needed.

In the case of the SM, the STIs and the gauge fixing conditions are not sufficient to fix the abelian sector of the theory [10, 4] completely. In addition, one has to use another functional equation which controls the renormalization of the abelian ghost fields. Including also the background fields the equation reads

$$\begin{aligned} & c_W \frac{\delta\Gamma}{\delta c_A} + s_W \frac{\delta\Gamma}{\delta c_Z} + \frac{ie}{2c_W}\left(\hat{G}^+ \frac{\delta\Gamma}{\delta\Omega^+} - \hat{G}^- \frac{\delta\Gamma}{\delta\Omega^-}\right) - \frac{e}{2c_W}\left(\hat{G}_0 \frac{\delta\Gamma}{\delta\Omega^H} - (\hat{H}+v)\frac{\delta\Gamma}{\delta\Omega^0}\right) = \\ & \quad \frac{e}{2c_W}\left(H^* G_0 - G_0^*(H+v)\right) + \frac{ie}{2c_W}\left(G^{+,*}G^- - G^{-,*}G^+\right) \\ & \quad + \sum_\alpha \left(\frac{1}{6}\bar{Q}^{L,*}_\alpha Q^L_\alpha + \frac{2}{3}\bar{u}^{R,*}_\alpha u^R_\alpha - \frac{1}{3}\bar{d}^{R,*}_\alpha d^R_\alpha - \frac{1}{2}\bar{L}^{L,*}_\alpha L^L_\alpha - \bar{e}^{R,*}_\alpha e^R_\alpha\right) + \text{h.c.} \\ & \quad + \left(s_W \partial^2 \bar{c}^Z + c_W \partial^2 \bar{c}^A\right) , \end{aligned} \tag{B.1}$$

where we used the notation of [7]. Notice that Eq. (B.1) is linear in Γ. However, by means of zero momentum subtraction this equation is spoiled. This is due to the fact that the UV power counting of the ghost fields c_A and c_Z and the one of Ω are different and, consequently, over-subtractions are generated. Fortunately, the algebraic analysis of this problem is simple.

The Faddeev-Popov equations of motion should be analyzed along the same lines. The complete analysis has been given in [10, 4, 22]. In the application discussed in Section 4 of the present paper, a particular set of Faddeev-Popov equations of motion has been used (cf. Eq. (4.18)) and their renormalization was analyzed.

The Lagrange multiplier equations (2.4), the Faddeev-Popov equations of motion and the Abelian Anti-ghost equation (B.1) are linear differential functional equations which can be solved by simple redefinitions of the functional Γ_0 and of the anti-fields. This is because the set of the functional differential operators associated with those supplementary constraints[10] forms a sub-algebra of the complete algebra including the WTIs and STIs. The solution of the renormalized supplementary equations is called the *reduced functional* [13, 15, 36]. At tree level it reads

$$\begin{aligned} \Gamma_0^{\text{red.func.}} \;=\; & \Gamma_0 - \int d^4x \Big[b_C \mathcal{F}^C(V,\Phi,b) + \bar{c}\,\partial^2 c \\ & + \Omega^\alpha_\mu \left(\nabla^\mu \bar{c}\right)_\alpha + \Omega^a_\mu \left(\nabla^\mu \bar{c}\right)_a + \Omega^i \bar{c}_\alpha t^\alpha_{ij}\left(\Phi + \hat{\Phi} + v\right)^j \\ & + \Phi^{*,i} c\, t^0_{ij}\left(\Phi + \hat{\Phi} + v\right)^j + \left(\bar{\psi}^{*,I} c\, T^0_{IJ}\psi^J + \text{h.c.}\right)\Big] , \end{aligned} \tag{B.2}$$

where C, α and a are the indices for the adjoint representation of $SU(3)\times SU(2)\times U(1)$, $SU(2)$ and $SU(3)$, respectively. The ghost c belongs to the $U(1)$ sector. It can also be written in terms of the rotated fields $c = c_W c_A + s_W c_Z$. The reduced functional $\Gamma_0^{\text{red.func.}}$

[10]E.g., the functional operator $\delta/\delta b_C$ acting on Γ is associated with Eq. (2.4).

does not depend on b_C, on the anti-ghosts $\bar{c}^C$, on the abelian ghost c or on the Ω fields. In Eq. (B.2), we used the compact notation introduced in Section 2 and used in Appendix C. The explicit form of the second line can be found in Eq. (A.3) of Appendix A and the explicit form of the third line can be read off from Eq. (A.1) of Ref. [7] by selecting the contributions of the abelian ghost. $\Gamma_0^{\text{red.func.}}$ depends only on the following combinations

$$\begin{aligned} \tilde{V}^{*,C}_\mu &= V^{*,C}_\mu + \left(\hat{\nabla}_\mu \bar{c}\right)^C , \\ \tilde{\Phi}^{*,i} &= \tilde{\Phi}^{*,i} + \bar{c}_\alpha t^{\alpha i}{}_j \left(\hat{\Phi} + v\right)^j + \bar{c}\, t^{0i}{}_j \left(\hat{\Phi} + v\right)^j , \end{aligned} \tag{B.3}$$

of the anti-fields $V^{*,C}_\mu$ and $\Phi^{*,i}$ where t^0_{ij} is the $U(1)$ generator in the representation of the scalar fields. At higher orders, some suitable normalization conditions should be taken into account in order to avoid spurious off-shell IR problems [10, 4]. Note, that the superscript "red.func." is omitted in the main text of the paper.

C Background gauge invariant counterterms for the STIs

In the present appendix, we list and classify all possible counterterms Ξ^S needed to restore the STIs. We assume that the WTIs for the background gauge invariance are already recovered. Furthermore, we assume that all other constraints such as the Nakanishi-Lautrup identities, the Faddeev-Popov equations and the anti-ghost equation (cf. Appendix B) are satisfied. This implies that we can consider the simple factors of the gauge group separately from the abelian one, and the dependence upon anti-ghost, Lagrangians multiplier and abelian ghost field is already taken into account.

In the description of the general counterterms, we follow the previous classification into anti-field dependent and independent counterterms, namely $\Xi^S_\#[\phi, \hat{\phi}, \phi^*, \Omega]$ and $\Xi^S_O[\phi, \hat{\phi}]$, respectively. In addition, we organize the anti-field dependent counterterms according to the highest ghost number.

We use the following notation for the counterterms

$$\int \mathrm{d}^4x\, \xi_{\phi^1\phi^2\ldots\phi^n,i}\, T_i^{a_1 a_2 \ldots a_n}\ \phi^1_{a_1}\phi^2_{a_2}\ldots\phi^n_{a_n} ,$$

where $\phi^i_{a_i}$ denotes the fields and their derivatives and $T_i^{a_1 a_2\ldots a_n}$ contains field-independent Lorentz and gauge group invariant (in the adjoint or in the matter representation) tensors whose independent component are parameterized by the index i. $\xi_{\phi^1\phi^2\ldots\phi^n,i}$ are the coefficients of the counterterms. Eventually, $\xi^{(n)}_{\phi^1\phi^2\ldots\phi^n,i}$ denotes the n^{th} contribution to the coefficient $\xi_{\phi^1\phi^2\ldots\phi^n,i}$. In Section 4 also the notation $\xi^{W,(n)}_{\phi^1\phi^2\ldots\phi^n,i}$, $\xi^{S,(n)}_{\phi^1\phi^2\ldots\phi^n,i}$ and $\xi^{N,(n)}_{\phi^1\phi^2\ldots\phi^n,i}$ is introduced to distinguish between the counterterms arising from the WTIs, the STIs and the normalization conditions, respectively.

Anti-field dependent terms $\Xi^S_{\#}[\phi, \hat{\phi}, \phi^*, \Omega]$

1. *Anti-ghost fields*

   The most negative ghost number is carried by the anti-fields of the ghost fields, c_3^* and $c_\pm^*$, therefore the counterterms of the type

$$\int \mathrm{d}^4x \left[\xi_{c^*c^2,1}\, \epsilon_{\alpha\beta\gamma}\, c^{*,\alpha} c^\beta c^\gamma + \xi_{c^*c^2,2}\, f_{abc}\, c^{*,a} c^b c^c\right] , \tag{C.1}$$

   are the most general background gauge invariant contribution. Note that due to the Abelian Anti-ghost equation (B.1), the dependence on the abelian ghost field has already been fixed. Here $\epsilon_{\alpha\beta\gamma}$ and f_{abc} are the structure constants of the $su(2)$ and $su(3)$ algebras, respectively.

   Notice that the parameters $\xi_{c^*c^2,i}$ $(i = 1, 2)$ correspond to the wave function renormalization of the $SU(2)$ and $SU(3)$ ghost fields, respectively. Therefore they are fixed by normalization conditions of the type (4.20).

2. *Couplings of anti-fields with ghost fields and background fields*

   Having fixed the anti-fields of the ghosts, we now turn to the anti-fields of the quantum fields. Therefore one has to select the couplings of the anti-fields for the gauge fields, $W_\mu^{*,3}$ and $W_\mu^{*,\pm}$, and of the anti-fields of the scalar fields, H^*, G_0^* and $G_\pm^*$, with the ghost fields. A generic counterterm can be expressed by the equation

$$\int \mathrm{d}^4x \left[\xi_{V^*c,1}\, V^{*,\alpha\mu} \left(\hat{\nabla}_\mu c\right)_\alpha + \xi_{V^*c,2}\, V^{*,a\mu} \left(\hat{\nabla}_\mu c\right)_a + \xi_{\Phi^*c}\, \Phi^{*,i} c_\alpha t^\alpha_{ij} (\hat{\Phi} + v)^j\right] , \tag{C.2}$$

   where t^α_{ij} are $SU(2)$ generators in the scalar representation. Notice again that, due to Eq. (B.1), the coupling of the scalar fields with the abelian ghost field are already determined. Therefore, in Eq. (C.2) only the coupling between $SU(2)$-ghost fields and scalars has been taken into account. The covariant derivatives are defined by $\left(\hat{\nabla}_\mu c\right)^a = \partial_\mu c^a - f^a_{\ bc} \hat{V}^b_\mu c^c$ and $\left(\hat{\nabla}_\mu c\right)^\alpha = \partial_\mu c^\alpha - \epsilon^\alpha_{\ \beta\gamma} \hat{V}^\beta_\mu c^\gamma$.

   The parameters $\xi_{V^*c,i}$ $(i = 1, 2)$ and ξ_{Φ^*c} amount to a wave function renormalization of the $SU(2)$ and $SU(3)$ quantum gauge fields and of the quantum scalar multiplet Φ^i. Consequently, they are fixed by normalization conditions like (4.20) instead of STIs. This simplifies further the task of the computation.

3. *Couplings of anti-fields with ghost fields and quantum fields*

   The gauge fields V^α_μ and V^a_μ transform as vectors of the adjoint representation under background gauge transformations. Thus, among the anti-field dependent counterterms, we also have to list the following contributions

$$\begin{aligned} \int \mathrm{d}^4x \Big[&\xi_{V^*Vc,1} \epsilon_{\alpha\beta\gamma}\, V^{*,\alpha\mu} V^\beta_\mu c^\gamma + \xi_{V^*Vc,2}\, f_{abc} V^{*,a\mu} V^b_\mu c^c \\ &+ \xi_{V^*Vc,3} d_{abc}\, V^{*,a\mu} V^b_\mu c^c + \xi_{\Phi^*c}\, \Phi^{*,i} c_\alpha t^\alpha_{ij} \Phi^j \\ &+ \sum_\psi \Big(\xi_{\bar{\psi}^*\psi,1} \bar{\psi}^*_I T^{IJ}_\alpha \psi_J c^\alpha + \xi_{\bar{\psi}\psi^*,1} \bar{\psi}_I T^{IJ,\dagger}_\alpha \psi^*_J c^\alpha \\ &+ \xi_{\bar{\psi}^*\psi,2} \bar{\psi}^*_I T^{IJ}_a \psi_J c^a + \xi_{\bar{\psi}\psi^*,2} \bar{\psi}_I T^{IJ,\dagger}_a \psi^*_J c^a + \text{h.c.}\Big)\Big] , \end{aligned} \tag{C.3}$$

where $T^{IJ}_\alpha = T^{L,IJ}_\alpha P_L + T^{R,IJ}_\alpha P_R$ and T^{IJ}_a are the generators for $SU(2)$ and $SU(3)$ gauge transformations, respectively. Notice that $T^\dagger$ is the pseudo-hermitian conjugate of T. In order to take into account the mixings among fermion generations in the counterterm of the type $\xi_{\bar\psi^*\psi,2}\bar\psi^*_I T^{IJ}_a \psi_J c^a$ a summation is understood. For instance,

$$\sum_\psi \int \mathrm{d}^4x\, \xi_{\bar\psi^*\psi,2} \bar\psi^*_I T^{IJ}_a \psi_J c^a \quad = \sum_{q,q'=u,c,t,d,s,b} \int \mathrm{d}^4x\, \xi_{\bar q^* q',2}\, \bar q^* \frac{i\,\lambda_a}{2} q'\, c^a\,, \qquad \text{(C.4)}$$

where λ_a are the $SU(3)$ Gell-Mann matrices and $\xi_{\bar q^* q',1}$ is a complex matrix.

4. *Couplings of anti-fields with* Ω

A well-known feature of the BFM is the fact that multiplets are renormalized by the same constant. As a consequence, the renormalization of the background and of the quantum fields is related to each other. The counterterms that control these relations at the quantum levels are

$$\int \mathrm{d}^4x \left[\xi_{V^*\Omega,1} V^{*,\alpha\mu}\Omega_{\alpha\mu} + \xi_{V^*\Omega,2} V^{*,a\mu}\Omega_{a\mu} + \xi_{\Phi^*\Omega}\Phi^{*,i}\Omega_i\right]. \qquad \text{(C.5)}$$

Anti-field independent terms $\Xi^S_O[\phi,\hat\phi]$

1. *Gauge sector*

We have to recall that abelian quantum gauge field V_μ has no background partner. More precisely, combining the WTI for the abelian factor and the anti-ghost equation (cf. Eq. (B.1) in Appendix B), the couplings with background abelian gauge fields $\hat V_\mu$ are completely fixed [10, 4]. In the following formulae, only the background fields $\hat V^\alpha_\mu$ and $\hat V^a_\mu$, for the $SU(2)$ and $SU(3)$ part of the gauge group, are taken into account. The most general counterterm containing only gauge fields reads

$$\begin{aligned}
\int \mathrm{d}^4x \Big[&\xi_{F^2,1}\, \hat F^{\alpha,\mu\nu}\hat F_{\alpha,\mu\nu} + \xi_{F^2,2}\, \hat F^{a,\mu\nu}\hat F_{a,\mu\nu} + \xi_{F^2,3}\, F^{\mu\nu}F_{\mu\nu} \\
&+ \xi_{FVV,1}\, \epsilon_{\alpha\beta\gamma}\hat F^{\alpha,\mu\nu}V^\beta_\mu V^\gamma_\nu + \xi_{FVV,2}\, f_{abc}\hat F^{a,\mu\nu}V^b_\mu V^c_\nu \\
&+ \xi_{F\nabla V,1}\, \hat F^{\alpha,\mu\nu}\left(\hat\nabla_\mu V_\nu\right)_\alpha + \xi_{F\nabla V,2}\, \hat F^{a,\mu\nu}\left(\hat\nabla_\mu V_\nu\right)_a \\
&+ \xi_{\nabla V^2,1}\left(\hat\nabla_\mu V_\nu\right)^\alpha\left(\hat\nabla^\mu V^\nu\right)_\alpha + \xi_{\nabla V^2,2}\left(\hat\nabla^\mu V_\mu\right)^\alpha\left(\hat\nabla^\nu V_\nu\right)_\alpha \\
&+ \xi_{\epsilon FVV,1}\epsilon_{\alpha\beta\gamma}\epsilon^{\mu\nu\rho\sigma}\hat F_{\alpha,\mu\nu}V^b_\beta V^\gamma_\sigma + \xi_{\epsilon FVV,2} f_{abc}\epsilon^{\mu\nu\rho\sigma}\hat F_{a,\mu\nu}V^b_b V^c_\sigma \\
&+ \xi_{\nabla V^2,3}\left(\hat\nabla_\mu V_\nu\right)^a\left(\hat\nabla^\mu V^\nu\right)_a + \xi_{\nabla V^2,4}\left(\hat\nabla^\mu V_\mu\right)^a\left(\hat\nabla^\nu V_\nu\right)_a \\
&+ \xi_{V^2\nabla V,1}\,\epsilon_{\alpha\beta\gamma}\left(\hat\nabla^\mu V^\nu\right)^\alpha V^\beta_\mu V^\gamma_\nu + \xi_{V^2\nabla V,2}\, f_{abc}\left(\hat\nabla^\mu V^\nu\right)^a V^b_\mu V^c_\nu \\
&+ \xi_{V^2\nabla V,3}\, d_{abc}\left(\hat\nabla^\mu V^\nu\right)^a V^b_\mu V^c_\nu + \xi_{V^2\nabla V,4}\, d_{abc}\left(\hat\nabla^\mu V_\mu\right)^a V^b_\nu V^{c,\nu} \\
&+ \xi_{V^4,1}\, V^{\beta,\mu}V^{\gamma,\nu}V_{\beta,\mu}V_{\gamma,\nu} + \xi_{V^4,2}\, V^{\beta,\mu}V^{\gamma,\nu}V_{\beta,\nu}V_{\gamma,\mu} \\
&+ \xi_{V^4,3}\, V^{b,\mu}V^{a,\nu}V_{b,\mu}V_{a,\nu} + \xi_{V^4,4}\, V^{b,\mu}V^{a,\nu}V_{b,\nu}V_{a,\mu}
\end{aligned}$$

$$
\begin{aligned}
&+ \xi_{V^4,5}\, d_{ab}^{\ x} d_{xcd} V^{b,\mu} V^{a,\nu} V_{c,\mu} V_{d,\nu} \\
&+ \xi_{V^4,6}\, V^{a,\mu} V^{a,\nu} V_{\alpha,\mu} V_{\alpha,\nu} + \xi_{V^4,7}\, V^{a,\mu} V^{a,\mu} V_{\alpha,\nu} V_{\alpha,\nu} \\
&+ \xi_{V^4,8}\, f_{ab}^{\ x} f_{xcd} V^{b,\mu} V^{a,\nu} V_{c,\mu} V_{d,\nu} + \xi_{V^4,9}\, \epsilon_{\alpha\beta}^{\ x} \epsilon_{x\gamma\delta} V^{\beta,\mu} V^{\alpha,\nu} V_{\gamma,\mu} V_{\delta,\nu} \\
&+ \xi_{V^2,1}\, V^{\beta,\mu} V_{\beta,\mu} + \xi_{V^2,2}\, V^{b,\mu} V_{b,\mu} \Big], \qquad \text{(C.6)}
\end{aligned}
$$

where $\hat{F}^{\alpha,\mu\nu}$ and $\hat{F}^{a,\mu\nu}$ are the $SU(2)$ and $SU(3)$ background gauge field strengths, respectively. $F^{\mu\nu}$ is the abelian quantum gauge field strength and d^{abc} is the totally symmetric tensor in the adjoint representation of $su(3)$.

It is clear that at the one-loop order only the quantum field independent counterterms contribute. This means that only coefficients $\xi_{F^2,1}, \xi_{F^2,1}$ and $\xi_{F^2,3}$ in the first line of Eq. (C.6) are needed. At two loops, the inspection of diagrams reveals that also the counterterms quadratic in the quantum fields, namely $\xi_{FV^2,1}, \ldots, \xi_{\nabla V^2,4}, \xi_{V^2,1}$ and $\xi_{V^2,1}$, are necessary. At higher orders, all the other coefficients become important.

2. *Mixed scalar and gauge sector*

Due to the absence of the background partner to the abelian gauge field V_μ, the covariant derivative of scalars Φ_i and fermions ψ_I is defined with respect to the abelian quantum gauge field and the $SU(2) \times SU(3)$ background gauge fields: $\hat{\nabla}_\mu \Phi_i = \partial_\mu \Phi_i - V_\mu t^0_{ij} \Phi^j - \hat{V}_{\alpha\mu} t^\alpha_{ij} \Phi^j$ and $\hat{\nabla}_\mu \psi_I = \partial_\mu \psi_I - V_\mu T^0_{IJ} \psi^J - \hat{V}_{\alpha\mu} T^\alpha_{IJ} \psi^J - \hat{V}_{a\mu} T^a_{IJ} \psi^J$. Here t^0_{ij} and T^0_{IJ} are the hypercharge generators in the scalar and fermion representations.

The general counterterm for the kinetic terms for scalars and their interaction with gauge fields is described by the following expression

$$
\begin{aligned}
\int \mathrm{d}^4x \Big[&\xi_{\nabla\hat\Phi\nabla\hat\Phi} \left(\hat\nabla^\mu(\hat\Phi + v)\right)^i \left(\hat\nabla_\mu(\hat\Phi + v)\right)_i + \xi_{\nabla\hat\Phi\nabla\Phi} \left(\hat\nabla^\mu(\hat\Phi + v)\right)^i \left(\hat\nabla_\mu \Phi\right)_i \\
&+ \xi_{\nabla\hat\Phi V\hat\Phi} \left(\hat\nabla^\mu(\hat\Phi + v)\right)^i t^\alpha_{ij} V_{\alpha,\mu} \left(\hat\Phi + v\right)^j + \xi_{\nabla\hat\Phi V\Phi} \left(\hat\nabla^\mu(\hat\Phi + v)\right)^i t^\alpha_{ij} V_{\alpha,\mu} \Phi^j \\
&+ \xi_{\nabla\Phi\nabla\Phi} \left(\hat\nabla^\mu \Phi\right)^i \left(\hat\nabla_\mu \Phi\right)_i + \xi_{\nabla\Phi V\hat\Phi} \left(\hat\nabla^\mu \Phi\right)^i t^\alpha_{ij} V_{\alpha,\mu} \left(\hat\Phi + v\right)^j \\
&+ \xi_{\nabla\Phi V\Phi} \left(\hat\nabla^\mu \Phi\right)^i t^\alpha_{ij} V_{\alpha,\mu} \Phi^j + \xi_{V^2\Phi\hat\Phi,1}\, t^\alpha_{ij} V_{\alpha,\mu} \Phi^j t^{\beta,i}_{\ \ k} V^\mu_\beta \left(\hat\Phi + v\right)^k \\
&+ \xi_{V^2\hat\Phi^2,1}\, t^\alpha_{ij} V_{\alpha,\mu} \left(\hat\Phi + v\right)^j t^{\beta,i}_{\ \ k} V^\mu_\beta \left(\hat\Phi + v\right)^k + \xi_{V^2\Phi^2,1}\, t^\alpha_{ij} V_{\alpha,\mu} \Phi^j t^{\beta,i}_{\ \ k} V^\mu_\beta \Phi^k \\
&+ \xi_{V^2\hat\Phi^2,2}\, V_{\alpha,\mu} V^{\alpha,\mu} \left(\hat\Phi + v\right)_j \left(\hat\Phi + v\right)^j + \xi_{V^2\Phi\hat\Phi,2}\, V_{\alpha,\mu} V^{\alpha,\mu} \Phi_j \left(\hat\Phi + v\right)^j \\
&+ \xi_{V^2\Phi^2,2}\, V_{\alpha,\mu} V^{\alpha,\mu} \Phi_j \Phi^j + \xi_{V^2\hat\Phi^2,3}\, V_{a,\mu} V^{a,\mu} \left(\hat\Phi + v\right)_j \left(\hat\Phi + v\right)^j \\
&+ \xi_{V^2\Phi\hat\Phi,3}\, V_{a,\mu} V^{a,\mu}\, \Phi_j \left(\hat\Phi + v\right)^j + \xi_{V^2\Phi^2,3}\, V_{a,\mu} V^{a,\mu}\, \Phi_j \Phi^j \Big]. \qquad \text{(C.7)}
\end{aligned}
$$

Note that the last three terms are a consequence of the background gauge invariance in the $SU(3)$ part of the gauge group.

3. *Scalar sector*

To complete the counterterms for the scalar sector, we must list the ones that reconstruct the correct scalar potential

$$\int \mathrm{d}^4x \Big[\xi_{\Phi^2}\,\Phi_j\Phi^j + \xi_{\hat\Phi^2}\,\left(\hat\Phi+v\right)_j\left(\hat\Phi+v\right)^j + \xi_{\Phi\hat\Phi}\,\Phi_j\left(\hat\Phi+v\right)^j + \xi_{\Phi^4}\,\left(\Phi_j\Phi^j\right)^2 + \xi_{\hat\Phi^4}\,\left[\left(\hat\Phi+v\right)_j\left(\hat\Phi+v\right)^j\right]^2 + \xi_{\Phi\hat\Phi^2,1}\,\left[\Phi_j\left(\hat\Phi+v\right)^j\right]^2 + \xi_{\Phi\hat\Phi^2,2}\,\Phi_j\Phi^j\left(\hat\Phi+v\right)_k\left(\hat\Phi+v\right)^k + \xi_{\Phi^3\hat\Phi}\,\Phi_j\left(\hat\Phi+v\right)^j\Phi_k\Phi^k + \xi_{\hat\Phi^3\Phi}\,\Phi_j\left(\hat\Phi+v\right)^j\left(\hat\Phi+v\right)_k\left(\hat\Phi+v\right)^k\Big]\,. \tag{C.8}$$

Note that at actually one- and two-loop order only few of these terms are needed.

4. *Fermion sector*

Finally, we have to discuss the fermionic terms. Again, it is easy to establish the most general background gauge invariant contributions

$$\sum_{\psi\psi'}\int \mathrm{d}^4x\Big[\xi_{\bar\psi\nabla\psi'}\bar\psi_I\hat{\nabla}^{IJ}\psi'_J + \xi_{\bar\psi\psi'V,1}\bar\psi_I T^{IJ}_\alpha\,\not{V}^\alpha\psi'_J + \xi_{\bar\psi\psi'V,2}\bar\psi_I T^{IJ}_a\,\not{V}^a\psi'_J + \xi_{\bar\psi\psi'\hat\Phi,m}Y^{IJ,j}_m\bar\psi_I\psi'_J\left(\hat\Phi+v\right)_j + \xi_{\bar\psi\psi'\Phi,m}Y^{IJ,j}_m\bar\psi_I\psi'_J\Phi_j + \mathrm{h.c.}\Big]\,. \tag{C.9}$$

Here, the fermionic indices run over the $SU(2)$ isospin, the color and the flavours of fermions. In order to simplify the notation, we introduced $Y^{IJ,j}_m$ to denote m independent couplings between scalars and fermions. These tensors satisfy the relations $T_{\alpha,K}^{\ \ I}Y^{KJ,j}_m + Y^{IK,j}_m T_{\alpha,K}^{\ \ J} + t_{\alpha,r}^{\ \ i}Y^{IJ,r}_m = 0$, $T_{0,K}^{\ \ I}Y^{KJ,j}_m + Y^{IK,j}_m T_{0,K}^{\ \ J} + t_{0,r}^{\ \ i}Y^{IJ,r}_m = 0$ and $T_{a,K}^{\ \ I}Y^{KJ,j}_m + Y^{IK,j}_m T_{a,K}^{\ \ J} = 0$ for every m.

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