Large scale geometry and evolution of a universe with radiation pressure and cosmological constant.

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Abstract

In view of new experimental results that strongly suggest a non-zero cosmological constant, it becomes interesting to revisit the Friedman-Lemaître model of evolution of a universe with cosmological constant and radiation pressure. In this paper, we discuss the explicit solutions for that model, and perform numerical explorations for reasonable values of cosmological parameters. We also analyse the behaviour of red-shifts in such models and the description of "very large scale geometrical features" when analysed by distant observers.

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1 INTRODUCTION

According to present experimental constraints, the universe, nowadays, is almost spatially flat; moreover, the effect of the radiation term, in the Friedman's equation, is also negligeble. However, one should distinguish between the description of the universe, as it appears "now",and the description the description of how it was and how it will be.

Most papers dealing with the determination of the cosmological parameters, nowadays, deal with the universe as it appears "experimentally", now. Many other articles deal with the complicated issues related with the mechanism(s) of inflation, reheating etc.

In the present article, starting with recent experimental constraints ([6],[7],[4],[8]) on the density parameters Ω_m, Ω_Λ, we want to study analytically the whole history of the universe, i.e., how it was (*after* the first few minutes) and how it wil be. A detailed analytic discussion of Friedman universes with cosmological constant and radiation pressure was performed years ago ([2], [3]), but at that time, the cosmological constant was believed to be zero by many, so that the above work – although justified by a particular phenomenological analysis ([10]) – could be considered only as only a purely mathematical exercise.

At the light of recent experimental results, part of the work carried out in ([2], [3]) becomes phenomenologically relevant : we do not plan to give a thorough discussion of analytical solutions in all possible cases, since this was done already by us, but restrict our attention to those cases that are compatible with recent observations and present an updated analytic discussion.

We are not concerned, in this paper, with the problem of inflation (there are several theoretical scenarios ...) as it is clear that, at such early periods, Friedman's equation is anyway not expected to provide a good mathematical model of the evolution of the universe. It looks however reasonable to suppose that there was a time, in some remote past, where the universe was already almost homogeneous and still rather hot. One can then use Friedman's equation to extrapolate, back in time, to this early epoch. Putting artificially the radiation term to zero is an unnecessary simplification that would prevent one to perform useful extrapolation towards this remote past. One can actually keep that term at no cost since cosmological models with a non-zero cosmological constant and radiation pressure can be studied analytically. Moreover, the experimental constraints on the radiation term are quite well known thanks to the measurement of the Cosmic Microwave Background radiation (CMB).

A similar comment can be made about the value of the constant parameter ($k = \pm 1$ or 0) that allows one to distinguish between spatially closed ($k = 1$) and open ($k = 0$ or -1) universes: it may be that the reduced cosmological curvature density Ω_k is very small, *nowadays*, and experimentally compatible with zero; however, this quantity is function of time, and setting artifically the constant k to 0 prevents one from studying some possibly interesting features in the history of the universe (for instance, the existence of an inflexion point at some particular time during the expansion, requires $k = 1$).

For the above reasons we shall not drop, in general, the radiation term and will keep k as an unknown discrete parameter with values ± 1 or 0. Our main purpose, however, is to study the effects of a non zero (and positive) cosmological constant on the analytic solutions describing the dynamics of our universe and on the large scale geometry.

In the coming section, we shall recall the relations between different cosmological quantities of interest. One should remark that experimentally available parameters do not necessary coincide with the quantities that provide a good mathematical description for the solutions of Friedman's equation.

In the following section we shall start with this equation (written with a "good" set of variables), and shall analyse possible histories of our universe, both qualitatively (using a well known associated mechanical model [12]) and analytically (using mathematical techniques (that can be traced back to [1]) explained in [2] and used in [2], [3]).

Finally, in the last section, we investigate the influence of the selection of a cosmological model matters, on "changes of redshift charts". We answer the following question (following again [2]): if some observer, located somewhere in the universe, plots the redshift of every celestial object against its direction, what is the redshift that would be measured, for the same objects, by another observer, at the same moment of time (we have an homogeneous cosmology!) but situated somewhere else – and possibly very far – in the spatial universe ? These transformation laws involve not only the geometry of our universe (relative location of the two points) but also its dynamics (in particular the value of the cosmological constant).

If the distribution of matter in the universe posesses some particular large scale features (for instance a "symmetry" of some kind), our solution of the above problem becomes phenomenologically relevant. Indeed, there is no reason to believe that our position (our galaxy) is ideally located with respect to such putative elements of symmetry, and it would therefore be necessary to perform a change of position to recognize such features. For instance, the absence of matter in a three dimensional shell centered around some particular point would not appear as such when analysed from a galaxy situated far away from this special point (this was actually the hypothesis made in [10]).

2 RELATIONS BETWEEN COSMOLOGICAL PARAMETERS

2.1 Geometrical Background for Friedman-Lemaître Models

Most of the present section is well known and can be found in standard textbooks. The dimensionless temperature $T(\tau)$ which seems preferable to the radius $R(t)$) if one wants to perform an analytic discussion of Friedman's equation, was introduced by [2]. Our purpose here is mostly to provide a summary of useful formulae and present our notations.

2.1.1 Units

We chall work in a system of units in which both $\hbar$ and c are set to 1. All quantities with dimention are thus given as powers of length [cm].

In our units, energies are in $[cm^{-1}]$. Since temperatures (to be denoted by $\tilde{T}$ in this paper) appear only in products $k_B\tilde{T}$ (where k_B is the Boltzmann constant), we can take take the temperature as dimensionless, and k_B as an energy.

2.1.2 Metric and Topological Considerations

As it is well known, these models assume that the universe is, in first approximation, homogeneous and isotropic.

The metric There are three possibilities:

- $k = 1$, $ds^2 = -dt^2 + R^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]$
- $k = -1$, $ds^2 = -dt^2 + R^2[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]$
- $k = 0$, $ds^2 = -dt^2 + R^2[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)]$

The topology The first case $k = 1$, is sometimes called the "compact case" (closed universe), since the spatial universe is then compact; the global topology for space-time can be taken as $S^3 \times \mathbb{R}$ but one should remember that the above expression for the metric is only local, so that one could take, in place of S^3, any quotient S^3/Γ of the 3-sphere by a finite group Γ operating without fixed point (for instance one of the binary tetrahedral, octahedral or icosahedral subgroups of $SU(2)$). In the case of S^3, the full isometry group is $SU(2) \times SU(2)$ since S^3 itself is homeomorphic with the group of unimodular and unitary matrices.

In cases $k = 0$ or $k = -1$, the space component is non compact (open) and the universe is respectively called "flat" or "hyperbolic".

On intuitive grounds, and independently of experimental fits, one could argue that the model with $k = 1$ is preferable to the others since Friedman's equation was derived by assuming homogeneity of the stress-energy, something that is hard to achieve in an open universe if the total quantity of matter is finite.

2.1.3 Friedman's Equation

This equation is obtained by writing Einstein's equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ where $T_{\mu\nu}$ is the stress energy tensor of a perfect fluid and where $G_{\mu\nu}$ is the Einstein tensor associated with the above metrics.

The average local energy splits into three parts :

- Vacuum contribution $\rho_{vac} = \frac{\Lambda}{8\pi G}$ where Λ is the cosmological constant,
- Radiation contribution ρ_{rad} such that $R^4(t)\rho_{rad}(t) = const. = \frac{3}{8\pi G}C_r$,
- Averaged matter contribution ρ_m such that $R^3(t)\rho_m(t) = const. = \frac{3}{8\pi G}C_m$,

The evolution of $R(t)$ is governed by the Friedman's equation

$$\frac{1}{R^2}(\frac{dR}{dt})^2 = \frac{C_r}{R^4} + \frac{C_m}{R^3} - \frac{k}{R^2} + \frac{\Lambda}{3}$$

It is convenient to introduce the *conformal time* τ, defined, up to an additive constant by

$$d\tau = \frac{dt}{R}$$

This variable is natural, both geometrically (it gives three-dimensional geodesic distances) and analytically (as we shall see later). The quantities C_r, C_m and Λ have dimensions L^2, L, L^{-2} respectively. In order to replace R by a dimensionless quantity, we introduce the characteristic length scale of matter

$$\Lambda_c = \frac{4}{9C_m^2}$$

and replace $R(\tau)$ by the *reduced dimensionless temperature* [2]

$$T(\tau) = \frac{1}{\Lambda_c^{1/2}R(\tau)} \tag{1}$$

With this new variable, Friedman's equation becomes

$$(dT/d\tau)^2 = \alpha T^4 + 23T^3 - kT^2 + \lambda 3 \tag{2}$$

where

$$\lambda = \Lambda\Lambda_c$$

and

$$\alpha = C_r\Lambda_c$$

are two constant dimensionless parameters.

The reason for calling T the "reduced dimensionless temperature" is that it is proportionnal to the temperature $\tilde{T}$ of the black body radiation. Indeed, $\rho_{rad} = 4\sigma\tilde{T}^4$ where σ is the Stefan-Boltzmann constant. Since $\rho_{rad} = 38\pi G\alpha\Lambda_c T^4$, one finds

$$\tilde{T}^4 = 38\pi G\alpha\Lambda_c 4\sigma T^4 \tag{3}$$

This may be the right place to recall that Lemaître ([1]) did, long ago, an analytic study of the solutions of Friedman's equations; his discussion, made in terms of R and t involves, of course, elliptic functions, but it is only when we express the cosmological quantities in terms of $T(\tau)$ (the reduced temperature, as a function of conformal time) that these quantities can be written themselves as elliptic functions with respect to a particular lattice (the associated Weierstrass $\mathcal{P}$ function appears then naturally). The fact that such a direct link with the theory of Weierstrass can be established should be clear from the fact that, with our parametrization, the right hand side of equation 2 is a polynomial of degree four.

The Hubble function decribing the rate of expansion is defined, as usual, by $H = R^{-1}(dR/dt)$ and can be written, in terms of $T(\tau)$ as

$$H(\tau) = -\Lambda_c^{12} dT d\tau$$

Another useful quantity is the *deceleration function* $q = -R(dR/dt)^{-2}(d^2R/dt^2)$ that can be expressed in terms of $T(\tau)$ as follows:

$$q = -\lambda 3 + 13T^3 + \alpha T^4 \lambda 3 - kT^2 + 23T^3 + \alpha T^4$$

Actually, the choice of the sign (and of the name) turns out to be a historical mistake, since the recent bounds on the cosmological constant lead to a negative value for q (an accelerating expanding universe).

By multiplying Friedman's equation by Λ_c/H^2, one obtains the famous relation :

$$1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda \tag{4}$$

with

$$\begin{aligned} \Omega_k &= -kT^2 \Lambda_c H^2 && (5) \\ \Omega_m &= 23T^3 \Lambda_c H^2 && (6) \\ \Omega_\Lambda &= \lambda 3 \Lambda_c H^2 && (7) \\ \Omega_r &= \alpha T^4 \Lambda_c H^2 && (8) \end{aligned}$$

Notice that the famous "cosmic triangle" of [4] – a kind of cosmic Dalitz' plot, becomes a triangle only if one decides to forget the radiation term (then, indeed, $1 = \Omega_m + \Omega_k + \Omega_\Lambda$). In terms of the usual temperature $\tilde{T}$, the radiation terms reads:

$$\Omega_r = 32\pi G 3\sigma \tilde{T}^4 H^2 \tag{9}$$

2.1.4 Description of the History of the Universe by the Reduced Temperature Function $T(\tau)$

- Rather than using $R(t)$, we describe [2] the history of the universe by the function $T(\tau)$. This function can be intuitively thought of as a dimensionless inverse radius; it is proportionnal to the usual temperature $\tilde{T}$. Because of the phenomenon of expansion, these quantities T and $\tilde{T}$ decrease with τ. The argument τ itself, a dimensionless arc length, measures time : for instance, if the spatial universe is (hyper)spherical, τ is a measure (in radians) of the length of the arc spanned by a photon that was produced at the big bang. A given solution (a given "history" of a – dimensionless – universe) is fully characterized by the solution of a differential equation depending on the two dimensionless parameters α and λ. Of course, one has also to specify some initial value data (one takes $T(\tau) \to \infty$ when $\tau \to 0$). As we shall see in a later section, $T(\tau)$ is a particular elliptic function; the history of our universe is then also fully encoded by the two Weierstrass invariants (g_2, g_3), or, equivalently, by the two periods ω_1, ω_2 of this elliptic function. The Weierstrass invariants are given, in terms of parameters k, α and λ, by the algebraic relations

$$g_2 = k^2 12 + \alpha\lambda 3. \tag{10}$$

$$g_3 = 16^3(k^3 - 2\lambda) - \alpha\lambda k18. \tag{11}$$

The formulae giving the two periods ω_1, ω_2 in terms of the cosmological parameters k, α and λ (or g_1, g_2) involve elliptic integrals and will be given later.

- Dimensional quantities are obtained from $T(\tau)$ thanks to the measurement of a single function having dimensions of a length. Such a function is usually the Hubble function $H = H(\tau)$. The behaviour of $R(t)$ is given by the two parametric equations $R(\tau)$ and $t(\tau)$. As we shall see later, in all cases of physical interest, $t(\tau)$ reaches a logarithmic singularity for a finite conformal time τ_f (the universe expands forever, but as $t \to \infty$ the arc length associated with the path of a single photon goes to a finite value τ_f).
- Finally, it remains to know "when" we are, i.e., the age of the universe. This can be expressed in terms of the dimensionless quantity τ_o (a particular value of τ) or, more conventionally, in terms of $t_o = t(\tau_o)$. Experimentally, one measures the Hubble "constant" $H_o = H(\tau_o)$, i.e., the value of the function $H(\tau)$, now.

2.1.5 Cosmological Quantities

We just give here a list of most cosmological quantities of interest.

Dimensionless Quantities We have the constant parameters α, λ, $k = 0, \pm 1$, the time-dependent dimensionless densities $\Omega_k(\tau)$, $\Omega_\Lambda(\tau)$, $\Omega_m(\tau)$, $\Omega_r(\tau)$ and the reduced temperature function $T(\tau)$. Of interest also is the deceleration function $q(\tau)$. Of course, the conformal time τ itself is a non-constant (!) dimensionless quantity. The black-body temperature $\tilde{T}$ (temperature of the blackbody microwave radiation, experimentally expressed in degree Kelvin) is itself dimensionless – see our remark at the beginning of this section. It is proportional to the dimensionless reduced temperature T (or to the inverse radius R).

Dimensional quantities

- The Hubble function $H(\tau)$. This quantity is usually chosen to fix the length scale.
- The critical length scale Λ_c. It can be thought of as giving a measure of the "total mass" of the universe. More precisely, if the universe is spatially closed and has the topology and metric of a 3-sphere, its total mass is $M = 4\pi^2 R^3 \rho_m$ so that $\Lambda_c = (\pi 2GM)^2$.
- The radius R. Intuitivelly, it is a measure of the "mesh" of our spatial universe.
- The age t of the universe (cosmic time).

2.1.6 Expression of the Cosmological Quantities in Terms of the Hubble Function H and of the Dimensionless Densities $\Omega_\Lambda, \Omega_k, \Omega_m, \Omega_r$

Since most experimental results are expressed in terms of the dimensionless densities $\Omega_\Lambda, \Omega_k, \Omega_m, \Omega_r$, we express all other cosmological quantities of interest in terms of them. Typically, dimensional quantities also involve the value of the Hubble function $H(\tau)$. All these formulae can be obtained by straightforward algebraic manipulations.

Constant dimensionless quantities (parameters)

$$\lambda = 274\Omega_\Lambda\Omega_m^2|\Omega_k^3| \tag{12}$$

$$\alpha = 49\Omega_r|\Omega_k|\Omega_m^2 \tag{13}$$

Time-dependent dimensionless quantities (assuming $k \neq 0$)

$$T = -k32\Omega_m\Omega_k = 32\Omega_m|\Omega_k| \tag{14}$$

$$q = \Omega_m 2 - \Omega_\Lambda + \Omega_r = 3\Omega_m 2 + \Omega_k + 2\Omega_r - 1 \tag{15}$$

The temperature of the CMB radiation:

$$\tilde{T}^4 = 38\pi G 14\sigma\Omega_r H^2$$

The conformal time τ itself can be found numerically, once a value of T is known, by solving the (non algebraic) equation expressing T as a function of τ (see the next section).

Constant dimensional quantities (parameters)

$$\Lambda = 3H^2\Omega_\Lambda \tag{16}$$

$$\Lambda_c = 49|\Omega_k|^3\Omega_m^2 H^2 \tag{17}$$

$$C_m = \Omega_m|\Omega_k|^{3/2}H$$

$$C_r = \Omega_r|\Omega_k|^2 H^2$$

Time-dependent dimensional quantities (assuming $k \neq 0$) Besides the Hubble "constant" H itself, one has to consider also the radius

$$R = 1T{\Lambda_c}^{1/2} \tag{18}$$

the cosmic time (the age of the universe corresponding to conformal time τ) given by

$$t(\tau) = {\Lambda_c}^{-1/2}\int_0^\tau d\tau' T(\tau') \tag{19}$$

2.2 Dimensions of the Cosmological Quantities

In our system of units, all quantities are either dimensionless or have a dimension which is some power of a lengh ($[cm]$). We gather all the relevant information as follows:

$$t \sim R \sim C_m \sim [cm]$$

$$G \sim C_r \sim [cm^2]$$

$$H \sim k_B \sim energy \sim [cm^{-1}]$$

$$\Lambda \sim \Lambda_c \sim [cm^{-2}]$$

$$\rho_{vac} \sim \rho_{rad} \sim \rho_m \sim \sigma \sim [cm^{-4}]$$

Here σ denotes the Stefan constant (remember that degrees Kelvin are dimensionless). Finally we list the dimensionless quantities:

$$\Omega_m \sim \Omega_\Lambda \sim \Omega_k \sim \Omega_r \sim q \sim T \sim \tilde{T} \sim \tau \sim \alpha \sim \lambda \sim [cm^0 = 1]$$

3 EXPERIMENTAL CONSTRAINTS

3.1 Experimental Constraints from High-redshift Supernovae and Cosmic Microwave Background Anisotropies

In general we shall add an index o to refer to *present* values of time-dependent quantities (like H_o, Ω_k^o, Ω_r^o, Ω_m^o, Ω_Λ^o, q_o, T_o or $\tilde{T}_o$). Remember that $k, \alpha, \lambda, \Lambda, \Lambda_c$ are constant parameters.

The analysis given by [8] (based on experimental results on the latest cosmic microwave background anisotropy and on the distant Type Ia supernovae data [5] [6]) gives $0.13 < \Omega_m^o < 0.43$ and $+0.40 < \Omega_\Lambda^o < +0.80$. In order to stay on the safe side, we do not restrict ourselves to the more severe bounds that one can obtain by performing a combined likelihood analysis that would use both sets of data coming from anisotropy of CMB radiation and distant type Ia supernovae.

Moreover, many experiments, nowadays, give a present value of the Hubble function close to $H_o = h\,100\,km\,sec^{-1}\,Mpc^{-1}$, with $h \simeq 0.66$, so that $H_o^{-1} \simeq 14.2 \times 10^9\,yr = 13.25 \times 10^{27}\,cm$. This is the value which we shall use in our estimations.

3.2 Implications of Experimental Constraints on Friedman-Lemaître Parameters

With $\hbar = c = 1$, the value of the Stefan-Boltzmann constant is $\sigma = \pi^2 k_B^4 60 = 59.8 cm^{-4}$, k_B is the Boltzmann constant and the Newton constant is $G = 2.61 \times 10^{-66} cm^2$. Using the above value for H_o and formula 9 (with $\tilde{T}_o = 2.73K$), we obtain $\Omega_r^o \simeq 5.10 \times 10^{-5}$.

Using now the "cosmic triangle relation" (equation 4) together with the experimental bounds on Ω_m^o and Ω_Λ^o previously recalled, one finds $-0.23 < \Omega_k^o < 0.47$. Remember that, with the present conventions, $\Omega_k < 0$ when $k = +1$.

The formula 16 given in the last section leads to $+0.68 \times 10^{-56}\,cm^{-2} < \Lambda < +1.36 \times 10^{-56}\,cm^{-2}$.

In order to obtain good estimates for the remaining quantities, one would need a better measurement of the curvature density Ω_k^o. Indeed, formulae 12, 13, together with the previously given bounds on Ω_k^o only imply the following for the dimensionless constant parameters: $\alpha < 6.3 \times 10^{-4}$ and $\lambda > 0.44$.

Notice that λ is quite sensitive to the independent values of Ω_m^o and Ω_Λ^o (indeed, Ω_k, appearing in the denominator of equation 12, can be very small); for instance, the values $\Omega_m^o = 0.13$, $\Omega_\Lambda^o = 0.40$ lead to $\lambda = 0.44$ but $\Omega_m^o = 0.13$, $\Omega_\Lambda^o = 0.80$ lead to $\lambda = 266.68$. By the way, one should stress the fact that the behaviour of analytic solutions of Friedman's equations is essentially determined by the value of the (dimensionless) reduced cosmological constant λ, which can be rather "big", even when the genuine cosmological constant Λ (which is a dimensional quantity) is itself very "small".

As already mentionned, present experimental constraints only imply $\lambda > 0.44$, but one should remember that $\Omega_k \simeq 1 - \Omega_m - \Omega_\Lambda$ should be negative for a spatially closed universe ($k = +1$); therefore, if, on top of experimental constraints on Ω_m and Ω_Λ we *assume* that we live in a spatially closed universe (a –reasonable– hypothesis that we shall make later, for illustration purposes) then $\Omega_m + \Omega_\Lambda > 1$ and the constraint on λ, as given by equation 12 is more tight; an elementary variational calculation shows then that the smallest possible value of λ, taking into account both the experimental constraints and the hypothesis $k = +1$, is obtained when both Ω_m and Ω_Λ saturate their experimental bounds (values respectively equal to 0.43 and 0.8), so that $\Omega_k \geq -0.23$ and $\lambda \geq 82$.

One should not think that the value of λ could be arbitrary large: we shall see a little later (next subsection) that, for experimental reasons, it has also to be bounded from above (condition $\lambda_- < \lambda < \lambda_+$).

Using equations 14, 15, one obtains easily the bounds $T_o > 0.41$, $-0.75 < q_o < -0.18$ and $\Lambda_c < 1.6 \times 10^{-56} cm^{-2}$ for the present values of the reduced temperature T and deceleration function q and for the critical length scale Λ_c.

In view of the experimental results (Ω_r^o small and Ω_k^o compatible with zero), one maybe tempted to make the simplifying hypothesis $1 = \Omega_m^o + \Omega_\Lambda^o$, i.e., set *both* terms Ω_k^o and Ω_r^o to zero in the relation $1 = \Omega_m + \Omega_k + \Omega_\Lambda + \Omega_r$; notice however that assuming the vanishing of $\Omega_k + \Omega_r$ at all times (without assuming the vanishing of each term, individually) is totally impossible, as these densities are not constant and it is easy to see, from the definition of these quantities, that such a relation can only hold at at a single moment; it is obvious that the radiation contribution, Ω_r^o, although small, is not *strictly* zero. Moreover, putting artificially the constant k to zero is certainly compatible with the present experimental data, but one should be aware of the fact that the curvature density Ω_k is not a constant quantity and that setting it to zero at all times is an artificial simplification that was probably not justified when the universe was younger ...

3.3 A Curious Coincidence

The parameters C_m and C_r appearing in Friedman's equation, and measuring respectively the matter contribution and radiation contribution to the average local energy, are, *a priori* , independent quantities. For this reason, the relation between $\tilde{T}$ (the temperature of the cosmological background radiation) and T (the reduced temperature) involves C_m and C_r.

$$\tilde{T}^4 = \frac{3}{8\pi G} \frac{\alpha \Lambda_c}{4\sigma} T^4 = \frac{3}{8\pi G} \frac{C_r \frac{2^4}{3^4 C_m^4}}{4\sigma} T^4$$

One may consider he special case of a universe for which the two functions $T(\tau)$ and $\tilde{T}(\tau)$ coincide; this amounts setting

$$\frac{C_r}{C_m^4} = 54\pi G\sigma = 54\pi G \frac{\pi^2 k_B^4}{60} = \frac{9}{10}\pi^3 G k_B^4$$

In this case, the present values of the densities Ω_m^o and Ω_Λ^o are no longer independent: $\Omega_r^o = 5.1 \times 10^{-5}$ is, as usual, exactly known since $\tilde{T}_o = 2.73K$ is known; again, as usual, we have $\Omega_m^o = \frac{2}{3}|\Omega_k^o| T_o$ but if we measure Ω_m^o (for instance) and *assume* $T_o = \tilde{T}_o$, we find the value of $|\Omega_k^o|$. Choosing then, for instance, $k = +1$ (a closed spatial universe), so that $\Omega_k^o = -|\Omega_k^o|$, we deduce Ω_Λ from the equation $\Omega_k^o + \Omega_m^o + \Omega_\Lambda^o + \Omega_r^o = 1$.

We do not see any theoretical reason that would justify to set $T = \tilde{T}$, since these two quantities could very well be different by several orders of magnitude (even with the rather small value of the radiation density Ω_r^o), nevertheless ... by a curious coincidence, if one takes the two densities Ω_m^o and Ω_Λ^o to be numerically equal to their highest possible values compatible with present experimental bounds (0.43 and 0.8), one finds a value of T_o equal to 2.80; this is very near the experimental value 2.73 of the temperature $\tilde{T}_o$. We have no explanation for this "cosmological miracle".

4 ANALYTIC BEHAVIOUR OF SOLUTIONS

4.1 Qualitative Behaviour of Solutions

Equation (2) can be written

$$(dTd\tau)^2 + V_{\alpha,k}(T) = \lambda 3$$

with

$$V_{\alpha,k}(T) = -\alpha T^4 - 23T^3 + kT^2$$

This is the equation of a one-dimensional mechanical system with "coordinate" T, potential $V_{\alpha,k}(T)$ (shown in Fig 1) and total energy $\lambda/3$. The kinetic energy beeing non negative, the associated mechanical system describes a horizontal line in the $(V(T), T)$ plane but never penetrates under the curve $V_{\alpha,k}(T)$ (that would correspond to τ imaginary). The length of the vertical line segment between a point belonging to the curve and a point with same value of T but belonging to the horizontal line $\lambda/3$ (described by the associated mechanical system) is a measure of $(dTd\tau)^2$.

For a given value of α (the radiation parameter), and for $k = \pm 1$, the curve $V_{\alpha,k}(T)$ has typically two bumps (two local maxima). For $k = 1$ (closed universe), the right maximum occurs for a positive value of T and $V(T)$ whereas, for $k = -1$ (hyperbolic case), this maximum is shifted to $T = 0$ and $V(T) = 0$. For $k = 0$ (flat case), the right maximum disappears and we are left with an inflection point at $T = 0, V(T) = 0$.

In this paper,we almost always suppose that α is non zero; however, one should notice that if $\alpha = 0$, the curve $V(T)$ becomes a cubic (Fig. 2) and the left hand side maximum disappears (it moves to $-\infty$ as α goes to 0).

Notice that figure 1 describes only the qualitative features of our space-time history, since, for reasonable values of α and λ (i.e. values compatible with experimental constraints), the vertical coordinate of the left hand side maximum should be at least 1000 times higher than the vertical coordinate of the right hand side maximum.

The experimental constraints also show not only that λ is positive (since Ω_Λ is) but that it is bigger than $\lambda_-/3$ (the vertical coordinate of the right maximum) and, at the same time much smaller than $\lambda_+/3$ (the vertical coordinate of the left maximum). Indeed, for small values of α, the positions of the extrema (cf. next section) are given by $\lambda_- \simeq k(1 - 3\alpha k)$ and $\lambda_+ \simeq 116\alpha^3(1 + 12k\alpha)$; the previous bounds on λ show then directly that $\lambda > \lambda_-$; moreover, the inequality $\lambda < \lambda_+$ is equivalent to $\Omega_r^3 < 27256\Omega_m^4/\Omega_\Lambda$, and this clearly holds. Unfortunately, this does not lead to a stringent constraint on λ istself since the bound $\alpha < 6.3 \times 10^{-4}$ only implies $\lambda_+ > 2.5 \times 10^8$.

The associated mechanical system is therefore such that it describes an horizontal line like the one displayed on Fig. 1. Typically, a given universe starts from the right of the picture ($T \to \infty$ corresponding to the big bang), moves to the left (slowing down), until it reaches the vertical axis ($T = 0$). This actually takes place in a finite conformal time τ_f but, as we shall see, it corresponds to a cosmic time $t \to \infty$, so that, for the universe in which we live, history stops there. However, the solution can be continued for $T < 0$ (a negative radius R) until the system bumps against $V(T)$ to come back to right infinity; the system then jumps to left infinity, follows the horizontal line (to the right) till it bumps on $V(T)$ again, and comes back. This round trip of the associated mechanical system is done in a (conformal) time $2\omega_r$ – a period of the corresponding elliptic function. Let us stress the fact that only the first part of the motion (from right infinity to the intersection with the vertical axis) is physically relevant for ordinary matter.

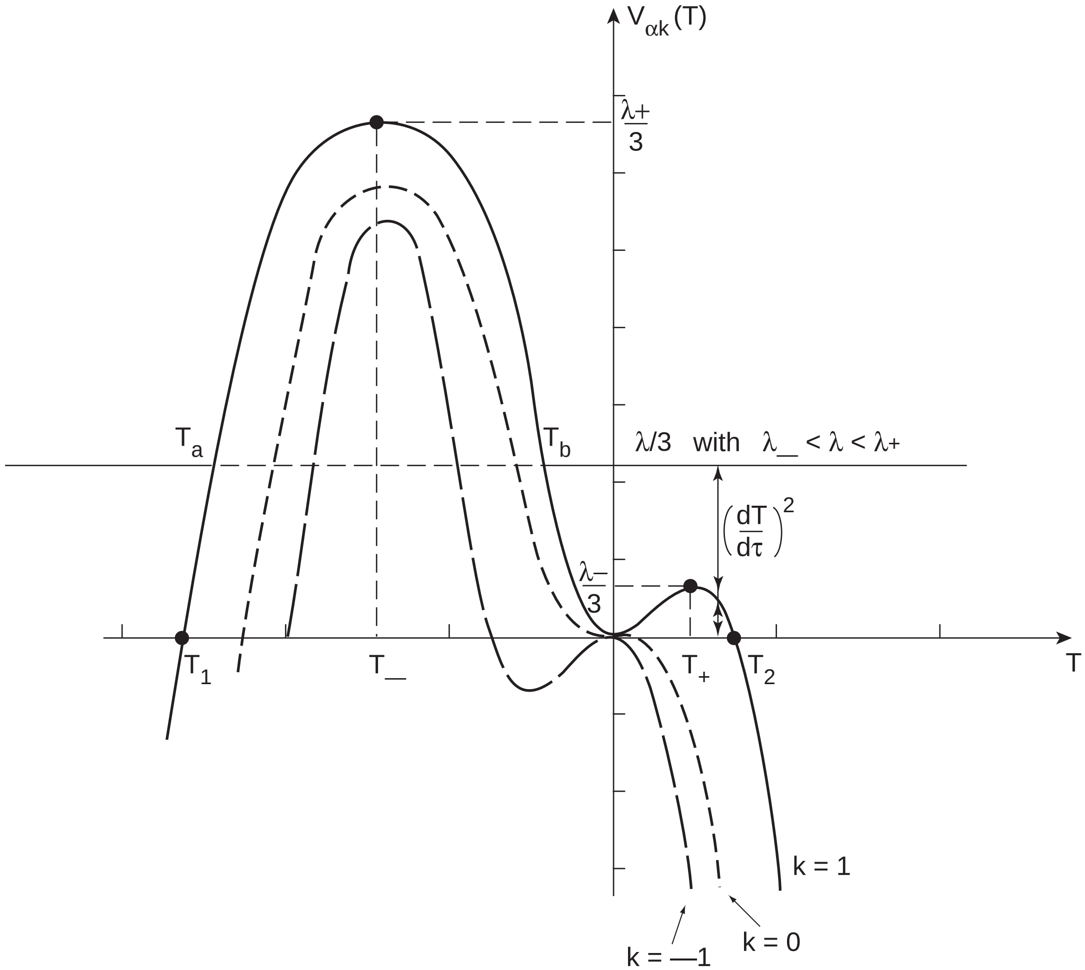

Figure 1: Potential for the associated mechanical system. Case $\alpha \neq 0$

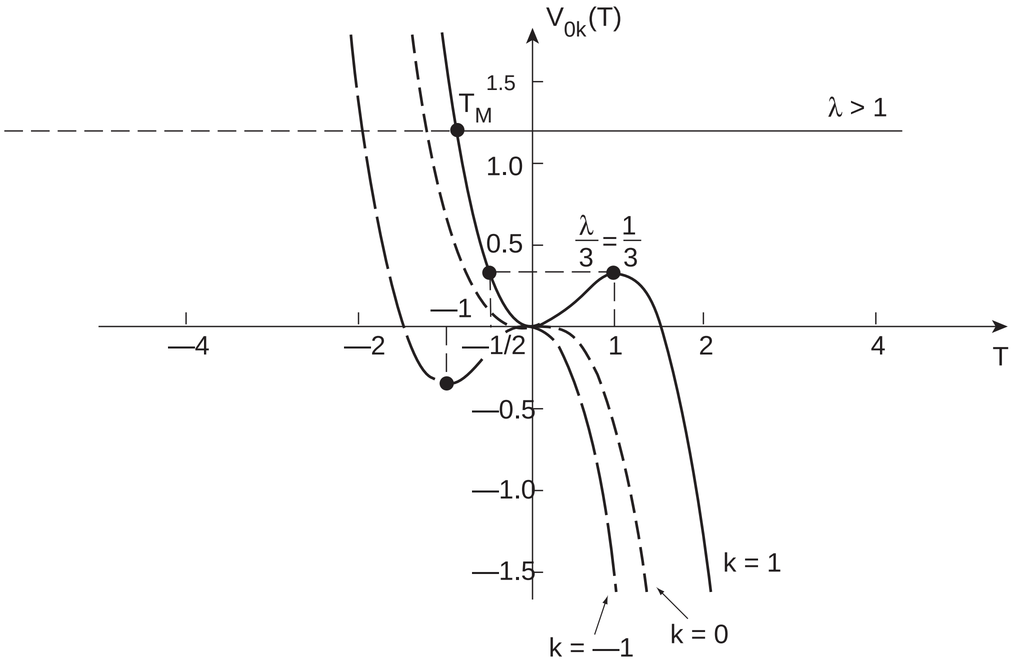

Figure 2: Potential for the associated mechanical system. Case $\alpha = 0$

Since the radius R is proportionnal to $1/T$ (see equation 1), the discussion, in terms of R is of course different: the system starts with $R = 0$ (big bang), and expands forever; moreover the expansion speeds up in all three cases $k = \pm 1$, $k = 0$. The only particularity of the case $k = 1$ is that there is an inflexion point (coming from the existence of a positive right hand side maximum for the curve $V(T)$) : the expansion speeds up anyway, but there is a time τ_I for which the rate of expansion vanishes.

4.2 The Elliptic Curve Associated with a Given Cosmology

4.2.1 General Features

Let us call $Q(T)$ the fourth degree polynomial that appears at the right hand side of Friedman's equation (eq. 2). Let T_j be any one of the (possibly complex) roots of the equation $Q(T) = 0$, then the fractional linear transformation

$$y = Q'(T)41T - T_j + Q''(T)24 \tag{20}$$

brings (2) to the form

$$(dyd\tau)^2 = P(y) \doteq 4y^3 - g_2 y - g_3 \tag{21}$$

where g_2 and g_3 are the two Weierstrass invariants given in eqs. (10,11). The solution to the previous equation is well known: $y = \mathcal{P}(\tau; g_2, g_3)$ where $\mathcal{P}$ is the elliptic Weierstrass function associated with the invariants g_2 and g_3. One should remember that, given a lattice L in the complex plane, an elliptic function f with respect to L is non constant meromorphic function that is bi-periodic, with respect to L (so that $f(\tau + u) = f(\tau)$ whenever u belongs to the lattice L). It is known (since Liouville) that if a is an arbitrary complex number (including infinity), the number of solutions of the equation $f(\tau) = a$ is independent of a, if multiplicities are properly counted; this number is called the *order*

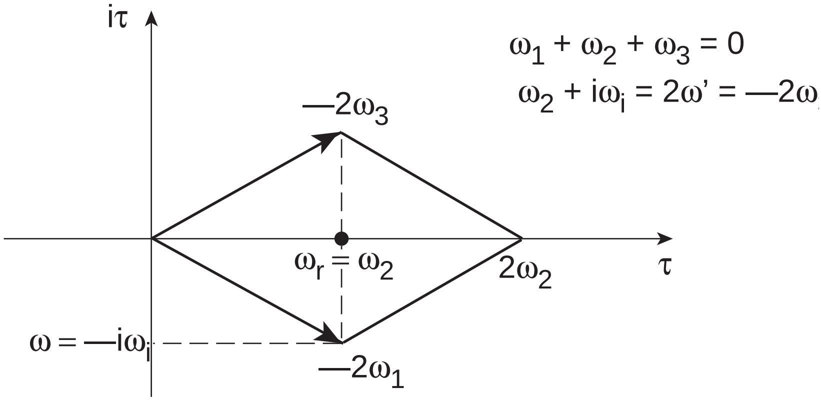

Figure 3: An elementary periodicity cell

of f. It is useful to know that any rational function of an elliptic function is also elliptic (with respect to the same lattice) but that its order will coincide with the order of f only if the transformation is fractional linear (like the transformation (20)). Finally, one should remember that the order of an elliptic functions is at least two and that the Weierstrass $\mathcal{P}$ function corresponding to a given lattice is defined as *the* elliptic function of order 2 that has a pole of order 2 at the origin (and consequently at all other points of L) and is such that $1/\tau^2 - \mathcal{P}(\tau)$ vanishes at $\tau = 0$. The two elementary periods generating the lattice L can be expressed in terms of g_2 and g_3, but conversely, the Weierstrass invariants can be expressed in terms of the elementary periods of the lattice L.

These old theorems of analysis lead directly to the fact that $T(\tau)$ is an elliptic function of order 2 and that other quantities that are rational functions of T (like the Hubble function H, the deceleration function q or all the cosmological densities Ω_m, Ω_r , Ω_k, Ω_Λ) are also elliptic with respect to the same lattice (but they are not of order 2). In this sense, one can say that our universe is fully described by the elliptic curve $x^2 = 4y^3 - g_2 y - g_3$ characterized by the two Weierstrass invariants g_2, g_3, or, alternatively, by two elementary periods generating the lattice L.

In this (physical) case, the elementary periodicity cell is a rhombus (figure 3) with vertices $\{0, \omega_r - i\omega_i, 2\omega_r, \omega_r + i\omega_i\}$. Another standard notations for periods are $2\omega_2 \doteq 2\omega_r$, $-2\omega_3 \doteq \omega_r + i\omega_i$ and

$$-2\omega_1 \doteq \omega_r - i\omega_i$$

(so that $\omega_1 + \omega_2 + \omega_3 = 0$). The transformation expressing the half-periods $\omega_{1,2,3}$ in terms of the Weierstrass invariants g_2 and g_3 (or conversely) can be obtained, either from a direct numerical evaluation of elliptic integrals (see below) or from fast algorithms described in [11]; one can also use the Mathematica function $WeierstrassHalfPeriods[\{g_2, g_3\}]$ as well as the opposite transformation $WeierstrassInvariants[\{\omega_1, \omega_2\}]$. In the case $\alpha = 0$, the value $\tau = \omega_r$ turns out to be a global minimum of $T(\tau)$, therefore another possibility to get ω_r in that case is to find numerically the first zero of the equation $dT(\tau)/d\tau = 0$.

4.2.2 Neglecting Radiation : the Case $\alpha = 0$

When $\alpha = 0$, the two poles of $T(\tau)$ coincide, so that the dimensionless temperature, up to shift and rescaling, coincides with the Weierstrass $\mathcal{P}$-function iself (the fractional linear transformation 20 takes the simple form $T = 6y + k/2$). Notice that $\mathcal{P}$ is an even function of τ.

In that case

$$T(\tau) = 6[\mathcal{P}(\tau; g_2, g_3) + k/12] \tag{22}$$

A typical plot of $T(\tau)$ for experimentally reasonable values of the cosmological parameters is given in figures 4 (closed universe: $k = 1$) and 5 (open universe: $k = -1$).

In both cases, the physically relevant part of the curve is given by the interval $0 < \tau < \tau_f$ since $\tau = 0$ corresponds to the big bang (infinite temperature) and $\tau = \tau_f$ to the end of conformal time: the cosmic time $t = t(\tau)$ reaches a logarithmic singularity when $\tau \to \tau_f$, so that when $t \to \infty$, the arc length described by a photon born with the big bang tends toward the finite value τ_f; for values of λ compatible with recent observations, this limit is strictly less than 2π so that it will never be possible to see the "back of our head", even if the universe is closed and if we wait infinitely many years . . . .

Notice that both curves show the existence of an inflexion point denoted by τ_I on the graphs; however, in the case $k = -1$ (open universe), this inflexion point is located after the end of conformal time ($\tau > \tau_f$) and is therefore physically irrelevant. The existence of such a point is of interest only in the case of a spatially closed universe. Notice that in that case, the experimental observations tell us that the present value τ_o of τ (i.e., todays' date) is bigger than τ_I (and, of course, smaller than τ_f).

The values $\tau_f < \tau < \tau_g$ correspond to a classicaly forbidden region (negative dimensionless temperature). The values $\tau_g < \tau < 2\omega_r$ correspond to universe in contraction, ending with a big crush ... Finally, one should remember that T is a (doubly) periodic function, so that the same analysis can be performed in all the intervals $[2p\omega_r, 2(p+1)\omega_r]$.

Let us repeat that τ is a *conformal* time and that $t \to \infty$ when $\tau \to \tau_f$, so that, in both cases $k = \pm 1$ we are indeed in a ever expanding universe starting with a big bang. The curve obtained for $k = 0$ has the same qualitative features, but for the fact that the inflection point moves to the end of conformal time: $\tau_I = \tau_f$ in that case.

The shape of $T(\tau)$, as given by figures 4 and 5 is in full agreement with the qualitative discussion given in the previous section. Such plots were already given in [2] where Weierstrass functions had been numerically calculated from the algorithms obtained in the same reference. The same plots can now be obtained easily using for instance Mathematica, (the last versions of this program incorporate routines for the Weierstrass functions $\mathcal{P}, \zeta$ and σ). Note, however, that intensive calculations involving such functions should probably make use of the extremely fast algorithms described in reference [11]; these algorithms use, for the numerical evaluation of these functions, a duplication formula known for the Weierstrass $\mathcal{P}$ function. Moreover, evaluation of elliptic integrals (for instance caculation of periods in terms of invariants) can also be done by an iterative procedure (based on classical properties arithmetic-geometric mean), this is also described in reference [11].

4.2.3 Not Neglecting Radiation : the Case $\alpha \neq 0$

This case, technically a little bit more difficult, is detailed in Appendix 1.

4.3 Determination of all Cosmological Parameters: a Fictitious Case

Here we summarize the procedure that should (ideally!) be followed, in order to specify all cosmological parameters of interest.

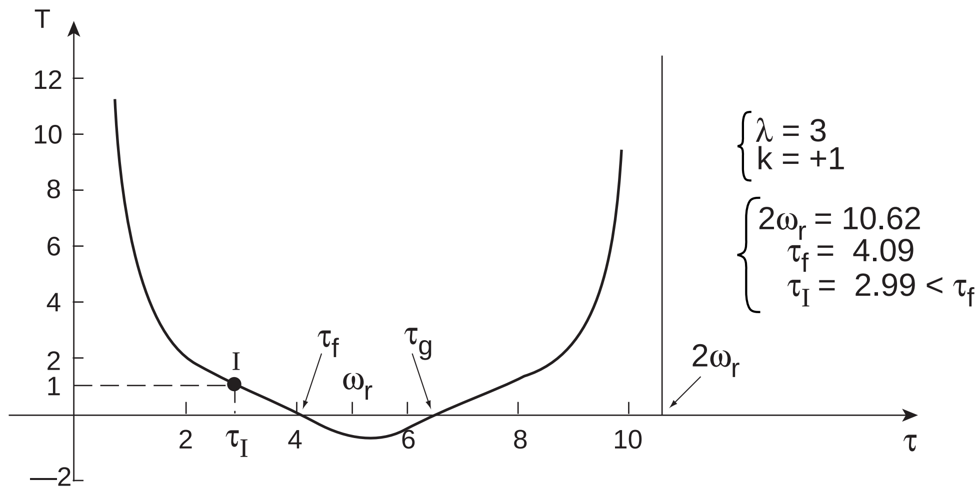

Figure 4: Evolution of the reduced temperature. Closed case.

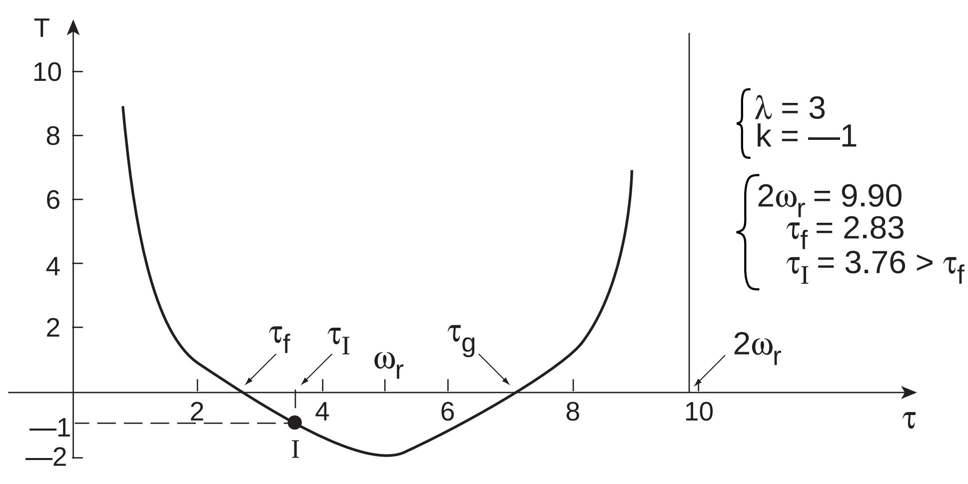

Figure 5: Evolution of the reduced temperature. Open case.

4.3.1 A Simple Procedure

- From experimental results on the present values of the Hubble function H_o and of the temperature $\tilde{T}_o$ of the cosmic background radiation, calculate the radiation density Ω_r^o (use formula 9).
- Improve the experimental bounds on the present values of matter density Ω_m^o *and* cosmological constant density Ω_Λ (so, without assuming that k is zero, of course).
- From the "Cosmic triangle relation" (eq. 4) obtain the present value of the curvature density function Ω_k^o. The sign of this quantity determines also the *opposite* of the constant $k = \pm 1$ (or 0).
- From these results on densities Ω's, compute the dimensionless constant parameters α and λ (use equations 13 and 12).
- From α and λ compute the two Weierstrass invariants g_2 and g_3 that characterize the elliptic curve associated with our universe (use equations 10 and 11).
- The present value of $T_o = T(\tau_0)$ of the reduced temperature is obtained from equation 14.
- If one decides to neglect radiation ($\alpha \simeq 0$), one can plot directly the reduced temperature in terms of the conformal time $T(\tau) = 6[\mathcal{P}(\tau; g_2, g_3) + 1/12]$, where $\mathcal{P}$ is the Weierstrass elliptic function, by using, for instance, Mathematica. The present value τ_o of conformal time is obtained by solving numerically the equation $T_o = 6[\mathcal{P}(\tau_o; g_2, g_3) + 1/12]$. Other interesting values like τ_f (the end of conformal time) or τ_I (inflection point, only interesting if $k = +1$) can be determined numerically; for instance τ_f is obtained, with Mathematica, thanks to the function FindRoot (remember that τ_f is the first positive zero of $T(\tau)$). The value of the half-periof ω_r can be found by evaluation of an elliptic integral but it is simpler to determine it by solving numerically the equation $\frac{dT(\tau)}{d\tau} = 0$ since $\tau = \omega_r$ corresponds to a global minimum. Another possibility, still with Mathematica, is to use the function $WeierstrassHalfPeriods[\{g_2, g_3\}]$ (this function returns a set of two elementary periods, and twice our ω_r coincides with the real part of one of them).
- If one decides to keep $\alpha \neq 0$, one has to use the results of Appendix 1 and determine τ_f and τ_g first; these two real zeros of $T(\tau)$ (within a periodicity cell) are given by the elliptic integrals given in Appendix 1; they have to be evaluated numerically (the value τ_g was just equal to $2\omega_r - \tau_f$ in the case with no radiation, but here, it has to be determined separately). The dimensionless temperature $T(\tau)$ is then given, in terms of the Weierstrass ζ function, by formula 32.
- In all cases, the present value of the deceleration function (a bad name since it is experimentally negative!) q_o is given by equation 15.
- The value of Hubble constant allows one to determine all the dimensional quantities, in particular the critical length parameter Λ_c (formula 17), the cosmological constant Λ $(= \lambda\Lambda_c)$, and the present values of the the cosmic scale (or "radius") R_o (formula 18) and of the cosmic time t_o (formula 19).

4.3.2 An Example

Here we follow the previous procedure assuming fully determined values for the Hubble constant and density parameters Ω_m and Ω_Λ. Of course, such precise values are not yet experimentally available and we have to make a random choice (compatible with

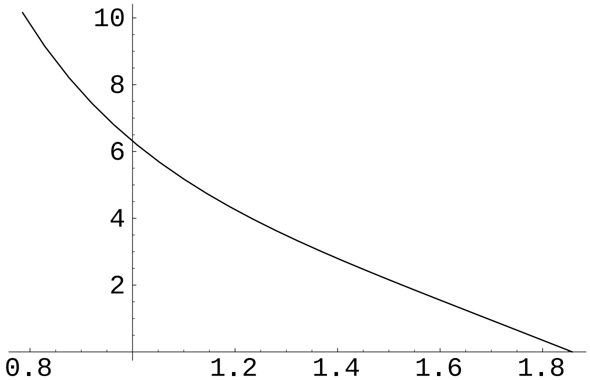

Figure 6: Evolution of the reduced temperature in a neighborhood of τ_o

observational bounds) in order to illustrate the previous “simple procedure” leading to the determination of all cosmological parameters of interest. In other words, what we are describing here is only a possible scenario. For simplicity reasons; we shall neglect the influence of radiation (so $\alpha \simeq 0$).

We take experimental densities $\Omega_m = 0.4$ and $\Omega_\Lambda = 0.8$. Then $\Omega_k = -0.2$ and $k = +1$. The reduced cosmological constant is $\lambda = 108$. The Weierstrass invariants are $g_2 = 1/12$ and $g_3 = -1.00463$. The evolution of the dimensionless temperature is $T(\tau) = 6\mathcal{P}(\tau, \{g_2, g_3\}) + 1/12$ and its present value turns out to be equal to $T_o = 3$; this is a curious coincidence (see section 3.3) since T_o has no reason, *a priori*, to be equal to the temperature of the cosmological black body radiation $\tilde{T}_o$. The present value of the conformal time, obtained by solving numerically the equation $T(\tau_o) = T_o$ is $\tau_o = 1.369$. The end of conformal time, obtained by solving numerically the equation $T(\tau_f) = 0$ is $\tau_f = 1.858$. The (real) half-period ω_r of the function $T(\tau)$ may be obtained by solving numerically the equation $T(\tau)' = 0$ is $\omega_r = 2.662$. The conformal time for which $T(\tau)$ has an inflexion point is obtained by solving numerically the equation $T(\tau)'' = 0$ and is $\tau_I = 1.691$. Notice that in that particular universe, we have $\tau_o < \tau_I$ $(< \tau_f)$, so that we have not reached the inflexion point, yet. Using now the value of the Hubble constant $H_o = (13.25 \times 10^{27})^{-1} cm^{-1}$, we find a characteristic length scale of matter equal to $\Lambda_c = 1.266 \times 10^{-58} cm^{-2}$ so that the cosmological constant itself is equal to $\Lambda = 1.367 \times 10^{-56} cm^{-2}$. Finally we find the cosmic scale $R_o \doteq R(\tau_o) = 2.96 \times 10^{28}\, cm = 3.14 \times 10^{10}\, yrs$ and the cosmic time $t_o = t(\tau_o) = 1.24 \times 10^{28} cm$. Figure 6 gives the behaviour of $T(\tau)$ in a neighborhood of τ_o In this example we neglected the influence of radiation described by the dimensionless parameter α (given by equation 13); using the experimental value of the density parameter $\Omega_r = 5.1 \times 10^{-5}$, one finds $\alpha = 2.77 \times 10^{-5}$. This does not modify T_o but modifies the values of the Weierstrass invariants g_2 and g_3; also, the behaviour of $T(\tau)$ is not the same. Taking this value into account together with the results of Appendix 1 leads to a slight modification of the values of τ_o and therefore of t_o.

5 INFLUENCE OF THE COSMOLOGICAL CONSTANT ON THE REDSHIFT. LARGE SCALE STRUCTURES AND GEOMETRY OF THE UNIVERSE.

5.1 Cosmological Constant Dependence of the Redshift Function

We shall now discuss the redshift, z, as function of τ_o and of the difference

$$\delta = \tau_o - \tau$$

where τ is parameter time (conformal time) τ of the emitter and parameter time, and τ_o the parameter time of the observer. The redshift is given by

$$z = \frac{R(\tau_o)}{R(\tau)} - 1 = \frac{T(\tau)}{T(\tau_o)} - 1$$

Assuming, as before, that we are in the situation $\lambda > 1, \alpha = 0$, which is compatible with recent observations, we obtain immediately, from equation 22, the expression

$$z = C[\mathcal{P}(\tau_o - \delta) + \frac{1}{12}] - 1 \tag{23}$$

with

$$C = [\mathcal{P}(\tau_o) + \frac{1}{12}]^{-1} \tag{24}$$

Remember that the Weierstrass elliptic function $\mathcal{P}$ is characterized by the two Weierstrass invariants $g_2 = \frac{1}{12}$ and $g_3 = \frac{1}{6^3}(k^3 - 2\lambda)$. In order to illustrate the influence of the cosmological constant on the behaviour of the redshift function, we continue our previous example (section 4.3.2), therefore taking $k = +1$ (a spatially closed universe), $\tau_o = 1.3694$, $\lambda = 108$ and plot z as a function of δ (see figure 7). The value $\delta = \tau_o$ (i.e., $\tau = 0$) corresponds to a photon that would have been emitted at the Big Bang whereas $\delta = 0$ (i.e., $\tau = \tau_o$) corresponds to a photon emitted by the observer himself (no redshift). The dashed curve (the lower curve) gives the corresponding redshift when the cosmological constant is absent ($\lambda = 0$). It is clear from these two curves that the influence of the cosmological constant becomes stronger and stronger when the geodesic distance δ increases.

5.2 Observer Dependence of Redshift Values and Large Scale Geometry of the universe

As already announced in the introduction, we now want to consider the problem of comparison of "redshift charts": Given two observers, far apart in the universe, how do we compare the redshift that they will record if they look at the same astronomical object ?

Actually, one may consider (at least) two very different situations. The first is to assume that the two observers perform their measurement at the same time (a meaningful notion since we are in a homogeneous space-time), in that case, it is clear that the photons that they are receiving were not emitted at the same time by the cosmological source that they are studying. The other situation is to assume that both are interested in the same cosmological event (a star belonging to a distant galaxy turning to a supernova, for example) and our two observers record the redshifted light coming from this event when the light reaches them (but in general this light will not reach them at the same time).

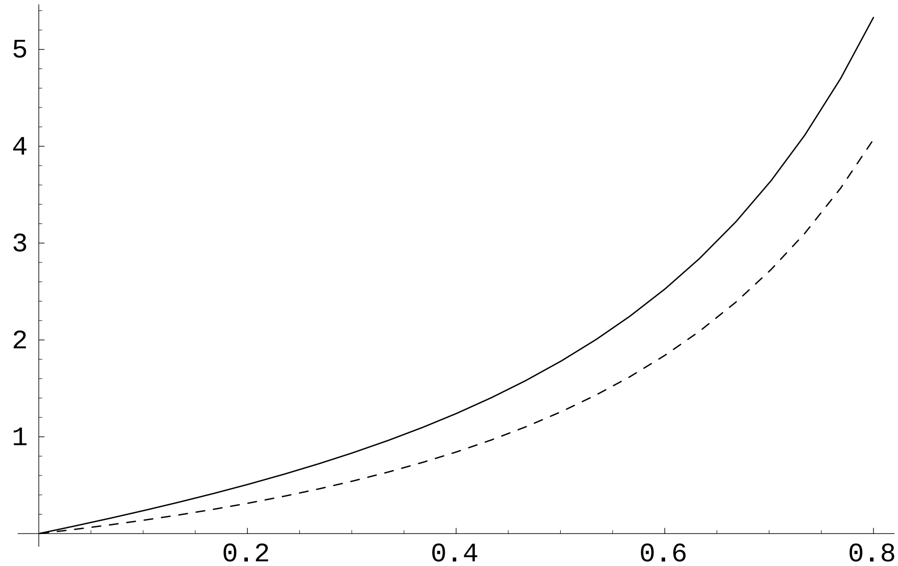

Figure 7: Cosmological constant dependence of the redshift function

5.2.1 Comparison of Measurements Made at the Same Time

We first suppose that both observers perform their observation at the same time. It is natural to specify events by a pair (σ, S) where S is a point in space (Sun for instance) and σ a particular value of the conformal time of the universe. Astronomers on earth (S), nowadays (τ_o), may decide to observe and record the redshift z of an astronomical object X (a quasar, say). At the same time τ_o, astronomers located in the neighborhood of a star P belonging to a distant galaxy may decide to observe the same astronomical object X and record its redshift Z. The problem is to compare z and Z. Of course both z and Z are given by the previous general formula expressing redshift as a function of the time difference between emission and reception, but this value is not the same for S and for P. In order to proceed, we need a brief discussion involving the geometry of our three-dimensional space manifold.

For definitness, we suppose that we are in the closed case ($k = 1$), i.e., the spatial universe is a three-sphere S^3. The reader will have no difficulties to generalize our formulae to the open case (essentially by replacing trigonometric functions by the corresponding hyperbolic ones). The fact that universe expands is taken into account by the evolution of the reduced temperature as a function of conformal time τ and, as far as geometry is concerned, we may analyse the situation on a fixed three-sphere of radius 1. The choice of the point P – we shall call it the "Pole" but it is an arbitrary point – allows one to define a notion of equator and of cosmic latitude: the equator, with respect to P, is the two-sphere (a usual sphere) of maximal radius, centered on P and the cosmic latitude $\ell(S)$ of the Sun S is just the length of the arc of great circle between S and the equator; this great circle is the geodesic going through the two points P and S (that we suppose not antipodal!). The astronomical object X under simultaneous study of P and S is also characterized by a cosmic latitude $\ell(X)$. It is obvious that $\ell(P) = \frac{\pi}{2}$. It is also clear that the (conformal) time difference $\delta = \tau_o - \tau$ between observation (by S at time τ_o) and emission of light (by X at time τ) is nothing else than a measure of the arc of geodesic defined by the two points S and X on the unit three-sphere. We have therefore a "triangle" PXS whose "edges" are arcs of great circles and the lengths of these three arcs are $b = \frac{\pi}{2} - \ell(S)$

(from P to S), $a = \frac{\pi}{2} - \ell(X)$ (from P to X) and $c = \delta$ (from S to X). The last piece of information that we need is a measure α of the angle[1], as seen from Sun (S) between the direction of P and the direction of X. It is a priori clear (draw a triangle !) that there is a relation between $\ell(S)$, $\ell(X)$, δ and α, a relation that generalizes the well known formula $a^2 = b^2 + c^2 - 2bc\cos\alpha$ valid for an arbitrary triangle in euclidean space. Here is the formula that we need:

$$\sin\ell(X) = \cos\delta\sin\ell(S) - \sin\delta\cos\alpha\cos\ell(S) \tag{25}$$

The proof of this formula is given in Appendix 2 and uses the fact the three-sphere S^3 carries a group structure: it can be identified with $SU(2)$ or with the unit sphere in the non-commutative field of quaternions $\mathbb{H}$. The uninterested reader may take the above formula for granted but the technique used in our proof is of independent interest and may be used by the reader to solve problems of similar nature.

The formula 25 gives in particular the (conformal) time difference δ between S and X, when all the points of our spatial universe are characterized in terms of cosmic latitude with respect to an arbitrary reference point P (the pole). The equation giving δ should be solved numerically in general, but notice that the equation simplifies, and give rises to an analytic expression, when we choose X on the equatorial two-sphere defined by P. In that particular case, $\ell(X) = 0$ and

$$\delta = \arctan(\frac{\tan\ell(S)}{\cos(\alpha)}) \tag{26}$$

Coming back to our problem of comparing redshifts, we find that the redshift of the object X, as observed by P at conformal time τ_o, is

$$Z = C[\mathcal{P}(\tau_o - (\pi/2 - \ell(X))) + \frac{1}{12}] - 1$$

with C still given by equation 24, whereas the redshift of the same object X, as observed by S (Sun) at the same time τ_o is given by equation 23, with δ determined by equation 25; as before, $\ell(X)$ and $\ell(S)$ refer to the cosmic latitude of X and S with respect to the reference point P (pole) and α is the angle between the sighting directions SP and SX.

In the particular case of an astronomical object X belonging to the two-sphere which is equatorial with respect to the reference point P, the formulae can be simplified: The redshift of X as, recorded by P is then

$$Z = \frac{[\mathcal{P}(\tau_o - \pi/2) + \frac{1}{12}]}{[\mathcal{P}(\tau_o) + \frac{1}{12}]} - 1 \tag{27}$$

whereas, as recorded by S, it is

$$z = \frac{[\mathcal{P}(\tau_o - \arctan(\frac{\tan\ell(S)}{\cos(\alpha)})) + \frac{1}{12}]}{[\mathcal{P}(\tau_o) + \frac{1}{12}]} - 1 \tag{28}$$

The redshift of equatorial objects X, as seen from P, can be numerically computed from equation 27, and it is a direction-independent quantity. The redshift of the same equatorial objects, as seen from S is clearly direction dependent. The existence of a direction-independent significative gap in the distribution of quasars, as seen from a particular point

[1]This symbol α has of course nothing to do with the one that was previously used to denote the radiation parameter

P of the Universe, around the value Z of the redshift, would certainly be an example of a (remarkable) large scale structure, but such a gap would become direction dependent as seen from another point S (the Sun); the previously given formulae would then be necessary to perform the necessary change of redshift charts allowing one to recognize the existence of these features. [2]

Notice however that, in this particular example (X equatorial with respect to P) we have to suppose $\tau_o > \pi/2$ since, in the opposite case, the light coming from X cannot be recorded by P. Such an hypothesis is not necessarily satisfied and actually is *not* satisfied when we use the values given in our numerical example of section 4.3.2 since we found, in that case, $\tau_o = 1.369 < \pi/2$. In such a situation, if we want to compare measurements between S and P made at the same time, we have to look at an object X which is not equatorial with respect to P but is such that its light can reach both S and P at time τ_o; the first condition (observability from P) gives $\pi/2 - \ell(X) < \tau_o$ i.e., $\ell(X) > \pi/2 - \tau_o$ whereas the second one (observability from S) reads $\delta < \tau_o$, but δ is still given by 25, so that this condition translates as a condition on the angle α between the sighting directions of X and of P as seen from S : this angle should be small enough with a maximal value obtained by replacing δ by τ_o in equation 25 and solving for α.

Another possibility to illustrate the above formulae is to compare the redshifts of X measured at time τ_o from two positions S_1 and S_2 that should have small enough cosmic latitudes since the two conditions on δ_1 and δ_2 (the geodesic distances between X and these two positions) are $\delta_1 < \tau_o$ and $\delta_2 < \tau_o$. To simplify the calculation we suppose that S_1 and S_2 belong to the same meridian (going through P) of our spatial hypersphere, and call α_1 (resp. α_2) the angle made between the sighting directions of X and of P, as seen from S_1 (resp. from S_2). Using the technique explained in Appendix 2 (write $\overline{S_2}X = \overline{S_2}S_1\overline{S_1}X$), the reader will have no difficulty to prove the following formula that generalizes equation 25:

$$\delta_2 = \arccos(\cos(\ell(S_2) - \ell(S_1)) \cos\delta_1 - \sin(\ell(S_2) - \ell(S_1)) \sin\delta_1 \cos\alpha_1 \qquad (29)$$

To simplify even further the calculation, we take X equatorial with respect to the reference point P; therefore the previous inequalities on δ_1 and δ_2, using equation 26, give, in turn, the following two conditions on the cosmic latitudes of the two observers: $\tan\ell(S_1) < \tan\tau_o \cos\alpha_1$ and $\tan\ell(S_2) < \tan\tau_o \cos\alpha_2$. The redshift z_1 of X as measured by S_1 is still given by equation 23 but we can use the particular equation 26, valid for equatorial objects, therefore $z_1 = C[\mathcal{P}(\tau_o - \delta_1) + \frac{1}{12}] - 1$ with $\delta_1 = \arctan(\frac{\tan\ell(S_1)}{\cos(\alpha_1)})$ whereas $z_2 = C[\mathcal{P}(\tau_o - \delta_2) + \frac{1}{12}] - 1$ and δ_2 given by equation 29.

In order to illustrate these results in the study of large scale structures, we choose, for the cosmological parameters, the particular values given in section 4.3.2; furthermore we fix the values of $\ell(S_1)$ (take $\pi/6$) and of α_1 (take $\pi/4$). Then δ_1, is given by 26, and the redshift of X, as measured by S_1 is $z_1 = 3.42$. The redshift z_2 of X, as measured by S_2 is given by the above formula, but since everything else is fixed, it becomes a function of the cosmic latitude $\ell(S_2)$ only. For definitness we choose P, S_1 and S_2 in the same hemisphere. The following figure (8) gives z_2 as a function of $\ell(S_2)$ in the range $[\pi/6 - \pi/12, \pi/6 + \pi/12]$; of course this curve intersects the horizontal line $z_1 = 3.42$ when $\ell(S_2) = \ell(S_1) = \pi/6$.

[2] existence of such a gap was investigated in references [10]

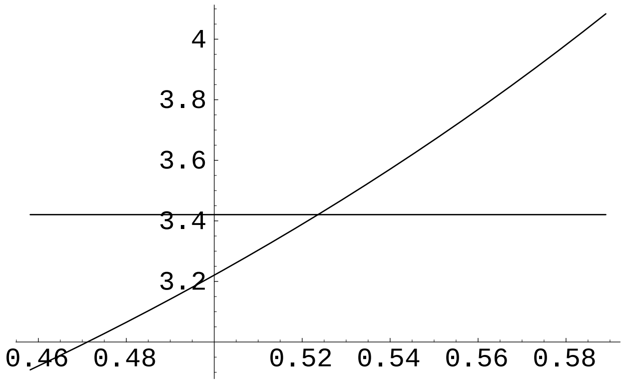

Figure 8: Dependence of the redshift function on the cosmic latitude of an observer

5.2.2 Comparison of Measurements Relative to a Single Event

We now examine the situation where both observers (take P and S) are interested in the same cosmological event (a star belonging to a distant galaxy turning to a supernova, for example) and record the redshifted light coming from this event when the light reaches them (both measurements are not usually performed at the same time). Here we do not have to assume that $\tau_o > \pi/2$, even if X is equatorial with respect to P since the light emitted by X will reach P anyway, in some future. Keeping the same notations as before, the cosmological event of interest took place at conformal time $\tau_o - \delta$, and the light will reach P at time $\tau_o - \delta + \pi/2 - \ell(X)$. The redshift measured by S is, as before, given by equation 23 i.e.,

$$z = \frac{[\mathcal{P}(\tau_o - \delta) + \frac{1}{12}]}{[\mathcal{P}(\tau_o) + \frac{1}{12}]} - 1$$

but the redshift measured by P is

$$Z = \frac{[\mathcal{P}(\tau_o - \delta) + \frac{1}{12}]}{[\mathcal{P}(\tau_o - \delta + \pi/2 - \ell(X)) + \frac{1}{12}]} - 1$$

A The Case With Radiation ($\alpha \neq 0$). Analytic Solution.

General features When $\alpha \neq 0$, the two poles of $T(\tau)$ are distinct (and each of them is of first order). The three roots e_1, e_2, e_3 of the equation $P(y) = 0$ (21) add up to zero. They are given by $-A + B2 \pm i\sqrt{3}2(A - B)$ and $A + B$, where

$$A = 1/2[g_3 + \sqrt{-\Delta/27}]^{1/3} \quad (30)$$
$$B = 1/2[g_3 - \sqrt{-\Delta/27}]^{1/3} \quad (31)$$

and where Δ is the discriminant

$$\Delta = g_2^3 - 27g_3^2 = 3^{-3}\alpha^3\lambda(\lambda - \lambda_+)(\lambda - \lambda_-)$$

with

$$\lambda_\pm = 132\alpha^3(24k^2\alpha^2 + 12k\alpha + 1 \pm (8k\alpha + 1)^{3/2})$$

As $\alpha \to 0$, we have

$$\Delta \to 2^{-4}3^{-3}\lambda(1 - \lambda)$$

indeed, $\lambda_+ \to \infty$ and $\lambda_- \to 1$ and, to first order in α, we have

$$\lambda_+ \simeq 116\alpha^3(1 + 12k\alpha)$$
$$\lambda_- \simeq k(1 - 3\alpha k)$$

If $\Delta = 0$ – something that is nowadays ruled-out experimentally but that used to be called the "physical case" ! – two zeroes of $P(\tau)$ coincide (then also, two zeroes of $Q(\tau)$); in this situation one of the two periods becomes infinite and the elliptic function degenerates to trigonometric or hyperbolic functions. The analytic study, in that situation, is well known.

We know assume that $\alpha \neq 0$ and $\lambda_- < \lambda < \lambda_+$ This is the situation that seems to be in agreement with the recent experiments (see the discussion given previously). For such values of λ, the polynomial $Q(T)$ has two negative roots $T_a < T_b < 0$ (see fig 1). Here $g_3 < 0$ and the discriminant Δ is (strictly) negative, as well. The three roots of the equation $P(y) = 0$ can be written

$$e_1 = a - ib$$
$$e_2 = -2a$$
$$e_3 = a + ib$$

with $a > 0$ and $b > 0$.

In the case $\alpha \neq 0$, it is of course still possible to express T in terms of y (using the fractionally linear transformation 20, with $y = \mathcal{P}(\tau; g_2, g_3)$, the Weierstrass $\mathcal{P}$-function. This is actually not very convenient and it is much better to use either the Weiertrass ζ-function $\zeta(\tau)$ or the Weierstrass σ-function $\sigma(\tau)$.

The function $\zeta(\tau)$ – which is an odd function of τ is not elliptic but it is defined by

$$d\zeta d\tau = -\mathcal{P}(\tau)$$

and the requirement that $1/\tau - \zeta(\tau)$ vanishes at $\tau = 0$. Although ζ is not elliptic, the difference $\zeta(\tau - u) - \zeta(\tau - v)$ is elliptic of order 2, with poles at u and v, when $u \neq v$

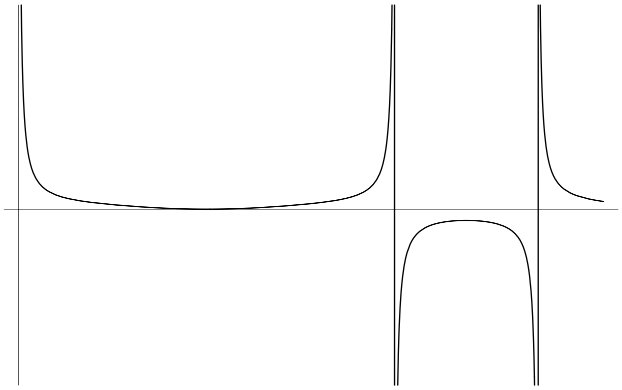

Figure 9: Typical evolution of the reduced temperature (case with radiation)

are two arbitrary complex numbers. In terms of ζ, the quantities $R(\tau)$ and $T(\tau)$ read immediately:

$$\sqrt{\Lambda 3}R(\tau) = \sqrt{\lambda 31}T(\tau) = \zeta(\tau - \tau_g) + \zeta(\tau_g) - \zeta(\tau - \tau_f) - \zeta(\tau_f) \qquad (32)$$

where the two zeros of $T(\tau)$, $0 < \tau_f < \tau_g < 2\omega_r$ are given by (use figure 1) :

$$\tau_f = \int_0^\infty dT \sqrt{Q(T)}$$

$$\tau_g = \tau_f + 2\int_{T_b}^0 dT \sqrt{Q(T)}$$

The quantities T_a and T_b are defined on the graph given on figure 1; they are numerically determined by solving the equation $V_{\alpha k}(T) = \frac{\lambda}{3}$ (for instance use the function FindRoot of Mathematica). The curve $T(\tau)$, in the closed case, has the shape given by figure 9 in full agreement with the qualitative discussion given in section 4; of course, only the first branch of this curve, from the big bang ($\tau = 0$) to the first zero τ_f of $T(\tau)$ (the end of conformal time) is physically relevant. It is instructive to compare this function with with the $\alpha = 0$ case (no radiation), given in figure 4. In both cases, we see that the function $T(\tau)$ is negative between τ_f and τ_g (its next zero), but here, it becomes infinite again for a value $\tau_f + \tau_g$ of conformal time which is strictly smaller than $2\omega_r$; after this singularity, we have a negative branch (a novel feature of the case with radiation) in the interval $]\tau_f + \tau_g, 2\omega_r[$; the period, on the real axis is, as usual, denoted by $2\omega_r$. One shows that $2\omega_r = \tau_f + \tau_g + \tau_c$ where

$$\tau_c = 2\int_{-\infty}^{T_a} dT \sqrt{Q(T)}$$

Such plots were already given in [2] where Weierstrass functions had been numerically calculated from the algorithms obtained in the same reference.

The same reduced temperature can be expressed in terms of the Weierstrass function $\sigma(\tau)$. The even function $\sigma(\tau)$ is not elliptic either and is defined by

$$1\sigma(\tau)d\sigma d\tau = \zeta(\tau)$$

and the requirement that $\sigma(\tau)$ should be an entire function vanishing at $\tau = 0$. Although not elliptic, the quotient $\sigma(\tau - u_1)\sigma(\tau - u_2)\sigma(\tau - u_3)\sigma(\tau - u_4)$ is elliptic of order 2, with poles at u_3, u_4 and zeroes at u_1, u_2, whenever u_1, u_2, u_3, u_4 are complex numbers such that $u_1 + u_2 = u_3 + u_4$. In terms of σ, the quantity $T(\tau)$ reads immediately:

$$T(\tau) = c.\sigma(\tau - \tau_f)\sigma(\tau - \tau_g)\sigma(\tau)\sigma(\tau - \tau_f - \tau_g)$$

where the constant c is given by

$$c = T_b \sigma^2(\tau_g + \tau_f 2)\sigma^2(\tau_g - \tau_f 2)$$

B Geometrical Relations in a Geodesic Triangle

The unit three-sphere S^3 will be here identified with the unit sphere in the non-commutative field of quaternions $\mathbb{H} \simeq \mathbb{R}^4$. To each $X \in S^3 \subset \mathbb{R}^4$, $X = (X_0, X_1, X_2, X_3)$ we associate the quaternion

$$X = X_0 + X_1 i + X_2 j + X_3 k \tag{33}$$

We shall use the non commutative multiplication rules $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. The norm square of X is given by $|X|^2 = \overline{X}X = X_0^2 + X_1^2 + X_2^2 + X_3^2$ where $\overline{X}$ is the quaternionic conjugate of X (with $\overline{i} = -i, \overline{j} = -j, \overline{k} = -k$).

It is convenient to write X in a form analoguous to the representation $X = e^{i\delta} = \cos\delta + i\sin\delta$ of a complex number of unit norm; in this elementary situation, the angle δ is the geodesic distance (arc length) between the unit 1 and the complex X, whereas i (unit norm) can be considered as the tangent (sighting) direction from 1 to X along an arc of geodesic (since i represents the vector $(0,1)$ tangent to the unit circle at the point $(1,0)$). In the three dimensional situation, the unit 1 is a particular point of the three-sphere S^3 and we shall write, in the same way

$$X = e^{\hat{X}\delta} = \cos\delta + \hat{X}\sin\delta \tag{34}$$

where δ is the geodesic distance (arc length) between the unit 1 and the quaternion X, while $\hat{X}$ is a quaternion of square -1 and unit norm representing the three dimensional vector tangent to S^3 at the point 1 in the direction Y (sighting direction). More precisely, let $\hat{X}_\ell$, $\ell = 1, 2, 3$, be this three dimensional unit vector ($\hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 = 1$) and call

$$\hat{X} \doteq \hat{X}_1 i + \hat{X}_2 k + \hat{X}_3 k$$

Then $\hat{X}\hat{X} = -1$ and $\overline{\hat{X}}\hat{X} = +1$. Comparing equations 33 and 34 gives

$$\hat{X}_\ell = \frac{X_\ell}{\sin\delta} \tag{35}$$

and

$$\cos\delta = X_0 \tag{36}$$

Take $\overrightarrow{u}$ and $\overrightarrow{v}$ two three-dimensional vectors (not necessarily of unit norm), call $u = u_1 i + u_2 j + u_3 k$, $v = v_1 i + v_2 j + v_3 k$ the corresponding quaternions and $\hat{u}$, $\hat{v}$ the corresponding normalized unit vectors. An elementary calculation, using the previous multiplicative rules, leads to $uv = -\cos\alpha + \overrightarrow{u} \times \overrightarrow{v}$, but $|\overrightarrow{u} \times \overrightarrow{v}| = |\overrightarrow{u}||\overrightarrow{v}|\sin\alpha$, so that

$$u\,v = |\overrightarrow{u}||\overrightarrow{v}|(-\cos\alpha + \widehat{\overrightarrow{u} \times \overrightarrow{v}}\sin\alpha)$$

where α is the angle between vectors $\overrightarrow{u}$, $\overrightarrow{v}$, and $\widehat{\overrightarrow{u} \times \overrightarrow{v}}$ is the normalized vector product of these two three-dimensional vectors. For normalized vectors, we have in particular $(uv)_0 = -\cos\alpha$, or better,

$$(\overline{u}\,v)_0 = +\cos\alpha \tag{37}$$

Now, let P (a reference point "Pole"), S (Sun) and X (a quasar) three points of $S^3 \subset \mathbb{H} \simeq \mathbb{R}^4$ that we represent by quaternions also denoted by P, S and Y. We call $\ell(S)$, resp. $\ell(X)$, the latitude of S (resp. of X) with respect to P, counted positively whenever P and S (resp. X) are in the same hemisphere, and counted negatively otherwise.

Let $Y \doteq \overline{S}X$. From equations 35 and 36, we find that the geodesic distance δ between X and S (or between 1 and Y) is such that $\cos\delta = Y_0$ so,

$$\delta = arccos(\overline{S}X)_0 \tag{38}$$

and that the sighting direction from S to X is given by a unit vector $\hat{Y}$ with three components

$$\hat{Y}_\ell = \frac{\overline{S}X_\ell}{\sin\delta} \tag{39}$$

We have

$$Y = \overline{S}X = \cos\delta + \hat{Y}\sin\delta = \cos\delta + \widehat{\overline{S}X}\sin\delta$$

Notice that, in the same way,

$$\overline{P}S = \cos(\frac{\pi}{2} - \ell(S)) + \sin(\frac{\pi}{2} - \ell(S))\widehat{\overline{P}S} = \sin(\ell(S)) + \cos(\ell(S))\widehat{\overline{P}S}$$

and

$$\overline{P}X = \sin(\ell(X)) + \cos(\ell(X))\widehat{\overline{P}X}$$

Using the fact that $S\overline{S} = 1$, we remark that

$$\overline{P}X = \overline{P}S\overline{S}X$$

This last equality provides the clue relating the three sides of our geodesic triangle. It implies in particular $(\overline{P}X)_0 = (\overline{P}S\overline{S}X)_0$. The left hand side is given by $(\overline{P}X)_0 = \sin(\ell(X))$ and the right hand side comes from

$$\begin{aligned}
(\overline{P}S)(\overline{S}X) &= (\sin(\ell(S)) + \cos(\ell(S))\widehat{\overline{P}S})(\cos\delta + \sin\delta\,\widehat{\overline{S}X}) \\
&= \sin\ell(S)\,\cos\delta + \cos\ell(S)\,\sin\delta\widehat{\overline{P}S}\,\widehat{\overline{S}X} + \\
&\quad \sin\ell(S)\,\sin\delta\,\widehat{\overline{S}X} + \cos\ell(S)\,\cos\delta\widehat{\overline{P}S} \\
((\overline{P}S)(\overline{S}X))_0 &= \sin\ell(S)\,\cos\delta + \cos\ell(S)\,\sin\delta(\widehat{\overline{P}S}\widehat{\overline{S}X})_0
\end{aligned}$$

By equation 37, we know that the cosine of the angle α between vectors $\overrightarrow{SP}$ and $\overrightarrow{SX}$, respectively represented by the quaternions $\overline{S}P$ and $\overline{S}X$, is given by the 0-component of the quaternionic product $\overline{(\widehat{\overline{S}P})}(\widehat{\overline{S}X}) = (\widehat{\overline{P}S})(\widehat{\overline{S}X})$.

To conclude, one finds the following formula relating the lengths of the three sides of the geodesic triangle SXP, together with the angle α:

$$\sin(\ell(X)) = \sin\ell(S)\,\cos\delta - \cos\ell(S)\,\sin\delta\,\cos\alpha$$

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