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AN INTRODUCTION TO CHPT

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Abstract

These lectures provide an elementary introduction to Chiral Perturbation Theory, focused on the sector of pseudoscalar meson interactions. Basic concepts and technical methods of this approach are discussed on general grounds and with the help of a few specific examples.

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Prologue

Chiral Perturbation Theory (CHPT) is nothing but the low-energy limit of the Standard Model (SM) or, to be more precise, the effective quantum field theory describing hadronic interactions according to the SM, below the breaking scale of chiral symmetry ($E \ll \Lambda_\chi \sim 1 \text{ GeV}$). This theory, founded a long time ago by the pioneering works of Weinberg, ¹⁾ and Gasser and Leutwyler, ²⁾ is nowadays a rather mature subject. On one side several two-loop calculations have been performed in the purely mesonic sector, reaching, in some cases, a very high degree of precision. On the other side the original formulation has been successfully extended in several directions, including, for instance, heavy quark fields, bound-state dynamics, non-zero temperature effects, etc.

A complete overview of the subject would be a tremendous task, and we could certainly not provide it in the four hours that we were given for these lectures. The interested reader is referred to some excellent reviews ^{3, 4, 5)} for a broader survey of the subject. The purpose of these lectures is to provide a basic introduction to CHPT. We will therefore restrict our attention to the simplest case, namely the mesonic sector, discussing in detail the determination of the effective Lagrangians and the calculation of a few specific quantities.

The lectures are organized as follows: motivations, basic principles and the tree-level structure of CHPT are presented in the first lecture. Various aspects of loop calculations are discussed in the following two lectures. In particular, the renormalization at the one-loop level is presented in the second lecture, whereas the third lecture is devoted to studying the properties of unitarity and analyticity, and to showing how dispersion relations can be combined with the chiral expansion. Finally, the issue of non-leptonic weak interactions is introduced in the last lecture.

1 Generalities and lowest-order Lagrangians

1.1 Effective quantum field theories

Within the Standard Model the interactions between quarks and gluons, ruled by Quantum Chromo Dynamics (QCD), are highly non-perturbative at energies below the breaking scale of chiral symmetry. This makes very difficult any description of the low-energy hadronic world in terms of partonic degrees of freedom. On the other hand, the spectrum of the theory is rather simple at low energies, containing only the octet of light pseudoscalar mesons: π , K and η . Experimentally we also

know that, at very low energies, these pseudoscalar mesons interact weakly, both among themselves and with nucleons. It is then reasonable to expect that QCD can be treated perturbatively even at low energies, provided a suitable transformation of degrees of freedom is performed. This is exactly the goal of Chiral Perturbation Theory, where the pseudoscalar mesons are assumed to be fundamental degrees of freedom.

Having an intrinsic energy limitation and being the low-energy limit of a more fundamental theory, CHPT is a typical example of *effective* quantum field theory (EQFT), a widely used tool in modern physics. The basic principle of any EQFT is that, in a given energy range, only few degrees of freedom are relevant and need to be described by dynamical fields. The remaining degrees of freedom of the more general theory can be integrated out, leading to effects that are encoded in the coefficients of appropriate local operators.

Assuming this general point of view, all known quantum field theories, including the Standard Model, can be considered as effective. An important distinction, however, is provided by the degree of renormalizability. In general the requirement of renormalizability, understood in the classical sense, is not mandatory within an EQFT. ⁶⁾ Indeed if the theory is meant to be valid only for energies below a given cut-off Λ , and we perform an expansion of the physical amplitudes in powers of E/Λ , we can impose the weaker condition that, for any $n > 0$, the number of counterterms needed to regularize the amplitudes and contributing at $\mathcal{O}[(E/\Lambda)^n]$ is finite. This condition is sufficient to ensure a predictive power to the theory for $E < \Lambda$. In other words, within an EQFT we only need renormalizability order by order in the energy expansion. The classical requirement of renormalizability, is a stronger constraint, which prevents a naive derivation of the intrinsic cut-off of the theory. The SM belongs to this restricted subclass of EQFTs without an intrinsic cut-off, whereas CHPT belongs to the more general case.

1.2 Chiral symmetry

Neglecting light-quark masses, the QCD Lagrangian can be written as

$$\mathcal{L}_{QCD}^{(0)} = \sum_{q=u,d,s} \bar{q} \gamma^\mu \left(i \partial_\mu - g_s \frac{\lambda_a}{2} G_\mu^a \right) q - \frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} + \mathcal{O}(\text{heavy quarks}). \quad (1)$$

Apart from the $SU(3)_C$ local invariance, $\mathcal{L}_{QCD}^{(0)}$ has a global invariance under the group $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$, where N_f is the number of massless quarks ($N_f = 3$ in the case considered above). The $U(1)_V$ symmetry, which survives

also in the case of non-vanishing quark masses, is exactly conserved and its generator is the baryonic number. The $U(1)_A$ symmetry is explicitly broken at the quantum level by the Abelian anomaly.⁷⁾ Finally $G = SU(3)_L \times SU(3)_R$ is the group of chiral transformations:

$$\psi_{L,R} \xrightarrow{G} g_{L,R} \psi_{L,R}, \quad \text{where} \quad \psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad \text{and} \quad g_{L,R} \in G. \quad (2)$$

If the operator $\bar{\psi}\psi$ has a non-vanishing expectation value in the ground state ($\langle 0|\bar{\psi}\psi|0\rangle \neq 0$), or in the presence of a non-vanishing quark condensate, chiral symmetry is spontaneously broken. The subgroup that remains unbroken after the breaking of G is $H = SU(3)_V \equiv SU(3)_{L+R}$, the famous $SU(3)$ of the *eightfold way*,⁸⁾ or, in the limit where only two quarks are kept massless, the $SU(2)$ group of isospin transformations.

The fundamental idea of CHPT is that, in the chiral limit ($m_u = m_d = m_s = 0$ or, eventually, $m_u = m_d = 0$), the light pseudoscalar mesons are the Goldstone bosons generated by the spontaneous breaking of G into H (in the $SU(3)$ case, $m_u = m_d = m_s = 0$, the full octet of pseudoscalar mesons is identified with Goldstone fields, whereas in the $SU(2)$ case, $m_u = m_d = 0$, only the three pions are). These light particles have the correct quantum numbers to be associated with the generators of G/H , as required by the Goldstone theorem.⁹⁾ Moreover, since Goldstone fields can always be redefined so that they interact only through derivative couplings,⁹⁾ this hypothesis justifies the soft behaviour of pseudoscalar interactions at low energies. If pseudoscalar mesons were effectively Goldstone bosons, they would had been massless. This does not happen in the real world, owing to the light-quark mass terms, which explicitly break G . Since $m_{u,d,s} < \Lambda_\chi$, it is natural to expect that these breaking terms can be treated as small perturbations. The fact that pseudoscalar-meson masses are much smaller than the typical hadronic scale indicates that also this hypothesis is reasonable. Clearly this approximation works much better in the $SU(2)$ case ($M_\pi^2/M_\rho^2 \sim 0.03$) than in the $SU(3)$ one ($M_\pi^2/M_\rho^2 \sim 0.4$). Summarizing, the two basic assumptions of CHPT are the following:

1. *In the chiral limit the $SU(3)_L \times SU(3)_R$ symmetry of the QCD Lagrangian is spontaneously broken into $SU(3)_{L+R}$ and the pseudoscalar meson fields can be identified with the corresponding Goldstone bosons.*
2. *The mass terms of light quarks can be treated as small perturbations around the chiral limit.*

According to these hypotheses, in order to describe the QCD interactions of pseudoscalar mesons it is necessary to construct, in terms of Goldstone-boson fields, the most general Lagrangian invariant under G and add to it the explicit breaking terms (light-quark masses) that transform linearly under G .¹⁾ The Lagrangian built in this way necessarily contains an infinite number of operators. Nevertheless, as anticipated, only a finite number of operators contribute at $\mathcal{O}[(E/\Lambda_\chi)^n]$. Therefore the theory can reach an arbitrary degree of precision for processes occurring at $E < \Lambda_\chi$, provided a sufficient (but finite) number of couplings is fixed by experimental data.

1.3 Non-linear realization of chiral symmetry

Denoting by V_i the generators of H and by A_i the remaining generators of G , any element of G can be unambiguously decomposed as $g = e^{\xi_i A_i} e^{\eta_i V_i}$. The Goldstone-boson fields are associated to the coordinates ξ_i of the coset space G/H . To understand how these transform under G we consider the action of a generic element $g \in G$ on $u(\xi_i) = e^{\xi_i A_i}$:

$$g \in G \quad g e^{\xi_i A_i} = e^{\xi'_i(g, \xi) A_i} e^{\eta'_i(g, \xi) V_i} . \quad (3)$$

The transformation $u(\xi_i) \xrightarrow{G} u(\xi'_i)$ provides a non-linear realization of the group G .¹⁰⁾ This realization is not linear since V_i 's and A_i 's do not commute ($[V_i, A_j] \sim A_k$); however, it becomes linear if restricted to the subgroup H :

$$h_0 = e^{\eta_i^0 V_i} \in H , \quad h_0 e^{\xi_i A_i} = \left[e^{\eta_i^0 V_i} e^{\xi_i A_i} e^{-\eta_i^0 V_i} \right] e^{\eta_i^0 V_i} . \quad (4)$$

As shown by Callan, Coleman, Wess and Zumino,¹⁰⁾ this non-linear realization provides the most general tool to construct operators that transform linearly under G (or invariant operators) in terms of the Goldstone-boson fields generated by the spontaneous breakdown of G into H .

In the specific case of chiral symmetry an important additional information is provided by the automorphism induced on G by parity [$P : (A_i, V_i) \rightarrow (-A_i, V_i)$]. This implies that if $g_R : u(\xi_i) \rightarrow u(\xi'_i)$, then $g_L : u(-\xi_i) = u(\xi_i)^\dagger \rightarrow u(\xi'_i)^\dagger$. We can therefore write

$$\begin{aligned} u(\xi_i) &\xrightarrow{G} g_R u(\xi_i) h^{-1}(g, \xi_i) = h(g, \xi_i) u(\xi_i) g_L^{-1} , \\ u(\xi_i)^\dagger &\xrightarrow{G} g_L u(\xi_i)^\dagger h^{-1}(g, \xi_i) = h(g, \xi_i) u(\xi_i)^\dagger g_R^{-1} , \end{aligned} \quad (5)$$

where $h(g, \xi_i) = e^{\eta'(g, \xi_i) V}$. At this point we have all the ingredients to build generic operators transforming linearly under G , starting from their projection into H . For

instance, given a generic field Ψ transforming linearly under H as $\Psi \xrightarrow{H} e^{\eta_i V_i} \Psi e^{-\eta_i V_i}$, in the non-linear realization of G we find

$$\Psi \xrightarrow{G} h(g, \xi_i) \Psi h^{-1}(g, \xi_i) , \quad (6)$$

thus any product of the type $(u, u^\dagger) \times \Psi \times (u, u^\dagger)$ transforms linearly under G :

$$\begin{aligned} u \Psi u^\dagger &\xrightarrow{G} g_R (u \Psi u^\dagger) g_R^{-1} , \\ u^\dagger \Psi u &\xrightarrow{G} g_L (u^\dagger \Psi u) g_L^{-1} , \\ u \Psi u &\xrightarrow{G} g_R (u \Psi u) g_L^{-1} , \\ u^\dagger \Psi u^\dagger &\xrightarrow{G} g_L (u^\dagger \Psi u^\dagger) g_R^{-1} . \end{aligned} \quad (7)$$

The above procedure can be generalized to the case of fields Ψ' belonging to different representations of H . Furthermore, starting from the derivative of u and u^\dagger we can define the following operators:

$$u_\mu = i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger) = iu^\dagger \partial_\mu U u^\dagger = u_\mu^\dagger , \quad (8)$$

$$\Gamma_\mu = \frac{1}{2}(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger) = -\Gamma_\mu^\dagger . \quad (9)$$

It is easy to verify that both u_μ and the covariant derivative of Ψ ,

$$\nabla_\mu \Psi = \partial_\mu \Psi - [\Gamma_\mu, \Psi] , \quad (10)$$

transform according to Eq. (6). Thus, similarly to Eq. (7), with appropriate contractions with u and u^\dagger we can construct operators transforming linearly under G also starting from u_μ and $\nabla_\mu \Psi$. Concerning the construction of terms invariant under G , we note that given a generic operator Q transforming as $Q \xrightarrow{G} g_L Q g_L^{-1}$ or $Q \xrightarrow{G} g_R Q g_R^{-1}$, an invariant term is obtained by the trace of Q in flavour space, denoted in the following by $\langle Q \rangle$.

The choice of coordinates in the coset space G/H is not unique; however, in any given set of coordinates we can introduce a field $u(\xi_i)$ transforming as in Eq. (5).¹⁰⁾ The freedom in the choice of coordinates implies that the parametrization of u in terms of the pseudoscalar meson fields is not unique. In the following we shall adopt the exponential parametrization in the 3×3 flavour space, defined by

$$u^2 = U = e^{i\sqrt{2}\Phi/F} ,$$

$$\Phi = \frac{1}{\sqrt{2}} \sum_i \lambda_i \phi^i = \begin{bmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{bmatrix}, \quad (11)$$

where η_8 denotes the octet component of the η meson. The parameter F appearing in Eq. (11) is a dimensional constant ($\dim[F] = \dim[\Phi]$) that, as we shall see in the following, can be related to the decay constant of pseudoscalar mesons.

1.4 Lowest-order (strong) Lagrangian

In the absence of external fields it is impossible to build non-trivial invariant operators in terms of u and u^\dagger only, without their derivatives: it is necessary to have at least two derivatives acting on u or u^\dagger in order to build a non-trivial structure invariant under both chiral and Lorentz symmetries. If only two derivatives are considered this structure is unique:

$$\langle u_\mu u^\mu \rangle = \langle \partial_\mu U \partial^\mu U^\dagger \rangle. \quad (12)$$

Fixing the coupling constant of the operator $\langle u_\mu u^\mu \rangle$ so as to reproduce the correct kinetic term of spinless fields, leads to

$$\tilde{\mathcal{L}}_S^{(2)} = \frac{F^2}{4} \langle \partial_\mu U \partial^\mu U^\dagger \rangle. \quad (13)$$

This Lagrangian is the chiral realization of $\mathcal{L}_{QCD}^{(0)}$ at the lowest order in the derivative expansion.

Expanding $\tilde{\mathcal{L}}_S^{(2)}$ in powers of Φ leads to an infinite series of operators, whose couplings are all determined in terms of F :

$$\tilde{\mathcal{L}}_S^{(2)} = \frac{F^2}{4} \langle \partial_\mu U \partial^\mu U^\dagger \rangle = \frac{1}{2} \langle \partial_\mu \Phi \partial^\mu \Phi \rangle + \frac{1}{6F^2} \langle [\partial_\mu \Phi, \Phi] [\partial^\mu \Phi, \Phi] \rangle + \mathcal{O}(\Phi^6). \quad (14)$$

From $\tilde{\mathcal{L}}_S^{(2)}$ one can therefore determine the amplitude for any process of the type $\pi_1 \pi_2 \dots \pi_n \rightarrow \pi'_1 \pi'_2 \dots \pi'_m$ in the chiral limit. For example, considering the second term in Eq. (14) it is easy to show that

$$\mathcal{A}(\pi^+ \pi^0 \rightarrow \pi^+ \pi^0) \Big|_{[m_q=0]} = \frac{(p_+ - p'_+)^2}{F^2} + \mathcal{O}(p^4). \quad (15)$$

To parametrize the breaking terms induced by quark masses, and also to generate in a systematic way the Green functions of quark currents, it is convenient

to insert appropriate external sources both in the QCD Lagrangian and in its chiral realization. Following Gasser and Leutwyler,²⁾ we introduce the sources v_μ , a_μ , s and p , which transform as

$$\begin{aligned} r_\mu &= v_\mu + a_\mu \xrightarrow{G} g_R r_\mu g_R^{-1}, \\ l_\mu &= v_\mu - a_\mu \xrightarrow{G} g_L l_\mu g_L^{-1}, \\ s + ip &\xrightarrow{G} g_R (s + ip) g_L^{-1}, \\ s - ip &\xrightarrow{G} g_L (s - ip) g_R^{-1}, \end{aligned} \quad (16)$$

and we consider the Lagrangian

$$\mathcal{L}_{QCD}(v, a, s, p) = \mathcal{L}_{QCD}^{(0)} + \bar{\psi} \gamma^\mu (v_\mu + a_\mu \gamma_5) \psi - \bar{\psi} (s - ip \gamma_5) \psi. \quad (17)$$

In this way we reach two interesting results:⁵⁾

- The generating functional

$$e^{iZ(v, a, s, p)} = \int \mathcal{D}q \mathcal{D}\bar{q} \mathcal{D}G e^{i \int d^4x \mathcal{L}_{QCD}(v, a, s, p)} \quad (18)$$

is explicitly invariant under chiral transformations, but the explicit breaking of G can directly be obtained by calculating the Green functions, i.e. the functional derivatives of $Z(v, a, s, p)$, at

$$v_\mu = a_\mu = p = 0 \quad s = M_q = \text{diag}(m_u, m_d, m_s). \quad (19)$$

- The global symmetry G can be promoted to a local one by modifying the transformation laws of l_μ and r_μ in

$$\begin{aligned} r_\mu &= v_\mu + a_\mu \xrightarrow{G} g_R r_\mu g_R^{-1} + i g_R \partial_\mu g_R^{-1}, \\ l_\mu &= v_\mu - a_\mu \xrightarrow{G} g_L l_\mu g_L^{-1} + i g_L \partial_\mu g_L^{-1}. \end{aligned} \quad (20)$$

Then the gauge fields of electroweak interactions can be automatically included as external fields by the substitution

$$\begin{aligned} v_\mu &\rightarrow -eQ A_\mu - \frac{g}{2 \cos \theta_W} \left[Q \cos(2\theta_W) - \frac{1}{6} \right] Z_\mu - \frac{g}{2\sqrt{2}} (T_+ W_\mu^+ + \text{h.c.}), \\ a_\mu &\rightarrow +\frac{g}{2 \cos \theta_W} \left[Q - \frac{1}{6} \right] Z_\mu + \frac{g}{2\sqrt{2}} (T_+ W_\mu^+ + \text{h.c.}), \end{aligned} \quad (21)$$

where

$$Q = \frac{1}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (22)$$

and V_{ij} denote the elements of the Cabibbo–Kobayashi–Maskawa matrix.

As a consequence, Green functions for processes with external photons, Z or W bosons, can simply be obtained as functional derivatives of $Z(v, a, s, p)$.

The chiral realization of $Z(v, a, s, p)$ at the lowest order in the derivative expansion is given by the classical action obtained from $\tilde{\mathcal{L}}_S^{(2)}$, after having included the external sources in a chiral invariant way. Concerning spin-1 sources, this can be achieved by means of the minimal substitution

$$\partial_\mu U \rightarrow D_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu \quad (23)$$

or, in principle, introducing new operators written in terms of the tensors

$$\begin{aligned} F_L^{\mu\nu} &= \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu] , \\ F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu] . \end{aligned} \quad (24)$$

From Eq. (23) it appears convenient to assign the same power counting to derivatives of u and to the sources a^μ and v^μ , so that $D_\mu U$ becomes a homogeneous term of first order in the derivative (or chiral) expansion:

$$\begin{aligned} U &\sim \mathcal{O}(p^0) , \\ u^\mu, a^\mu, v^\mu &\sim \mathcal{O}(p^1) , \\ F_{L,R}^{\mu\nu} &\sim \mathcal{O}(p^2) . \end{aligned} \quad (25)$$

According to this assignment, the Lorentz-invariant terms containing the external tensors $F_{L,R}^{\mu\nu}$ are at least of $\mathcal{O}(p^4)$ and do not appear at the lowest order.

Regarding spin-0 sources, the most natural power-counting assignment is given by: ²⁾

$$s, p \sim \mathcal{O}(p^2) . \quad (26)$$

As we shall see in the following, this choice is welcome since it implies $M_\pi^2 \sim \mathcal{O}(p^2)$ and leads to the Gell-Mann–Okubo mass formula. ^{8, 11)}

We are now able to write down the most general Lagrangian invariant under G , of order p^2 , which includes pseudoscalar mesons and external sources. This is

$$\mathcal{L}_S^{(2)} = \frac{F^2}{4} \langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U \chi^\dagger \rangle , \quad (27)$$

where

$$\chi = 2B(s + ip) . \quad (28)$$

$\mathcal{L}_S^{(2)}$ is completely determined by chiral symmetry but for the couplings F and B , which have to be constrained from experimental data. These two couplings are in turn related to two fundamental order parameters of the spontaneous breakdown of chiral symmetry: the pion decay constant F_π , defined by $\langle 0|\bar{\psi}\gamma^\mu\gamma_5\psi|\pi^+(p)\rangle = i\sqrt{2}F_\pi p^\mu$, and the quark condensate $\langle 0|\bar{\psi}\psi|0\rangle$. Indeed, differentiating with respect to the external sources, we find

$$\langle 0|\bar{\psi}\gamma^\mu\gamma_5\psi|\pi^+(p)\rangle = \langle 0|\frac{\delta S^{(2)}}{\delta a_\mu}|\pi^+(p)\rangle = i\sqrt{2}F p^\mu, \quad (29)$$

$$\langle 0|\bar{\psi}\psi|0\rangle = -\langle 0|\frac{\delta S^{(2)}}{\delta s}|0\rangle = -F^2 B, \quad (30)$$

where $S^{(2)}$ is the classical action:

$$S^{(2)} = \int d^4x \mathcal{L}_S^{(2)}. \quad (31)$$

It is important to stress that relations (29–30) are exactly valid only in the chiral limit: in the real case ($m_q \neq 0$) they are modified by $\mathcal{O}(m_q^2) \sim \mathcal{O}(p^4)$ terms.

The pion decay constant is experimentally known from the process $\pi^+ \rightarrow \mu^+ \nu$: $F_\pi = 92.4$ MeV. In principle one could determine F also from the kaon decay constant, defined analogously to F_π and measured to be $F_K = 114$ MeV. The difference between F_π and F_K is a typical $\mathcal{O}(p^4)$ effect, which goes beyond lowest order. However, since $\mathcal{O}(p^4)$ effects are expected to be larger in the case of F_K , the most natural determination of F at $\mathcal{O}(p^2)$ is provided by F_π .

Contrary to the decay constants, the quark condensate is not directly related to any physical observable. It is the product $B \times m_q$ that can be constrained by means of experimental data. This appears in the quadratic terms of $\mathcal{L}_S^{(2)}$ and is therefore related to pseudoscalar meson masses:

$$\begin{aligned} M_{\pi^+}^2 &= (m_u + m_d)B, \\ M_{K^+}^2 &= (m_u + m_s)B, \\ M_{K^0}^2 &= (m_d + m_s)B. \end{aligned} \quad (32)$$

The analogous equation for $M_{\eta_8}^2$ contains no new parameter and gives rise to a consistency relation:

$$3M_{\eta_8}^2 = 4M_K^2 - M_\pi^2, \quad (33)$$

the famous Gell-Mann–Okubo mass formula,^{8, 11)} well satisfied by experimental data under the assumption $M_\eta = M_{\eta_8}$. The validity of the Gell-Mann–Okubo relation provides a significant a posteriori check that the $\mathcal{O}(m_q^2)$ corrections to Eqs. (32) are small and that the assignment $s \sim \mathcal{O}(p^2)$ is consistent.

1.5 Quark-mass ratios and isospin breaking

Similarly to the quark condensate, also the absolute value of light-quark masses cannot be determined within CHPT. Relations (32) provide, however, stringent constraints on quark mass ratios.

In addition to $\mathcal{O}(m_q^2)$ corrections, relations (32) can be affected by electromagnetic effects. At leading order in the chiral expansion, the latter can only depend on meson charges, and we can therefore write

$$\begin{aligned}
 M_{\pi^0}^2 &= (m_u + m_d)B + \mathcal{O}(m_q^2) , \\
 M_{K^0}^2 &= (m_d + m_s)B + \mathcal{O}(m_q^2) , \\
 M_{\pi^+}^2 &= (m_u + m_d)B + \alpha\Delta_{\text{e.m.}} + \mathcal{O}(m_q^2, \alpha m_q) , \\
 M_{K^+}^2 &= (m_u + m_s)B + \alpha\Delta_{\text{e.m.}} + \mathcal{O}(m_q^2, \alpha m_q) .
 \end{aligned} \tag{34}$$

Neglecting the small $\mathcal{O}(m_q^2, \alpha m_q)$ corrections and using the experimental values of pseudoscalar meson masses, one can extract from Eqs. (34) the following two ratios

$$\frac{m_d - m_u}{m_d + m_u} = \frac{(M_{K^0} - M_{K^+}) - (M_{\pi^0} - M_{\pi^+})}{M_{\pi^0}^0} = 0.22 + 0.07 = 0.29 , \tag{35}$$

$$\frac{m_s}{m_d + m_u} = \frac{(M_{K^+} - M_{\pi^+}) + (M_{K^0} - M_{\pi^0})}{M_{\pi^0}^0} = 25 . \tag{36}$$

Interestingly, the three light-quark masses turn out to be rather different: Eq. (35) indicates a sizable violation ($\sim 30\%$) of isospin symmetry and Eq. (36) shows that $SU(3)$ is not at all a good symmetry for quark masses. However, it is known that both symmetries, especially the isospin one, are usually well respected by strong interactions. For instance, considering the spectrum of vector mesons, it is found that isospin works at the (1–2)% level $[(M_\omega - M_\rho)/M_\rho \sim 1.5\%]$ and $SU(3)$ at the (10–20)% level $[(M_{K^*} - M_\rho)/M_\rho \sim 16\%]$.

The reason of this behaviour can be traced back to the smallness of quark masses with respect to the scale of chiral symmetry breaking. Indeed, although we cannot have a precise information about the absolute value of quark masses, by looking at the breaking of isospin and $SU(3)$ symmetries in the spectrum of hadrons we can infer the following hierarchy:

$$\begin{aligned}
 \Lambda_\chi &\sim 10^3 \text{ MeV} , \\
 m_s &\sim 10^2 \text{ MeV} , \\
 m_d, m_u &\sim 10^1 \text{ MeV} .
 \end{aligned} \tag{37}$$

The masses of all light hadrons, except for the would-be Goldstone bosons, remain different from zero in the chiral limit and are mainly determined by Λ_χ . Then isospin and $SU(3)$ symmetries are *accidental* consequences of the fact that $m_{u,d,s} \ll \Lambda_\chi$. The observation that both these symmetries are well respected in the hadronic world is a good a posteriori check of the basic assumptions of CHPT discussed in Section 1.2.

1.6 Lagrangian at order p^4

The Lagrangian (27) is only the first term of an infinite series. At the moment, in the purely strong sector the Lagrangian is known up to and including order p^6 .¹²⁾ Here we introduce the Lagrangian of order p^4 that was originally written down by Gasser and Leutwyler.²⁾ In order to derive this Lagrangian, one needs to list all possible chiral-invariant terms of order p^4 . It is not strictly necessary, but very useful, to also reduce this list to a minimum, using all possible trace identities (which depend on the number of light flavours) and also the classical equations of motion derived from the Lagrangian (27), in all possible ways. The interested reader is referred to the original articles,^{2, 12)} for more details on how this is done. Here we simply write down the Lagrangian in the case of three light flavours:

$$\begin{aligned}
\mathcal{L}_4 = & L_1 \langle D_\mu U^\dagger D^\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle \\
& + L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U \rangle + L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle \chi^\dagger U + \chi U^\dagger \rangle \\
& + L_5 \langle D_\mu U^\dagger D^\mu U (\chi^\dagger U + U^\dagger \chi) \rangle + L_6 \langle \chi^\dagger U + \chi U^\dagger \rangle^2 + L_7 \langle \chi^\dagger U - \chi U^\dagger \rangle^2 \\
& + L_8 \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle - i L_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle \\
& + L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu} \rangle + L_{11} \langle F_{R\mu\nu} F_R^{\mu\nu} + F_{L\mu\nu} F_L^{\mu\nu} \rangle + L_{12} \langle \chi^\dagger \chi \rangle , \quad (38)
\end{aligned}$$

where, for completeness, we have also written down the so-called ‘‘contact terms’’, those multiplying the constants L_{11} and L_{12} , which contain only external fields. We should add that at this order also the Lagrangian of Wess, Zumino and Witten¹³⁾ enters. This describes the effects due to the axial anomaly, which are therefore parity-violating. In the following we will not consider such effects.

2 One-loop graphs: renormalization

In the following two sections we will introduce the subject of loop calculations in CHPT. Technically, these loop calculations need no special introduction: any graduate student who has already made loop calculations (in QED for example) should be able to perform them also in CHPT also. On the other hand, at a conceptual

level, he/she may have doubts about the meaning of these loop calculations: how should one interpret the loops of pions, which contain contributions of pions of any virtuality, if one knows that at high energy scales these degrees of freedom are no longer relevant? The main aim of these two lectures is to answer these kinds of questions, and to illustrate the physical meaning of the loop contributions.

2.1 The scalar form factor of the pion

To illustrate the basic concepts of loop calculations in CHPT we will focus on one specific example: the scalar form factor of the pion, defined as

$$\langle \pi^i(p_1)\pi^j(p_2)|\hat{m}(\bar{u}u + \bar{d}d)|0\rangle =: \delta^{ij}\Gamma(t) \quad , \quad t = (p_1 + p_2)^2 \quad , \quad (39)$$

where $\hat{m} = (m_s + m_d)/2$. This matrix element is relevant to the decay $h \rightarrow \pi\pi$, which would have been the main decay mode for a light Higgs (of course this scenario is now experimentally excluded). The tree-level calculation of this matrix element is simple: in the Lagrangian (27) the coupling between the scalar source and two pions, which can be read from Eq. (28), does not involve derivatives and leads to

$$\Gamma(t) = 2\hat{m}B = M_\pi^2 + \mathcal{O}(p^4) \quad . \quad (40)$$

This result, which we worked out from the Lagrangian, is actually a consequence of a general theorem, due to Feynman and Hellman.¹⁴⁾ This states that the expectation value of the perturbation in an eigenstate of the total Hamiltonian determines the derivative of the energy level with respect to the strength of the perturbation:

$$\hat{m} \frac{\partial M_\pi^2}{\partial \hat{m}} = \langle \pi | \hat{m} \bar{q} q | \pi \rangle = \Gamma(0) \quad . \quad (41)$$

The value of the form factor at zero momentum transfer is fixed by this theorem, and a simple power counting implies that at leading order the scalar form factor is a constant – at order p^2 the theorem completely fixes the form factor. On the other hand no general principle forbids a dependence of the form factor on t (to the contrary, they imply it, as we will see), and to generate this we necessarily have to go beyond leading order.

2.2 $SU(2)$ Lagrangian at order p^4

Before starting the loop calculation, let us have a look at what happens to the form factor once we include tree-level contributions from higher-order terms in the Lagrangian. The Lagrangian at order p^4 has been discussed in Sect. 1.6 in the case

of three light flavours. For the case at hand, the role of kaons and etas is marginal, if we stick to the very low-energy region. It is more convenient to use a simpler Lagrangian, by expanding around $m_u = m_d = 0$ and keeping m_s at its physical value. In this case the chiral symmetry is $SU(2)_L \times SU(2)_R$. The Lagrangian of order p^2 remains unchanged – the only change is that the field U , and its logarithm Φ are now 2×2 matrices. At order p^4 the Lagrangian is simpler, because we can use more trace identities to reduce the number of independent terms. For two light flavours this is a sum of seven terms²

$$\mathcal{L}_S^{(4)} = \sum_{i=1}^7 l_i P_i + \dots \quad , \quad (42)$$

where

$$\begin{aligned} P_1 &= \frac{1}{4} \langle u_\mu u^\mu \rangle^2 \quad , \quad P_2 = \frac{1}{4} \langle u_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle \quad , \\ P_3 &= \frac{1}{16} \langle \chi_+ \rangle^2 \quad , \quad P_4 = \frac{i}{4} \langle u_\mu \chi_-^\mu \rangle \quad , \quad P_5 = -\frac{1}{2} \langle f_{-\mu\nu} f^{\mu\nu} \rangle \quad , \\ P_6 &= \frac{1}{4} \langle [u_\mu, u_\nu] f_+^{\mu\nu} \rangle \quad , \quad P_7 = -\frac{1}{16} \langle \chi_- \rangle^2 \quad , \end{aligned} \quad (43)$$

and we have used the compact notation:

$$\begin{aligned} \chi_\pm &= u^\dagger \chi u^\dagger \pm u \chi^\dagger u \\ \chi_\pm^\mu &= u^\dagger D^\mu \chi u^\dagger \pm u D^\mu \chi^\dagger u \\ f_\pm^{\mu\nu} &= u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u \quad . \end{aligned} \quad (44)$$

Only two of these seven terms contribute to the scalar form factor: the terms proportional to l_3 and l_4 . Their contribution reads:

$$\Gamma_{[l_3, l_4]}(t) = \frac{M^2}{F^2} [-4M^2 l_3 + t l_4] \quad ; \quad (45)$$

the calculation is recommended as an easy exercise, as is the calculation of the contribution of l_3 to the pion mass. Once these two calculations are completed, one can then check that the Feynman–Hellman theorem is respected also in this case.

The result in Eq. (45), a tree-level calculation with the next-to-leading order Lagrangian, is merely a statement about how the symmetry constrains this particular matrix element: at this order the scalar form factor can have at most a term linear in t . No symmetry relation exists between the constant term and the

²The ellipsis is for the so-called contact terms, i.e. those that depend only on the external sources.

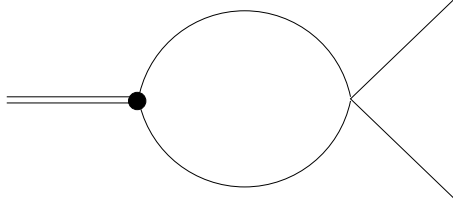


Figure 1: One-loop diagram for the scalar form factor of the pion. The double line stands for the scalar source, whereas single lines for pions.

coefficient of the linear term, hence we have two different constants. The constant term is related to the derivative of the pion mass with respect to the strength of the symmetry-breaking term in the Lagrangian, whereas the coefficient of the linear term is related to the correction to the pion decay constant (again, it is a very good exercise to calculate the latter with the Lagrangian in Eq. (42)).

2.3 Loop graphs

If we neglect the tadpole graphs and those renormalizing the external legs, there is basically only one graph for this process, the one shown in Fig. 1. Its structure is

$$\begin{aligned}
 \int \frac{d^4l}{(2\pi)^4} \frac{\{M^2, p^2, p \cdot l, l^2\}}{(l^2 - M^2)((p-l)^2 - M^2)} &\implies x \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2 - M^2)} \\
 &+ y \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2 - M^2)((p-l)^2 - M^2)} \\
 &\equiv xT(M^2) + yJ(t) \quad , \quad (46)
 \end{aligned}$$

where $p = p_1 + p_2$. We have indicated, in the first integral, all the terms that can appear in the numerator, and, after the arrow, the two possible structures to which the various terms can be reduced. The momenta and masses in the numerator come from the four-pion vertex on the right-hand side of the diagram. The power counting for this integral shows that it represents a correction of order p^2 to the leading-order term (the integration measure, which is of order p^4 , is compensated by the two propagators). This is true in general: one-loop graphs constructed from the Lagrangian (27) are always a correction of next-to-leading order in the chiral expansion. For example, it is easy to see that no matter how many vertices we add on the internal lines in the graph in Fig. 1, the graph remains of order p^2 : if we add a vertex of order p^2 we also get an extra propagator, which compensates it. The

general theorem was first proved by Weinberg.¹⁾ We will come back to this point later.

If we expand the tadpole integral $T(M^2)$ and the loop integral $J(t)$ in a Taylor series in their respective arguments:

$$\begin{aligned} T(M^2) &= a + bM^2 + \bar{T}(M^2) , \\ J(t) &= J(0) + \bar{J}(t) , \end{aligned} \tag{47}$$

only the first terms in the expansion are divergent, whereas both $\bar{T}(M^2)$ and $\bar{J}(t)$ are finite – this is easily seen by taking a sufficient number of derivatives on the loop integrals in Eq. (46). It is left as an exercise to the reader to show that the one-loop diagrams that we have neglected can only produce terms like $T(M^2)$. Therefore the contribution of the loop diagrams to the scalar form factor has the following structure:

$$\Gamma(t) \sim \frac{M^2}{F^2} \left[x_1 b M^2 + x_2 t J(0) + x_1 \bar{T}(M^2) + x_2 \bar{J}(t) \right] . \tag{48}$$

The divergent part of the loop diagrams has exactly the same structure as the counterterm contribution calculated above: to remove it we simply need to define the counterterms properly (in this case the constants l_3 and l_4).

The principles stated in the first lecture that guided the construction of the effective Lagrangian only appealed to symmetry arguments, and therefore allowed for an infinite number of terms. Once this is accepted, there is no problem of principle in carrying through the renormalization program: as anticipated, the difference between renormalizable and non-renormalizable theories is, in a sense, a technical detail.

2.4 Chiral logarithms

If we expand the form factor in a Taylor series in t , we can write it in the following form:

$$\Gamma(t) = \Gamma(0) \left[1 + \frac{1}{6} \langle r^2 \rangle_S^\pi t + O(t^2) \right] . \tag{49}$$

The coefficient of the linear term, properly normalized, is called the scalar radius of the pion. Its size is a way to represent the spatial extension of the pion when probed with a scalar source. We have stated above that the coupling constant that appears in this quantity, l_4 , also determines the first correction of the pion decay constant around the chiral limit. There is another piece of information on the scalar radius, which we can already gather from the simple sketch of the loop calculation given

above:

$$\langle r^2 \rangle_S^\pi \sim J(0) = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - M^2)^2} \sim \ln \frac{M^2}{\Lambda^2} , \quad (50)$$

namely that the scalar radius contains an infrared divergence. In the chiral limit this quantity diverges. This divergence should not be removed and does not represent a problem because it has a physical meaning, in the sense that when the pion becomes massless the cloud of pions surrounding any hadron (and therefore also the pion itself) extends to an infinite range. A quantity that measures the spatial extension of this cloud should indeed become infinite in the chiral limit. Notice that the scalar form factor is finite and remains finite also in the chiral limit – it is only the first derivative in t , calculated at $t = 0$ that diverges when the pion mass goes to zero.

These infrared divergences are present everywhere in pion physics, and in many cases they are among the most important physical effects (less so in the case of kaons). Their relevance was first pointed out by Li and Pagels.¹⁵⁾ The effective Lagrangian method provides a systematic way to calculate these effects. As we have seen in the above example, they arise from the infrared region in the loop integrals – precisely the region where we should fully trust the vertices of our effective Lagrangian.

One may be less at ease with the ultraviolet region of the loop integrals: there indeed has no justification the use of the effective Lagrangian. On the other hand, through the process of renormalization, that part of the integrals is completely removed and substituted with unknown constants, the counterterms. As is sometimes said, these parametrize our ignorance of the physics that lies above the range of applicability of the effective Lagrangian. In fact, we are not completely ignorant about the physics of strong interactions in the GeV range and above, and as we will see in the following lecture, it is not difficult to get a good estimate of the numerical value of the counterterms.

Until now the only part of the loop integrals that we have not analysed is the finite, analytically non-trivial part of the loop integral, the function $\bar{J}(t)$. This will be treated in full detail in the following lecture. Before doing that, however, we want to consider the problem of renormalization from a more general point of view.

2.5 Renormalization and chiral symmetry

The reader who has done the exercise of calculating the divergent part of the scalar form factor to one loop will have seen that, to renormalize it, the two counterterms

l_3 and l_4 have to be defined in the following way

$$l_3 = l_3^r(\mu) - \frac{1}{2}\lambda \quad , \quad l_4 = l_4^r(\mu) + 2\lambda \quad , \quad (51)$$

where λ is divergent and dependent on the regularization method (in dimensional regularization, for example, it is defined by $\lambda = (c\mu)^{d-4}/(d-4)$, with c an arbitrary constant that defines the regularization scheme).

If we look at the definition of the operators in front of l_3 and l_4 we see that (as usual) they contain an infinite number of pion fields. For example they both contain a term with one scalar source and 6 pion fields. Does it mean that if we had calculated that matrix element to one loop we would have found the same divergent part for l_3 and l_4 ? Or, to put it differently, that once we calculate the divergent part of the scalar form factor then we know the divergent part of all other matrix elements with a higher number of pion fields³?

The answer is yes. Chiral symmetry puts a strong constraint on the divergences: they have to be chiral-invariant terms. This conclusion is originally due to Weinberg,¹⁾ on the basis of a highly plausible, but still heuristic argument. It is now put on a solid basis by the work of Leutwyler:¹⁶⁾ he proved that to calculate hadronic Green functions with an effective Lagrangian, such that they respect the Ward identities implied by the chiral symmetry, one necessarily has to start from a chiral-invariant effective Lagrangian. The non-trivial part in this statement is that it takes into account also quantum effects: anomalous symmetries show that it is not always true that a symmetry that exists at the classical level survives the quantum corrections – or vice versa, that to have a symmetrical quantum theory one necessarily has to start from a symmetrical classical Lagrangian.

The theorem applies also to the divergent part of the quantum corrections: they have to be chiral-invariant. A general theorem of quantum field theory states that the divergent part of a loop graph is a polynomial in the external masses and momenta. These two theorems lead to the conclusion that the divergences, order by order, can be reabsorbed by the chiral-invariant counterterms.

While these general theorems are very important, performing a direct calculation of the divergences and the subsequent renormalization in an explicitly chiral-invariant form, is probably much more instructive. The rest of this lecture will be dedicated to an illustration of the tools necessary for this calculation and to a sketch of the procedure.

³Strictly speaking, these questions make full sense only if we suppose that these two are the only counterterms at order p^4 .

2.6 Background field method and heat kernel

To introduce these techniques, it is useful to consider a simpler theory than CHPT: a purely scalar $O(N)$ ϕ^4 theory. Its Lagrangian is the following:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0 + \sum_{n=1}^{\infty} \hbar^n \mathcal{L}_n , \\ \mathcal{L}_0 &= \frac{1}{2} \left(\partial_\mu \phi^i \partial^\mu \phi^i - M^2 \phi^i \phi^i \right) - \frac{g}{4} \left(\phi^i \phi^i \right)^2 - \phi^i f^i ,\end{aligned}\quad (52)$$

where \mathcal{L}_0 is the classical Lagrangian and \mathcal{L}_i , $i \geq 1$, the series of counterterms needed to renormalize the theory, and a summation over repeated indices is implied. So as to make the situation completely analogous to CHPT, where we have the external sources v , a , s and p , we have also introduced external sources f^i coupled to the fields ϕ^i . A path integral constructed with this Lagrangian is a function of the external sources f^i . By taking the appropriate functional derivatives with respect to the sources we can extract from the path integral all possible Green functions of the fields ϕ^i . The logarithm of the path integral is usually called the generating functional $Z\{f\}$:

$$\begin{aligned}e^{iZ\{f\}/\hbar} &= N \int [d\phi] e^{iS/\hbar} , & S &= \int dx \mathcal{L} , \\ Z &= Z_0 + \hbar Z_1 + \hbar^2 Z_2 + O(\hbar^3) , & Z\{0\} &= 0 .\end{aligned}\quad (53)$$

To calculate the classical part of the generating functional Z_0 and its quantum corrections Z_i , it is useful to expand the field ϕ around the solution of the classical equations of motion, $\bar{\phi}$:

$$\frac{\delta S_0}{\delta \phi^i} = 0 \quad \Rightarrow \quad (M^2 + \square) \bar{\phi}^i + g \bar{\phi}^2 \bar{\phi}^i + f^i = 0 . \quad (54)$$

We can then shift the integration variable:

$$\phi^i = \bar{\phi}^i + \xi^i , \quad [d\phi] = [d\xi] , \quad \xi \sim O(\hbar^{1/2}) , \quad (55)$$

and get, for the path integral:

$$e^{iZ\{f\}/\hbar} = N e^{i\bar{S}/\hbar} \int [d\xi] \exp \left\{ \frac{i}{\hbar} \int dx \left[\frac{1}{2} \xi^i D_{ij} \xi^j + O(\xi^3) \right] \right\} , \quad (56)$$

where

$$\begin{aligned}D_{ij} &= -\square \delta_{ij} + \sigma_{ij} , \\ \sigma_{ij} &= - \left(M^2 + g \bar{\phi}^2 \right) \delta_{ij} - 2g \bar{\phi}^i \bar{\phi}^j .\end{aligned}\quad (57)$$

The separation of the field into its classical part $\bar{\phi}$ and its quantum fluctuations ξ makes the calculation of the Taylor series of Z in \hbar more transparent. Indeed one immediately sees that the classical action S_n of the Lagrangian \mathcal{L}_n , evaluated at the solution of the classical equation of motion contributes only to the term Z_n of the generating functional. The path integral over the quantum fluctuations yields higher-order quantum corrections. For example Z_1 receives a contribution from the integral over the quadratic term in ξ , which was explicitly given in Eq. (56). The path integral of the exponential of a quadratic term is known, and can be given in closed form – it is the determinant of the differential operator D_{ij} :

$$Z_1 = \int dx \left[\frac{i}{2} \ln \left(D_{ij} D_{0ij}^{-1} \right) + \mathcal{L}_1 \right] , \quad (58)$$

where $D_0 = D|_{f=0} = -(\square + M^2)\delta_{ij}$. To acquire familiarity with the formal expression (58) it is useful to expand the operator D around $f = 0$: $D_{ij} = D_{0ij} + \eta_{ij}$, and correspondingly expand the logarithm around 1:

$$\ln \left(D_{ij} D_{0ij}^{-1} \right) = \eta_{ij} D_{0ij}^{-1} - \frac{1}{2} \eta_{ij} D_{0jk}^{-1} \eta_{kl} D_{0li}^{-1} + \dots . \quad (59)$$

It should now be clear that this logarithm simply represents the sum of all one-loop diagrams with any number of four- ϕ vertex insertions, as shown in Fig. 2. The reader with some experience of loop calculation will have immediately realized that only the first two terms in the expansion of the logarithm generate divergences: all the others are finite loop integrals. Indeed the divergent part of Z_1 can be calculated by working out two simple loop diagrams.

The situation would not be as simple in the case of CHPT: there, the presence of derivative couplings complicates the situation significantly. The calculation can nonetheless be done in close form with the help of the heat kernel method. Without giving many details let us see how it works in the case of the $O(N)$ theory.

A differential operator of the form:

$$D^2 = -d^2 + \sigma , \quad d_\mu = \partial_\mu + \gamma_\mu , \quad (60)$$

where σ and γ are matrices in general, is called an elliptic differential operator, and has the same form as the differential operator that appears in the equations describing the diffusion of the heat (which explains the origin of the name). The divergent part of the logarithm of such operators is known in terms of the operators σ and $\gamma_{\mu\nu}$, without knowing their explicit form. It is given by:

$$\int dx \ln \left(D_{ij} D_{0ij}^{-1} \right) = \frac{i}{(4\pi)^2(d-4)} \int dx \left[\frac{1}{6} \gamma_{\mu\nu} \gamma^{\mu\nu} + \sigma^2 \right] + \dots . \quad (61)$$

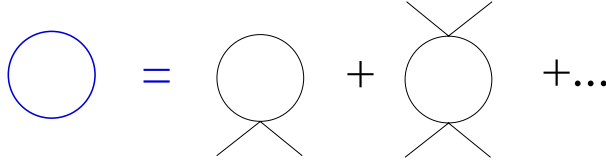


Figure 2: Graphical representation of the one-loop contributions to the generating functional. The logarithm of the differential operator, represented as the loop without external legs, is the sum of all diagrams with any number of external legs.

Details on how this result is obtained can be found in Ref. ¹⁷⁾. In the case of the $O(N)$ theory that we are considering, $\gamma_\mu = 0$ and we simply have to calculate the square of σ :

$$\sigma^2 = 2(N + 2)gM^2\phi^2 + (N + 8)g^2\phi^4 . \quad (62)$$

The calculation of the divergences is complete. The reader is urged to verify this result by an explicit calculation of the two divergent diagrams in Fig. 2.

2.7 Calculation of the one-loop divergences in CHPT

With the tools introduced in the preceding section we have reduced the problem of the calculation of one-loop divergences to a simple algebraic exercise. Even in the case of CHPT. The only step that requires some care is the choice of the field ξ , representing the fluctuations around the classical solution. Indeed this can be done in many different ways, but only a few of them will simplify and make the intermediate steps of the calculation transparent. One of the most convenient choices is the following: ²⁾

$$\begin{aligned} U &= e^{i\phi/F} , & \phi &= \bar{\phi} + \xi , \\ U &= \bar{u}e^{i\xi/F}\bar{u} . \end{aligned} \quad (63)$$

As we have seen in the first lecture, the transformation properties of the exponential of the ξ field are particularly well suited to check the chiral invariance properties during the various steps of the calculation.

The expansion of $S_2 = \int d^4x \mathcal{L}_S^{(2)}$ around the classical solution reads

$$\int dx \mathcal{L}_S^{(2)} = \int dx \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle$$

$$= \int dx \bar{\mathcal{L}}_S^{(2)} + \int dx \frac{1}{2} \xi^i \Delta_{ij} \xi^j + O(\xi^3) , \quad (64)$$

where

$$\begin{aligned} \Delta &= -d^2 + \sigma , \\ d_{\mu, ij} &= \partial_\mu \delta_{ij} + \gamma_{\mu, ij} , \\ \gamma_{\mu, ij} &= -\frac{1}{2} \langle [\lambda_i, \lambda_j] \Gamma_\mu \rangle , \\ \Gamma_\mu &= \frac{1}{2} \left\{ u^\dagger (\partial_\mu - i r_\mu) u + u (\partial_\mu - i l_\mu) u^\dagger \right\} , \\ \sigma_{ij} &= -\frac{1}{8} \langle [u_\mu, \lambda_i] [\lambda_j, u^\mu] + \{\lambda_i, \lambda_j\} \chi_+ \rangle . \end{aligned} \quad (65)$$

It is now a simple algebraic exercise to calculate the divergent part of the generating functional to one loop ($\epsilon = d - 4$):

$$\begin{aligned} -\frac{i}{2} \log \det \Delta &= \frac{1}{(4\pi)^2} \frac{1}{\epsilon} \int d^d x \left\{ \frac{N_f}{96} \langle ([u_\mu, u_\nu] - 2i f_{+\mu\nu})^2 \rangle + \frac{N_f}{16} \langle (u_\mu u^\mu + \chi_+)^2 \rangle \right. \\ &\quad \left. + \frac{1}{16} \langle u_\mu u^\mu + \chi_+ \rangle^2 + \frac{1}{8} \langle u_\mu u_\nu \rangle^2 \right. \\ &\quad \left. - \frac{1}{4N_f} \langle \chi_+^2 \rangle + \frac{1}{8N_f^2} \langle \chi_+ \rangle^2 \right\} + \dots . \end{aligned} \quad (66)$$

This expression is valid for a generic number of light flavours N_f , and is explicitly chiral-invariant, as it should be. To obtain the result in the interesting physical cases $N_f = 2, 3$ is not only necessary to trivially substitute N_f with the numerical value of interest, but also to reduce all the chiral-invariant terms appearing in Eq. (66) to a minimal set. This step involves the use of trace identities and of the equations of motion. Without giving the details of this final step, let us conclude this lecture by presenting the results in the case of $N_f = 2$. If we define the counterterms as

$$l_i = l_i^r(\mu) + \gamma_i \lambda(\mu) , \quad (67)$$

their coefficients for the divergent part are:

$$\begin{aligned} \gamma_1 &= \frac{1}{3}, & \gamma_2 &= \frac{2}{3}, & \gamma_3 &= -\frac{1}{2}, \\ \gamma_4 &= 2, & \gamma_5 &= -\frac{1}{6}, & \gamma_6 &= -\frac{1}{3}, & \gamma_7 &= 0 . \end{aligned} \quad (68)$$

3 One-loop graphs: analyticity and unitarity

According to the Watson theorem,¹⁸⁾ above threshold but below the inelasticity threshold, the phase of the scalar form factor is equal to the S -wave $\pi\pi$ phase shift

with isospin $I = 0$, δ_0^0 . As is well known, this theorem is a consequence of unitarity:

$$\text{Im}\bar{\Gamma}(t) = \sigma(t)\bar{\Gamma}(t)t_0^{0*}(t) = \bar{\Gamma}(t)e^{-i\delta_0^0} \sin \delta_0^0 \quad (69)$$

$$= |\bar{\Gamma}(t)| \sin \delta_0^0, \quad (70)$$

where $\sigma(t) = [1 - 4M_\pi^2/t]^{1/2}$ and $\bar{\Gamma}(t) = \Gamma(t)/\Gamma(0)$, and t_0^0 is the $I = 0$ S -wave of $\pi\pi$ scattering. The unitarity relation (70) shows that the leading-order expression $\bar{\Gamma}(t) = 1$ cannot be the whole story: if we want an accurate description of the form factor away from $t = 0$, we need to include higher orders and, in particular, loops – imaginary parts can only be generated by loop graphs.

Notice that since at leading order $\bar{\Gamma}(t)$ is $\mathcal{O}(1)$, and the phase δ_0^0 is $\mathcal{O}(p^2)$, the $\mathcal{O}(p^2)$ imaginary part (which is a next-to-leading order correction) is completely fixed by leading-order quantities:

$$\text{Im}\bar{\Gamma}^{(2)}(t) = \delta_0^{(2)}(t) = \sigma(t)\frac{2t - M_\pi^2}{32\pi F_\pi^2}. \quad (71)$$

The use of the effective Lagrangian method to calculate the form factor guarantees that this relation is satisfied. The complete expression for the one-loop scalar form factor reads as follows:

$$\bar{\Gamma}(t) = 1 + \frac{t}{16\pi^2 F_\pi^2}(\bar{l}_4 - 1) + \frac{2t - M_\pi^2}{2F_\pi^2}\bar{J}(t), \quad (72)$$

where $\bar{J}(t)$ is the subtracted one-loop integral (47). Its explicit expression reads:

$$\bar{J}(t) = \frac{1}{16\pi^2} \left[\sigma(t) \ln \frac{\sigma(t) - 1}{\sigma(t) + 1} + 2 \right]. \quad (73)$$

The reader can now easily verify that the imaginary part of the form factor at this order indeed satisfies (71).

3.1 Dispersion relation for the scalar form factor

An analytic function must be real on the real axis: the scalar form factor is non-analytic from threshold ($4M_\pi^2$) up to infinity. On the basis of very general arguments, which mainly use the causality principle, one can prove that as a function of t , the scalar form factor must be analytic everywhere else (see Ref. 21) for a general discussion of the analyticity properties of amplitudes and Green functions). Since

it is analytic everywhere else, the non-analyticity of the form factor on the real axis can be further characterized, and described as a discontinuity:

$$\bar{\Gamma}(t + i\varepsilon) = \bar{\Gamma}^*(t - i\varepsilon) = |\bar{\Gamma}(t)|e^{i\delta_0^0(t)} . \quad (74)$$

Given these analytic properties, we can write the following dispersion relation:

$$\bar{\Gamma}(t) = 1 + bt + \frac{t^2}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{\text{Im}\bar{\Gamma}(t')}{t' - t} , \quad (75)$$

where, for later convenience, we have subtracted the dispersive integral twice – we will come back to the issue of how many subtractions are necessary for the dispersive integral to converge. The dispersion relation shows that, if we know the subtraction constants (in this case only one, b) and the imaginary part on the real axis, we can reconstruct the scalar form factor everywhere on the complex plane.

It is no surprise that any perturbative calculation in a quantum field theory produces amplitudes and Green functions with the correct analytic properties. Using an effective field theory makes no difference: the form factor calculated to one loop in CHPT must have the correct analytic properties, and must satisfy (at the perturbative level) the dispersion relation (75). To convince ourselves that this is actually the case, let us first apply the chiral counting to the dispersion relation:

$$\begin{aligned} \bar{\Gamma}^{(0)} &= 1 \\ b &\sim \mathcal{O}(1) \\ \text{Im}\bar{\Gamma}(t') &= \mathcal{O}(p^2) . \end{aligned} \quad (76)$$

As we have seen the $\mathcal{O}(p^2)$ imaginary part is fully fixed by leading-order quantities, (71), and apart from an unconstrained polynomial term, the real part must be given by the dispersive integral over this known imaginary part. We leave it as an exercise to prove that this is true. For this it is useful to know that:

$$\bar{J}(t) = \frac{t}{16\pi^2} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'} \frac{\sigma(t')}{t' - t} . \quad (77)$$

3.2 High-energy contributions to the dispersive integrals

In the previous section we showed that the renormalization procedure removes the contributions to the loop integrals where the momentum squared of the pions is large. This was reassuring because we cannot hope that our effective Lagrangian describes highly virtual pions well. In the present section we are dealing with the

contribution to the loop integrals from real pions: the dispersive integrals. As we have seen above, these extend all the way up to infinity, as required by analyticity. In the perturbative expansion that we are considering, the imaginary part of the form factor, which is in the integrand, is evaluated only to leading non-trivial order. This description of the imaginary part can be reasonably accurate only in the low energy region: still, in the integral, it is used all the way up to infinity. How can we trust the dispersive integral?

In fact, we don't. At least not for the contribution that comes from the high-energy region. Suppose we decide to remove the part of the dispersive integral from $\Lambda = 1$ GeV to infinity. We should then subtract from the form factor a term like:

$$\frac{t^2}{\pi} \int_{\Lambda^2}^{\infty} \frac{\delta_0^{0(2)}(t')}{t'^2(t'-t)} = \frac{t^2}{\pi} \int_{\Lambda^2}^{\infty} \frac{\delta_0^{0(2)}(t')}{t'^3} \left(1 + \frac{t}{t'} + O(t^2)\right) . \quad (78)$$

The Taylor expansion inside the integral can be safely performed because the CHPT calculation of the form factor is valid only for $t \ll \Lambda^2$. In the chiral language, this part of the dispersive integral can be represented as a polynomial series starting at order p^4 , i.e. at an order which is beyond the accuracy at which we are presently working. This shows that worrying about these contributions to the dispersive integrals is futile: the only sensible way to improve the evaluation of the dispersive integral is to go one order higher in the chiral expansion. This would automatically give a representation of the form factor that contains the dispersive integral over the imaginary part correct up to order p^4 .

For those who are interested in the numerics, the first term in the Taylor expansion of the integral (78) is equal to:

$$\frac{t^2}{\pi} \int_{\Lambda^2}^{\infty} \frac{\delta_0^{0(2)}(t')}{t'^3} = 0.7 [t(\text{GeV}^2)^2] , \quad (79)$$

which means, for $t = (0.5 \text{ GeV})^2$ (which is about the upper limit of validity of the chiral expansion), a 4% correction to the leading-order result. Also numerically everything is well under control.

3.3 Estimate of the low-energy constants

The chiral representation to next-to-leading order satisfies a dispersion relation with two subtractions. The number of subtractions in this case is dictated by the behaviour at infinity of the leading-order chiral phase δ_0^0 . The latter, however, has nothing to do with the physical reality: it can be shown that, with form factors, one

subtraction is already sufficient for the dispersive integral to converge. To discuss the physical implications of this, it is useful to consider the vector form factor of the pion, which is defined as follows:

$$\langle \pi^i(p_1) | \bar{q} \tau^k \gamma_\mu q | \pi^j(p_2) \rangle = i \epsilon^{ikj} F_V(t) \quad , \quad q = \begin{pmatrix} u \\ d \end{pmatrix} . \quad (80)$$

It satisfies the following dispersion relation:

$$F_V(t) = 1 + \frac{t}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'} \frac{\text{Im} F_V(t')}{t' - t} . \quad (81)$$

This dispersion relation implies that the charge radius (the analogue of the scalar radius, for the vector form factor) is given by a sum rule:

$$\frac{1}{6} \langle r^2 \rangle_V^\pi = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} dt' \frac{\text{Im} F_V(t')}{t'^2} . \quad (82)$$

If one knows the imaginary part of the vector form factor one can calculate the charge radius.

Within the chiral representation the sum rule (82) does not make sense as it stands, because the integral on the right-hand side does not converge. In CHPT, the charge radius is given in terms of one of the low-energy constants:

$$\frac{1}{6} \langle r^2 \rangle_V^\pi = -\frac{1}{F_\pi^2} \left[l_6^r(\mu) - \frac{1}{96\pi^2} \left(\ln \frac{M_\pi^2}{\mu^2} + 1 \right) + O(M_\pi^2) \right] , \quad (83)$$

and, in this language, the sum rule (82) can be read as a way to determine $l_6^r(\mu)$. Notice that the fact that this constant appears at all in the chiral representation is a consequence of the non-convergence of the chiral dispersive integral: that integral would receive a sizeable contribution from the high-energy region, and there the chiral representation obviously fails, not even allowing the calculation to be made.

Having the sum rule at hand, we can attempt to estimate this particular low-energy constant, from an estimate of the behaviour of the imaginary part of the form factor. What do we know about it? Actually, this has been measured in various ways, and a lot of experimental information is available on this quantity. But rather than collecting all the available information to make an accurate evaluation of the sum rule, we want to make a simple, but instructive exercise. Since in this channel (two pions in the P wave and isospin $I = 1$) the ρ resonance contributes, we can try to give a rough estimate of its contribution to the sum rule. For this it is sufficient to use the narrow-width approximation:

$$\text{Im} F_V(t) \sim \pi \delta(t - M_\rho^2) \quad , \quad (84)$$

i	$10^3 L_i^r(M_\rho)$	V	A	S	S_1	η_1	Total
1	0.53 ± 0.25	0.6	0	-0.2	0.2^b	0	0.6
2	0.71 ± 0.27	1.2	0	0	0	0	1.2
3	-2.72 ± 1.12	-3.6	0	0.6	0	0	-3.0
4	-0.3 ± 0.5	0	0	-0.5	0.5^b	0	0.0
5	0.91 ± 0.15	0	0	1.4^a	0	0	1.4
6	-0.2 ± 0.3	0	0	-0.3	0.3^b	0	0.0
7	-0.32 ± 0.15	0	0	0	0	-0.3	-0.3
8	0.62 ± 0.2	0	0	0.9^a	0	0	0.9
9	6.9 ± 0.7	6.9^a	0	0	0	0	6.9
10	-5.5 ± 0.7	-10.0	4.0	0	0	0	-6.0

^{a)} Input.

^{b)} Estimate based on the limit $N_c \rightarrow \infty$.

Table 1: Contributions of the resonances V , A , S , S_1 and η_1 to the constants L_i^r in units of 10^{-3} . ¹⁹⁾ The phenomenological values in the second column are either from Ref. ^{2, 4)} (entries 4, 6, 9 and 10) or from Ref. ²⁰⁾ (all the remaining ones).

which, when inserted in the integral, gives

$$\frac{1}{6} \langle r^2 \rangle_V^\pi = \frac{f F_\rho}{F_\pi^2} \frac{1}{M_\rho^2}, \quad (85)$$

where the two coupling constants

$$F_\rho = 144 \text{ MeV}, \quad f = 69 \text{ MeV} \quad (86)$$

determine the strength of the coupling of the ρ resonance to the external vector field and to the pions.

These numbers give

$$l_6^r(M_\rho) \sim -13.3 \times 10^{-3}, \quad (87)$$

whereas if we extract it from the measured value of the charge radius, we obtain:

$$l_6^r(M_\rho) = -(13.5 \pm 2.5) \times 10^{-3}, \quad (88)$$

in very good agreement with (87). This is not a big surprise: we are simply comparing the experimental determination of the charge radius with the contribution that the ρ gives to it, and it is well known that the latter is largely dominant. It is however illuminating on the physical meaning of the low-energy constants in the

chiral expansion. Physically, these constants must encode the effects of the physics which occurs above the Goldstone-boson scale, and which does not appear explicitly in the effective Lagrangian. Intuitively one would expect that the resonances are the most important “high-energy” phenomenon that is neglected here. A simple estimate of their role should give account of at least the order of magnitude of these constants.

Actually a careful analysis of the contributions of all the lowest-lying resonances to the low-energy constants shows that they practically saturate the experimentally determined values.¹⁹⁾ This is clearly seen from Table 1.

3.4 Exact solution of the dispersion relation

If we assume that only the $\pi\pi$ channel is open all the way up to infinity, then the phase of the form factor on the cut is everywhere the $\pi\pi$ phase shift δ_0^0 . In this approximation, we can pose the following mathematical problem: suppose that

1. $F(t)$ is an analytic function of t in the whole complex plane, with the exception of a cut for $4M_\pi^2 \leq t < \infty$;
2. approaching the real axis from above, $e^{-i\delta(t)}F(t)$ is real on the real axis, where $\delta(t)$ is a known function.

Can $F(t)$ be determined in general? The solution to this problem is due to Omnès,²²⁾ and reads as follows:

$$F(t) = P(t)\Omega(t) = P(t) \exp \left\{ \frac{t}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'} \frac{\delta(t')}{t' - t} \right\} , \quad (89)$$

where $P(t)$ is a generic polynomial, and $\Omega(t)$ is called the Omnès function. $P(t)$ can only be constrained by the behaviour of the function $F(t)$ at infinity: if we know the behaviour of the phase δ , and also that of $F(t)$, we can fix at least the degree of the polynomial $P(t)$. Then, depending on the degree of this polynomial, we will need a number of inputs to fix its coefficients. Whatever the degree of the polynomial, the relevant conclusion here is that if we know the phase, we only need to determine a few constants to fix the function $F(t)$ completely.

Can we use this information and improve the chiral representation of the form factor combining it with the Omnès representation?

In the chiral representation at order p^4 we have seen that there appears the Omnès function expanded to next-to-leading order:

$$\bar{\Gamma}_{\text{CHPT}}^{(2)}(t) = 1 + b^{(0)}t + \Delta^{(2)}(t) , \quad \Delta^{(2)}(t) = \frac{t}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'} \frac{\delta^{(2)}(t')}{t' - t} . \quad (90)$$

Indeed the above expression can be seen as a chiral expansion of the Omnès solution (89):

$$\begin{aligned}\bar{\Gamma}(t) &= (1 + bt) \Omega(t) \sim \left(1 + [b^{(0)} + \mathcal{O}(M^2)]t\right) \left[1 + \Delta^{(2)}(t) + \mathcal{O}(p^4)\right] \\ &= 1 + b^{(0)}t + \Delta^{(2)}(t) + \mathcal{O}(p^4) \ ,\end{aligned}\tag{91}$$

and we could try to reconstruct the full Omnès solution from its expansion. An obvious improvement would be, for example:

$$\bar{\Gamma}^\alpha(t) = (1 + b^{(2)}t)e^{\Delta^{(2)}(t)} \ ,\tag{92}$$

where we have simply exponentiated the dispersive integral over the $\pi\pi$ phase; or we could use the chiral phase at next-to-leading order ²⁾ in the exponentiated dispersive integral:

$$\bar{\Gamma}^\beta(t) = (1 + b^{(2)}t)e^{\Delta^{(4)}(t)} \ ;\tag{93}$$

or, if we have a good phenomenological representation for the phase, we could use this to calculate the dispersive integral:

$$\bar{\Gamma}^\gamma(t) = (1 + b^{(2)}t)e^{\Delta_{\text{phys}}(t)} \ .\tag{94}$$

Also, in calculating the dispersive integral, we could in principle choose whether to use a cut-off or to extend the integration up to infinity. However, given the definition of the Omnès function (89), the number of subtractions is not sufficient to make the integral convergent, even for the p^2 phase: with a once-subtracted Omnès function using a cut-off is mandatory.

Finally, we have to mention another possible degree of freedom in combining the chiral and the Omnès representation: the choice of the subtraction point. In the present case there seems to be no choice, because we know that the form factor at $t = 0$ must be equal to 1. It is then natural to have both the polynomial and the Omnès function equal to 1 at $t = 0$. This choice fixes the subtraction point of the Omnès function. If the information on the value of the form factor at zero were missing, we could as well have chosen a different subtraction point of order M_π^2 , without changing the chiral counting of the final result.

There are various ways to combine the chiral and the Omnès representations. In all cases we improve the chiral representation since we are able to sum up higher-order terms that are fixed by unitarity. On the other hand the difference between one way and the other to implement this improvement is only one order higher than the chiral representation we started with. We can claim a real improvement

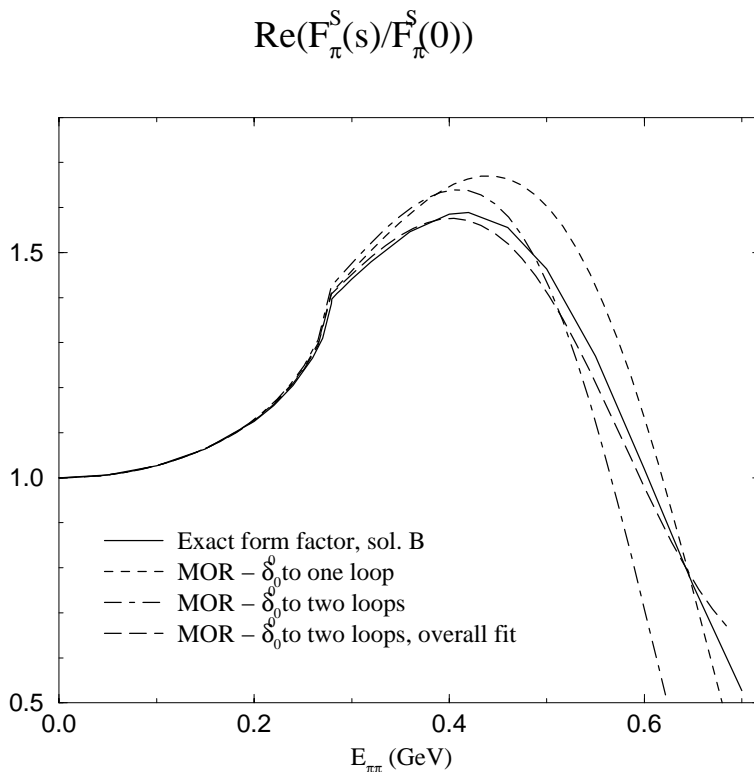


Figure 3: Comparison of the physical scalar form factor (solid line) with various “Modified Omnès representations” (MOR). The dashed and dot-dashed lines show the difference between the use of the phase to one and two loops, whereas the long-dashed line shows that by tuning some of the low-energy constants, the MOR can actually reproduce the form factor in a wide range of energy.

only if we can show that this arbitrariness can be constrained by physical arguments, and that its effect is numerically small. For example, If we had a good phenomenological representation for the phase, with very small uncertainties, it would certainly be preferable to use the representation $\bar{\Gamma}^\gamma(t)$ rather than $\bar{\Gamma}^\alpha(t)$, or $\bar{\Gamma}^\beta(t)$. We have already commented on the choice of the subtraction point: that is another example of an important information that helps in reducing the arbitrariness of this unitarization procedure. In the present case the only degree of freedom is in the choice of the linear term of the polynomial: Eq. (94) shows that combining the chiral and the Omnès representations amounts to fixing this subtraction constant with CHPT.

In the recent literature there are various examples of the application of such procedures: one combines the solution of a dispersion relation (be it exact, as in the

case of the form factor, or numerical, in other cases) with the chiral representation, and uses the latter only to determine the subtraction constant. The advantage of such a procedure is clear: the dispersive integrals, which extend all the way up to infinity, are certainly dominated by the low-energy region, but also receive sizeable contributions from the intermediate-energy region, between 0.5 and 1 GeV. Using the chiral representation to evaluate the contribution from that region is certainly not the best one can do. If one has better information on imaginary parts, or phases, in that region, one should definitely use it. As we have discussed here, this procedure is not free from arbitrariness, and all care should be used to constrain this to a minimum. The interested reader is referred to the original articles ²³⁾ for a more detailed treatment of the case of the pion form factors. A review of the marriage between CHPT and dispersion relations can be found in Ref. ²⁴⁾

3.5 Concluding remarks on lectures 2 and 3

In these two lectures we have analysed in detail the various contributions entering the next-to-leading order calculation of a typical amplitude in CHPT. To make the analysis concrete we have considered the scalar form factor.

We have seen that:

1. The renormalization procedure is straightforward – indeed not very different from the case of a renormalizable field theory.
2. The loop integrals generate infrared divergences that are physical: the effective-Lagrangian method is a systematic way to calculate them.
3. The finite, analytically non-trivial part of the loop integrals is dictated by the properties of unitarity and analyticity. As usual in quantum field theory these properties are built in, and automatically respected in loop calculations.
4. The finite part of the counterterms can be expressed in terms of sum rules and in most cases these are saturated by the lowest-lying resonances. The low-energy constants have a very clear physical meaning, as they embody the effect of the high-energy degrees of freedom that are not explicitly considered in the effective Lagrangian framework

4 Non-leptonic weak interactions

4.1 Partonic $|\Delta S| = 1$ effective Hamiltonian

As discussed in the first lecture, the chiral realization of $Z(v, a, s, p)$ allows us to calculate not only transition amplitudes of pure QCD, but also Green functions of weak and electromagnetic processes with external gauge fields, such as semileptonic kaon and pion decays. $Z(v, a, s, p)$ is not sufficient, however, to describe non-leptonic weak transitions, such as $K \rightarrow 2\pi$ and $K \rightarrow 3\pi$ decays. In this kind of processes the W boson is coupled to two quark currents and cannot be treated as an external field. The strong-interaction effects that renormalize the non-leptonic weak transition cannot be trivially factorized and should be evaluated up to distances of $\mathcal{O}(1/M_W)$.

The simplest strategy to describe non-leptonic weak interactions is based on a twofold EQFT approach. The first step, performed within perturbative QCD, is the construction of the partonic $|\Delta S| = 1$ effective Hamiltonian, or the expansion of the non-local product of weak currents into a series of local partonic operators renormalized at a scale $\mu \gtrsim 1$ GeV (see e.g. Buchalla *et al.* ²⁵) for a comprehensive review). This first step, based on Wilson's Operator Product Expansion, ²⁶) allows us to encode in appropriate coefficients the sizeable QCD corrections that renormalize the weak interaction at short distances, from M_W down to the lowest scale where perturbative QCD can still be applied, considerably simplifying the original problem.

Denoting by $J_\mu(x)$ the charged weak current and by $D_W^{\mu\nu}(x, M_W)$ the W propagator in spatial coordinates, the partonic $|\Delta S| = 1$ effective Hamiltonian at $\mathcal{O}(G_F)$,

$$\mathcal{H}_{eff}^{|\Delta S|=1} = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_i C_i(\mu) Q_i(\mu) + \text{h.c.} , \quad (95)$$

can formally be defined by the following equation

$$\begin{aligned} \mathcal{T}(I \rightarrow F) &= \frac{g^2}{8} \int d^4x D_W^{\mu\nu}(x, M_W) \langle F | T (J_\mu(x) J_\nu^\dagger(0)) | I \rangle + \mathcal{O}(G_F) \\ &= -\frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_i C_i(\mu) \langle F | Q_i(\mu) | I \rangle + \mathcal{O}(G_F) . \end{aligned} \quad (96)$$

As can easily be verified by expanding the W propagator in Fig. 4, in the absence of QCD interactions the sum in the right-hand side of Eq. (96) would be restricted to only two operators,

$$Q_\pm = 2 [\bar{s}_L \gamma^\mu u_L \bar{u}_L \gamma^\mu d_L \pm \bar{s}_L \gamma^\mu d_L \bar{u}_L \gamma^\mu u_L] , \quad (97)$$

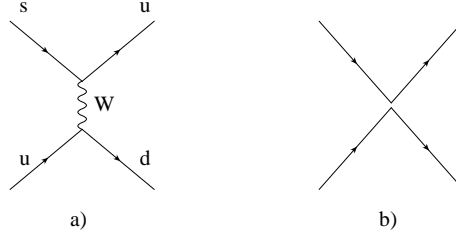


Figure 4: a) Tree-level Feynman diagram for $|\Delta S| = 1$ transitions, at the lowest order in G_F and without strong-interaction corrections; b) the same diagram in the effective theory.

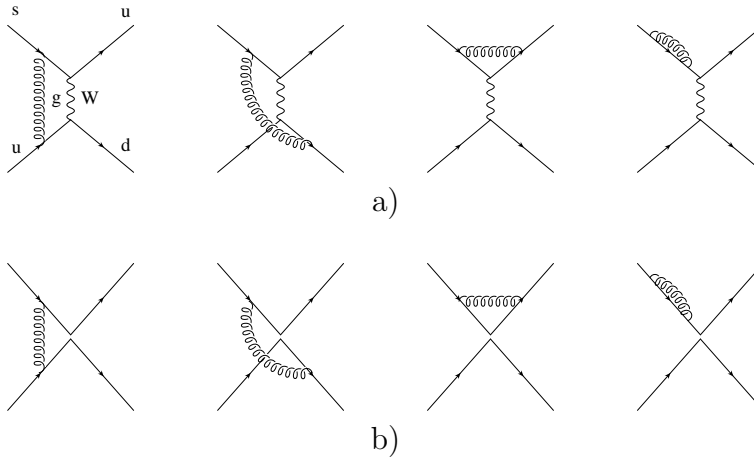


Figure 5: Leading-order QCD corrections to the diagram in Fig. 4: a) in the full theory; b) in the effective theory.

whose (Wilson) coefficients are given by

$$C_+ = C_- = 1 . \tag{98}$$

The situation becomes more complicated when the QCD corrections in Fig. 5 are taken into account. In order to evaluate these effects within perturbative QCD it is assumed that external quarks carry large momenta. Moreover, since some diagrams develop artificial divergences (absent in the real case) within the effective theory, it is necessary to regularize the effective operators by means of the Wilson coefficients, introducing an arbitrary renormalization scale μ . The requirement that the product $C_i(\mu) \times Q_i(\mu)$, and thus all physical observables, be independent of μ ,

fixes the renormalization-group evolution of the coefficients C_i as a function of μ unambiguously. The initial value of the C_i is fixed by matching the effective theory to the full one at high scales [i.e. for external momenta of $\mathcal{O}(M_W)$], where the perturbative calculation is rapidly convergent. Then, using renormalization-group equations the C_i are evolved down to a scale of $\mathcal{O}(1 \text{ GeV})$, as close as possible to the physical scale of the process (the kaon mass), but still high enough to trust perturbative QCD.

As a result of this procedure, all leading logarithms, i.e. all the terms of $\mathcal{O}[\alpha_s^n \log(M_W/\mu)^n]$, are encoded in the Wilson coefficients. At this level of accuracy the set of effective operators is still the one in Eq. (97); however, their coefficients have changed to ²⁷⁾

$$\begin{aligned} C_+(\mu) &= C_+(M_W) \left(\frac{\alpha_s(M_W)}{\alpha_s(\mu)} \right)^{\frac{1}{2\beta_0}} = \left(\frac{\alpha_s(M_W)}{\alpha_s(\mu)} \right)^{\frac{1}{2\beta_0}}, \\ C_-(\mu) &= C_-(M_W) \left(\frac{\alpha_s(M_W)}{\alpha_s(\mu)} \right)^{-\frac{1}{\beta_0}} = \left(\frac{\alpha_s(M_W)}{\alpha_s(\mu)} \right)^{-\frac{1}{\beta_0}}, \end{aligned} \quad (99)$$

where $\beta_0 = (33 - 2N_F/12)$ and N_F denote the number of dynamical quarks. The weight of the operator Q_- is therefore enhanced, whereas that of Q_+ is decreased.

This program can be iterated at the next-to-leading order, resumming also subleading logarithms. At this level the number of terms increases, with the inclusion of the so-called penguin operators. Finally, when also the small electroweak $\mathcal{O}(e^2 G_F)$ operators relevant to CP -violation studies are taken into account, the full basis of $\mathcal{H}_{eff}^{|\Delta S|=1}$ below the charm threshold contains, in addition to Q_{\pm} , the following set of operators: ²⁵⁾

$$\begin{aligned} Q_{3(5)} &= 4 \bar{s}_L^\alpha \gamma^\mu d_L^\alpha \sum_{q=u,d,s} \bar{q}_{L(R)}^\beta \gamma^\mu q_{L(R)}^\beta, \\ Q_{4(6)} &= 4 \bar{s}_L^\alpha \gamma^\mu d_L^\beta \sum_{q=u,d,s} \bar{q}_{L(R)}^\beta \gamma^\mu q_{L(R)}^\alpha, \\ Q_{7(9)} &= 6 \bar{s}_L^\alpha \gamma^\mu d_L^\alpha \sum_{q=u,d,s} e_q \bar{q}_{R(L)}^\beta \gamma^\mu q_{R(L)}^\beta, \\ Q_{8(10)} &= 6 \bar{s}_L^\alpha \gamma^\mu d_L^\beta \sum_{q=u,d,s} e_q \bar{q}_{R(L)}^\beta \gamma^\mu q_{R(L)}^\alpha, \end{aligned} \quad (100)$$

where α and β are colour indices and e_q denotes the electric charge of the quark q .

Thanks to the resummation of the subleading logarithms, the coefficients of the Q_i are known with high accuracy. ^{25, 28)} According to Eq. (96), the remaining problem to be addressed in order to calculate the transition amplitudes of physical

processes is the evaluation of the hadronic matrix elements of the Q_i . This issue is the goal of the second EQFT construction.

4.2 Lowest-order non-leptonic $|\Delta S| = 1$ chiral Lagrangian

Following the basic principles of CHPT, in order to compute non-leptonic weak transitions of pseudoscalar mesons we need to construct the chiral realization of $\mathcal{H}_{eff}^{|\Delta S|=1}$, or we need to consider the most general Lagrangian, written in terms of pseudo-Goldstone fields, transforming as $\mathcal{H}_{eff}^{|\Delta S|=1}$ under $SU(3)_L \times SU(3)_R$.

The operators of Eq. (100) transform linearly under $SU(3)_L \times SU(3)_R$ in the following way:

$$\begin{aligned} Q_-, Q_3, Q_4, Q_5, Q_6 & & (8_L, 1_R) , \\ Q_+, Q_9, Q_{10} & & (8_L, 1_R) + (27_L, 1_R) , \\ Q_7, Q_8 & & (8_L, 8_R) . \end{aligned} \tag{101}$$

Analogously to the case of light-quark masses, chiral operators transforming like the Q_i can be built by introducing appropriate external sources. As an example, in order to build the $(8_L, 1_R)$ operators we introduce the source

$$\hat{\lambda} \xrightarrow{G} g_L \hat{\lambda} g_L^{-1} \tag{102}$$

and we consider all possible operators invariant under G and linear in $\hat{\lambda}$. Then, fixing the source to the constant value

$$\hat{\lambda} \rightarrow \lambda = \frac{1}{2}(\lambda_6 - i\lambda_7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , \tag{103}$$

we automatically select the $|\Delta S| = 1$ component of all possible $(8_L, 1_R)$ terms. For $(27_L, 1_R)$ and $(8_L, 8_R)$ operators the procedure is very similar, the only change is the source structure.

Interestingly, within each group of transformations there is only one chiral realization at the lowest order. These are given by: 29, 30)

$$\begin{aligned} W_8^{(2)} &= \langle \lambda L_\mu L^\mu \rangle & (8_L, 1_R) & \mathcal{O}(p^2) , \\ W_{27}^{(2)} &= (L_\mu)_{23} (L^\mu)_{11} + \frac{2}{3} (L_\mu)_{21} (L^\mu)_{13} & (27_L, 1_R) & \mathcal{O}(p^2) , \\ W_{\underline{8}}^{(0)} &= F^2 \langle \lambda U^\dagger Q U \rangle & (8_L, 8_R) & \mathcal{O}(p^0) , \end{aligned} \tag{104}$$

where $L_\mu = u^\dagger u_\mu u$. While the singlets under $SU(3)_R$ are at least of $\mathcal{O}(p^2)$, the lowest-order realization of $(8_L, 8_R)$ operators starts at $\mathcal{O}(p^0)$. This difference does not create

a power-counting mismatch since at the quark level the $(8_L, 8_R)$ terms are suppressed by a factor e^2 with respect to the $SU(3)_R$ singlets. Thus we can consistently consider the chiral realization of the $|\Delta S| = 1$ Lagrangian at $\mathcal{O}(G_F p^2 e^0) + \mathcal{O}(G_F p^0 e^2)$:

$$\mathcal{L}_W^{(2)} = F^4 \left[G_8 W_8^{(2)} + G_{27} W_{27}^{(2)} + G_{\underline{8}} W_{\underline{8}}^{(0)} \right] + \text{h.c.} \quad (105)$$

The constants G_i appearing in $\mathcal{L}_W^{(2)}$ are not fixed by symmetry arguments, apart from the constraint $\Im(G_i) = 0$, which holds in the limit where CP is an exact symmetry. By construction we can only write

$$\mathcal{T}(I \rightarrow F) = -\frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_i C_i(\mu) \langle F | Q_i(\mu) | I \rangle = F^4 \sum_i G_i \langle F | W_i^{(2)} | I \rangle + \mathcal{O}(p^4) \quad (106)$$

and unfortunately in this equation we have unknown terms on both sides of the last identity: the $\langle F | Q_i(\mu) | I \rangle$ on the left and the G_i on the right. As we will show below, since the G_i are few, we can determine them all by comparison with experimental data in $K \rightarrow 2\pi$ decays, i.e. by measuring $\mathcal{T}(K \rightarrow 2\pi)$. In this case Eq. (106) lets us fix the matrix element of some combination of $Q_i(\mu)$ between a kaon and two pions and then, more interestingly, lets us *predict* the matrix element of the same combination of $Q_i(\mu)$ in other channels, e.g. between a kaon and three pions. In the future one could hope to replace, fully or in part, the use of experimental inputs with a theoretical determination by means of lattice-QCD of some $\langle F | Q_i(\mu) | I \rangle$. Note, however, that in both approaches, either using lattice QCD or experimental data, the role of CHPT is to relate the matrix elements of the $Q_i(\mu)$ in different channels and not to predict from scratch the $\langle F | Q_i(\mu) | I \rangle$ in a given channel.

Rough theoretical estimates of the G_i can be obtained by using simplifying assumptions that let us compute the $\langle F | Q_i(\mu) | I \rangle$ explicitly, although with large (and typically uncontrolled) uncertainties. These are very useful to understand the order of magnitude of the G_i . The most natural simplifying assumption to estimate the $\langle F | Q_i(\mu) | I \rangle$ is the factorization hypothesis, that can be formally justified within QCD in the limit of an infinite number of colours ($N_C \rightarrow \infty$).³¹ According to this hypothesis, the hadronization of operators such as Q_{\pm} , with a colour-singlet (current) \times (current) structure, is given by the product of the corresponding hadronized currents. Since the hadronization of a colour-singlet quark current is completely determined by the coupling of the external sources in the strong Lagrangian [see Eqs. (17)–(29)],

$$\bar{q}_L^i \gamma_{\mu} q_L^j \rightarrow \frac{iF^2}{2} \left(\partial_{\mu} U^{\dagger} U \right)_{ji} , \quad (107)$$

this hypothesis leads to a full determination of the hadronization of Q_{\pm} :

$$Q_- \rightarrow \frac{F^4}{2} W_8^{(2)}, \quad Q_+ \rightarrow F^4 \left[\frac{3}{5} W_{27}^{(2)} + \frac{1}{10} W_{27}^{(2)} \right]. \quad (108)$$

In this case there is no trace of the renormalization scale μ , and indeed it can be shown that the anomalous dimensions of Q_{\pm} vanish in the limit $N_C \rightarrow \infty$. Using Eq. (106) one then finds

$$\begin{aligned} G_8|_{\text{fact.}} &= -\frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \left(\frac{1}{2} C_- + \frac{1}{10} C_+ \right) \xrightarrow{N_C \rightarrow \infty} -\frac{3G_F}{5\sqrt{2}} V_{us}^* V_{ud}, \\ G_{27}|_{\text{fact.}} &= -\frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \left(\frac{3}{5} C_+ \right) \xrightarrow{N_C \rightarrow \infty} -\frac{3G_F}{5\sqrt{2}} V_{us}^* V_{ud}, \end{aligned} \quad (109)$$

where in the terms after the arrows we have employed the values of C_{\pm} in Eq. (98), which are consistent with the absence of anomalous dimensions for Q_{\pm} . As can be noted, the normalization of $\mathcal{L}_W^{(2)}$ is such that the G_i are expected to be $\mathcal{O}(G_F V_{us} C_j)$, where C_j are Wilson coefficients of partonic operators transforming as the corresponding chiral realizations.

As anticipated, the G_i can be experimentally determined by means of $K \rightarrow 2\pi$ data. As long as we are interested in the real parts or the absolute values of the decay amplitudes we can neglect G_8 . Indeed the Wilson coefficients of $(8_L, 8_R)$ operators are absolutely negligible with respect to the others – for the imaginary parts this is not the case. Introducing the isospin amplitudes A_0 and A_2

$$\begin{aligned} \mathcal{A}(K^0 \rightarrow \pi^+ \pi^-) &= A_0 e^{i\delta_0} + \frac{1}{\sqrt{2}} A_2 e^{i\delta_2}, \\ \mathcal{A}(K^0 \rightarrow \pi^0 \pi^0) &= A_0 e^{i\delta_0} - \sqrt{2} A_2 e^{i\delta_2}, \\ \mathcal{A}(K^+ \rightarrow \pi^+ \pi^0) &= \frac{3}{2} A_2 e^{i\delta_2}, \end{aligned} \quad (110)$$

it is easy to verify that

$$A_0 = \sqrt{2} F (M_K^2 - M_\pi^2) \left(G_8 + \frac{1}{9} G_{27} \right), \quad (111)$$

$$A_2 = \frac{10}{9} F G_{27} (M_K^2 - M_\pi^2). \quad (112)$$

The comparison with the experimental data then leads to:

$$|G_8| = 9.1 \times 10^{-6} \text{ GeV}^{-2}, \quad (113)$$

$$G_{27}/G_8 = 5.7 \times 10^{-2}. \quad (114)$$

Amplitude	$\mathcal{O}(p^2)$	$\mathcal{O}(p^4)$	Experimental fit
α_1	74.0	input	91.71 ± 0.32
β_1	-16.5	input	-25.68 ± 0.27
ζ_1	–	-0.47 ± 0.18	-0.47 ± 0.15
ξ_1	–	-1.58 ± 0.19	-1.51 ± 0.30
α_3	-4.1	input	-7.36 ± 0.47
β_3	-1.0	input	-2.43 ± 0.41
γ_3	1.8	input	2.26 ± 0.23
ζ_3	–	-0.011 ± 0.006	-0.21 ± 0.08
ξ_3	–	0.092 ± 0.030	-0.12 ± 0.17
ξ'_3	–	-0.033 ± 0.077	-0.21 ± 0.51

Table 2: Experimental results vs. CHPT predictions in $K \rightarrow 3\pi$ decays. ³²⁾

Interestingly, the experimental values of G_8 and G_{27} are substantially different from their naive estimates in Eq. (109): the absolute value of G_8 is about 8 times larger, whereas $|G_{27}|$ is reduced to 50%. This phenomenon, known as the “ $\Delta I = 1/2$ rule”, is in part explained by the running of the Wilson coefficients. Indeed if in Eq. (109) one uses the leading-log values of $C_{\pm}(\mu)$, with a renormalization scale $\mu \sim 1$ GeV, then the factorized estimate of $|G_8|$ increases by a factor of about 2 and the one of $|G_{27}|$ reduces to 70%. Although this is encouraging, it is clear that there still is a large non-perturbative effect hidden in the matrix elements of four-quark operators, especially the $(8_L, 1_R)$ ones. At the moment the best we can do is to measure this effect using $K \rightarrow 2\pi$ data.

Once the G_i have been fixed from $K \rightarrow 2\pi$, the theory is absolutely predictive in all the other non-leptonic channels. In Table 2 we show the comparison between theory and experiments in $K \rightarrow 3\pi$ amplitudes. The latter are classified according to the variation of isospin ($\Delta I = 1/2$ or $\Delta I = 3/2$) and to the dependence from Dalitz-plot variables, as summarized in Table 3. In agreement with naive power counting, the discrepancy between lowest-order chiral predictions and data, within the dominant amplitudes, turns out to be around 30%. On the other hand, $\mathcal{O}(p^2)$ operators have not enough derivatives to produce a non-vanishing result for the suppressed quadratic slopes. The comparison between data and lowest-order CHPT predictions is much more satisfactory in the case of $K \rightarrow 2\pi\gamma$ decays, where the $\mathcal{O}(p^2)$ amplitudes coincide with the bremsstrahlung from $K \rightarrow 2\pi$, which is known to be largely dominant. ³³⁾

	$\Delta I = 1/2$	$\Delta I = 3/2$	Phases
Constant terms	1 (α_1)	1 (α_3)	1
Linear slopes	1 (β_1)	2 (β_3, γ_3)	3
Quadratic slopes	2 (ζ_1, ξ_1)	3 (ζ_3, ξ_3, ξ_3')	-
$\mathcal{O}(p^4)$ free parameters	2	3	-

Table 3: Number of independent isospin amplitudes vs. $\mathcal{O}(p^4)$ free parameters in $K \rightarrow 3\pi$ decays.

4.3 $\mathcal{O}(p^4)$ counterterms

In order to obtain a more refined description of experimental data, able to include absorptive effects and subleading amplitudes (such as $K \rightarrow 3\pi$ quadratic slopes and $K \rightarrow \pi\pi\gamma$ direct-emission terms), it is necessary to go beyond the lowest order in the chiral expansion. Since we are interested only in contributions of order G_F , we can proceed analogously to the case of strong interactions with the simple substitution

$$\begin{aligned}\mathcal{L}_S^{(2)} &\rightarrow \mathcal{L}_S^{(2)} + \mathcal{L}_W^{(2)}, \\ \mathcal{L}_S^{(4)} &\rightarrow \mathcal{L}_S^{(4)} + \mathcal{L}_W^{(4)}.\end{aligned}\tag{115}$$

Here $\mathcal{L}_W^{(4)}$ denotes the most general $\mathcal{O}(p^4)$ Lagrangian transforming like $\mathcal{L}_W^{(2)}$ under chiral rotations and thus able to absorb all the one-loop divergences generated by $\mathcal{L}_S^{(2)} \times \mathcal{L}_W^{(2)}$.

The $\mathcal{O}(p^4)$ operators transforming like $(8_L, 1_R)$ and $(27_L, 1_R)$ were classified for the first time by Kambor *et al.*,³⁴⁾ about 10 years ago, whereas the $\mathcal{O}(e^2p^2)$ terms transforming as $(8_L, 8_R)$ have been analysed only very recently.³⁵⁾ The overall picture is certainly worse than in the case of strong interactions, since the number of independent operators is much larger: already within the dominant $(8_L, 1_R)$ sector there are 37 independent terms.³⁶⁾ Nonetheless the theory still has a considerable predictive power, since out of these terms only few combinations appear in observable processes.^{33, 36)} For instance, within $K \rightarrow 3\pi$ decays the number of independent counterterm combinations coincides with the number of leading (constant and linear) amplitudes. Thus a fit of the $\mathcal{O}(p^4)$ parameters from the leading amplitudes leads to unambiguous predictions for the quadratic slopes, as shown in Tables 2 and 3.

There are even cases of physical processes that receive $\mathcal{O}(p^4)$ contributions only from loop diagrams and not from counterterms, like the $K_S \rightarrow \gamma\gamma$ decay. The leading contributions to $\mathcal{A}(K_S \rightarrow \gamma\gamma)$ are the loop diagrams in Fig. 6. Since there

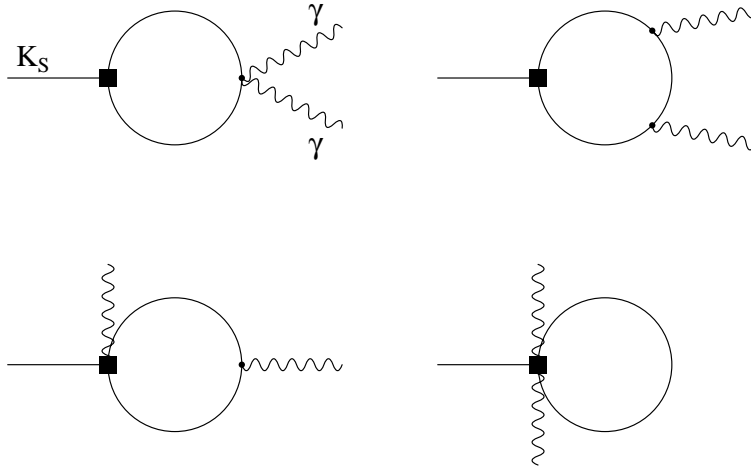


Figure 6: $\mathcal{O}(p^4)$ diagrams for the transition $K_S \rightarrow \gamma\gamma$. The black box denotes the weak vertex.

are no local contributions to this transition at $\mathcal{O}(p^4)$, the loop amplitude turns out to be finite³⁷⁾ and leads to the following parameter-free result:

$$\mathcal{B}(K_S \rightarrow \gamma\gamma)^{(4)} = 2.1 \times 10^{-6} . \quad (116)$$

The above prediction turns out to be in excellent agreement with the experimental data, $\mathcal{B}(K_S \rightarrow \gamma\gamma)^{\text{exp.}} = (2.5 \pm 0.5) \times 10^{-6}$, providing a very significant test of the quantum nature of this effective field theory.

4.4 Beyond $\mathcal{O}(p^4)$

Another process that receives $\mathcal{O}(p^4)$ contributions only from loop diagrams is $K_L \rightarrow \pi^0\gamma\gamma$. The diagrams describing this transition are very similar to those in Fig. 6; the only difference is an external pion leg attached to the weak vertex. In this case, however, the $\mathcal{O}(p^4)$ parameter-free prediction of the branching ratio

$$\mathcal{B}(K_L \rightarrow \pi^0\gamma\gamma)^{(4)} = 0.6 \times 10^{-6} \quad (117)$$

turns out to be in bad agreement with the experimental finding: $\mathcal{B}(K_L \rightarrow \gamma\gamma)^{\text{exp.}} = (1.7 \pm 0.1) \times 10^{-6}$. What is the reason for the big difference between $K_S \rightarrow \gamma\gamma$ and $K_L \rightarrow \pi^0\gamma\gamma$? The answer to this question can be traced back to what we learned in lecture 3.

The first difference is related to the absorptive parts of the amplitudes. Since the coupling G_8 has been fitted from $K_S \rightarrow 2\pi$, the imaginary part in $K_S \rightarrow \gamma\gamma$, computed at the one-loop level, perfectly reproduces the absorptive contribution due to the $K_S \rightarrow 2\pi$ intermediate state. On the contrary, because of the underestimate of $K \rightarrow 3\pi$ amplitudes at $\mathcal{O}(p^2)$ (see Table 2), the one-loop imaginary part of $K_L \rightarrow \pi^0\gamma\gamma$ underestimates the real absorptive contribution to this channel by 20%–30%. The second important difference between $K_S \rightarrow \gamma\gamma$ and $K_L \rightarrow \pi^0\gamma\gamma$ is induced by resonance contributions. Vector and axial-vector mesons, which are known to produce sizeable effects in the strong sector (see Table 1), do not appear in $K_S \rightarrow \gamma\gamma$ but can affect the $K_L \rightarrow \pi^0\gamma\gamma$ amplitude. In the latter case one can therefore expect a sizeable local $\mathcal{O}(p^6)$ counterterm, encoding the contribution induced by vector-meson exchange.

These two effects show that, contrary to $K_S \rightarrow \gamma\gamma$, a good description of $K_L \rightarrow \pi^0\gamma\gamma$ requires the inclusion of $\mathcal{O}(p^6)$ terms.³³⁾ A similar situation emerges in many other channels and can be interpreted as a general rule: unitarity corrections induced by pion loops and vector meson resonances provide a useful guide toward the estimate of the most significant higher-order corrections.

5 Conclusions

Solving strong interactions at low energy is a very difficult task, which so far nobody has been able to complete with analytical methods. On the other hand, at low energy, the approximate chiral symmetry does impose severe constraints on Green functions in the form of Ward identities. In studying the phenomenology of strong interactions at low energies it is extremely useful to take into account these symmetry constraints. The effective Lagrangian method is a tool to derive these constraints in an automatic manner – at the same time respecting also the general properties of unitarity and analyticity.

In this series of lectures we have introduced the basic concepts and technical tools for using this method. The last lecture was devoted to the study of a few examples of physically interesting decay channels of the kaons. In recent years a rich experimental activity at various facilities around the world has started, with the aim to study, with very high accuracy, all possible decay modes of the kaons, in some cases measuring extremely small branching ratios. The use of chiral perturbation theory for analysing this rich phenomenology is essential: it is not by chance that this lecture series was given at the Frascati Laboratories, which host one of the main

world facilities for this kind of physics.

Kaon physics is certainly not the only subfield of strong interactions where CHPT is useful and is applied successfully. A subject that we have not touched at all in these lectures, but where the application of CHPT is nowadays routine is the physics of baryons, going from the “easy” one-baryon sector (πN scattering, $\pi N \rightarrow \pi\pi N$, photo- and electro-production reactions, etc.) to the more complicated sectors involving two or more baryons.

In perspective, we believe that CHPT will become more and more used in connection with lattice calculations: as we have seen in the few examples of calculations discussed here, the dependence on the quark masses is always explicit. Whenever a quantity is calculated on the lattice one can use CHPT to determine the explicit dependence on the quark masses and make the extrapolation to the physical quark masses in a controlled manner. The “simple procedure” we described here hides in fact some serious technical difficulties – conceptually, however, it is obvious that merging the two methods offers clear advantages, and we will see more and more of them in the future. The technical difficulties are only welcome, because it means there will be work for theorists!

Let us finally conclude by mentioning that this method is also applicable to very different physical systems. In fact it can be applied to any physical system in which spontaneous symmetry breaking occurs, and which has an energy gap such that, at low energy, only the Goldstone modes dominate the physics. We find very interesting, for example, the applications to magnetic systems – in some cases, precisely the same effective Lagrangian that we have constructed for pions can be used to describe the behaviour of one of these. ³⁸⁾

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