CERN-TH/2001-015
January 22, 2001
hep-ph/0101248

Conformal Expansions: A Template for QCD Predictions

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The use of conformal expansions for predictions in quantum chromodynamics is discussed as a way to avoid renormalization scheme and scale ambiguities, as well as factorial growth of perturbative coefficients due to renormalons. Special emphasis is given to the properties of an assumed skeleton expansion and its relation to the Banks-Zaks expansion. The relation of BLM scale-setting to the skeleton expansion is also discussed and new criteria for the applicability of BLM scale-setting are presented.

Presented at the

5th International Symposium on Radiative Corrections
(RADCOR–2000)
Carmel CA, USA, 11–15 September, 2000

1 Introduction

The apparent freedom in choosing the renormalisation scale and scheme for perturbative calculations of observables in quantum chromodynamics (QCD) introduces theoretical uncertainties which, if taken literally, prohibit absolute predictions beyond a qualitative level. The renormalisation scheme dependence can be solved by using a fixed reference scheme or, equivalently, by relating measurements of different observables to each other. The $\overline{\rm MS}$ scheme is the simplest choice from a calculational point of view but the question then arises if there exists a preferred scheme which is optimal from a physics point of view. A closely related question is how to choose the renormalisation scale which is important since most QCD observables are only known to next-to-leading order (NLO) where the renormalisation scale dependence is still sizable.

Another problem with perturbative QCD predictions is that the series is in fact asymptotic, *i.e.* after a given order the higher order contributions start to increase and make the series divergent. The most prominent source for this asymptotic behaviour is due to so-called renormalons which make the higher order coefficients grow factorially [1].

This talk presents an alternative approach which avoids, or at least minimizes, the problems outlined above by using conformal expansions and the closely related skeleton expansion. The presentation is mainly based on [2] which also contains a complete list of references. The relation between the skeleton expansion and the Banks-Zaks expansion [3], as well as the BLM scale-setting method by Brodsky, Lepage, and Mackenzie [4] and its generalizations [5–8], is also discussed.

2 Conformal relations

For definiteness and simplicity the discussion will be limited to single-scale space-like observables in massless QCD, but the approach can also be generalised to time-like and multi-scale observables. The perturbative expansion for such a single-scale observable can be written as,

$$R(Q^2) = R_{\rm QPM}(Q^2) + R_0(Q^2)\frac{\alpha_s(\mu^2)}{\pi} + R_1(Q^2,\mu^2)\frac{\alpha_s^2(\mu^2)}{\pi^2} + R_2(Q^2,\mu^2,\beta_2)\frac{\alpha_s^3(\mu^2)}{\pi^3} + \cdots ,$$

where $Q^2 = -q^2$ is the (space-like) physical scale, μ^2 is the renormalisation scale, and β_2 is the next-to-next-to-leading order coefficient in the renormalisation group equation for the coupling,

$$\frac{da(\mu^2)}{d\log(\mu^2)} = -\beta_0 a^2(\mu^2) - \beta_1 a^3(\mu^2) - \beta_2 a^4(\mu^2) + \cdots .$$

where $a = \alpha_s/\pi$.

The truncation of the perturbative expansion at order N introduces a renormalisation scale and scheme uncertainty of order a^{N+1}. In addition the perturbative coefficients R_n will asymptotically grow factorially due to renormalons, $R_n \sim n!\beta_0^n$. This should be contrasted with the situation in the conformal (scale-invariant) limit where $da/d\log(\mu^2) = 0$. In this case there is no scale-ambiguity, and the coefficients R_n are free of factorial growth due to renormalons. The only remaining problem is the scheme uncertainty which can be circumvented by relating observables to each other instead of trying to make absolute predictions.

Before continuing it is useful to recall the concept of an effective charge [9] which collects all perturbative corrections to an observable. An observable $R(Q^2)$ can then be written in terms of the effective charge $a_R(Q^2)$ as,

$$R(Q^2) = R_{\rm QPM}(Q^2) + R_0(Q^2)a_R(Q^2)$$

where

$$a_R(Q^2) = a(\mu^2) + r_1(Q^2, \mu^2)a^2(\mu^2) + r_2(Q^2, \mu^2, \beta_2)a^3(\mu^2) + \cdots$$

and the perturbative coefficients $r_i = R_i/R_0$.

The most celebrated example of a conformal relation between observables is the Crewther relation [10,11,7] between the Adler D-function (a_D) and the polarized Bjorken sum-rule for deep inelastic scattering (a_{g_1}),

$$(1 + a_D)(1 - a_{g_1}) = 1\,.$$

Thus, the Crewther relation is simply a geometric series to all orders and there is no growth of higher order coefficients. The effective charges a_D and a_{g_1}, which appear in the relation, are defined by,

$$D(Q^2) = Q^2\frac{d\Pi(Q^2)}{dQ^2} \equiv N_C\sum_f e_f^2\left[1 + a_D(Q^2)\right]$$

$$\int_0^1\left[g_1^p(x, Q^2) - g_1^n(x, Q^2)\right]dx \equiv \frac{g_A}{6g_V}\left[1 - a_{g_1}(Q^2)\right]$$

where $\Pi(Q^2)$ is the hadronic correction to the vacuum polarisation of the photon, the spacelike continuation of $R_{e^+e^-}(s)$.

In general, conformal relations between two arbitrary observables A and B can be written as,

$$a_A = \sum_n c_n^{AB} a_B^n$$

where, as is evident, the conformal coefficients c_n^{AB} depend on which two observables that are related. Of course, in real life the QCD coupling is scale-dependent. Even so, the notion of conformal coefficients is still useful as will be shown below. The main advantage is that by identifying the conformal part of the ordinary perturbative coefficients it is possible to treat all the running coupling effects separately and thus keeping the coefficients free from factorial growth due to renormalons which are instead resummed in the running of the coupling.

3 The skeleton expansion

The skeleton expansion [12] organizes the perturbative series in terms of contributions to fundamental skeleton graphs. A skeleton graph is defined by the requirement that the fundamental vertices and propagators contain no substructure. One example of an ordinary Feynman diagram and the corresponding skeleton graph is shown in Fig. 1.

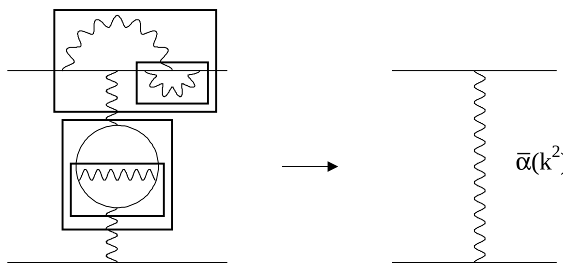

Figure 1: Example of an ordinary Feynman diagram (left) and the corresponding skeleton graph (right) in QED.

In QED, the skeleton expansion is straight-forward to construct thanks to the basic Ward identity, $Z_1 = Z_2$, from which it follows that charge renormalisation is given by photon propagator renormalisation (Z_3). The coupling $\bar{\alpha}$ that appears in the skeleton expansion is the Gell-Mann Low coupling which resums the Dyson series of the one-particle irreducible photon self-energy Π,

$$\bar{\alpha}(Q^2) = \frac{\alpha_0}{1 - \Pi(Q^2)} \, .$$

The radiative corrections to the one-photon exchange skeleton graph, such as illustrated in Fig. 1, can then be written as an integral over the running coupling,

$$\int \bar{a}(k^2) \, \phi_0 \left(\frac{k^2}{Q^2} \right) \frac{dk^2}{k^2}$$

where ϕ_0 is a momentum distribution function which has been normalised to 1 for convenience. (In the above example $\phi_0 \left(\frac{k^2}{Q^2} \right) = \delta(k^2 - Q^2)$.)

Adding the contributions from one-, two-, three-photon exchange etc., an effective charge can be written as

$$
\begin{aligned}
a_R(Q^2) &= \int \bar{a}(k^2)\,\phi_0\left(\frac{k^2}{Q^2}\right)\frac{dk^2}{k^2} + \bar{c}_1 \int \bar{a}(k_1^2)\bar{a}(k_2^2)\,\phi_1\left(\frac{k_1^2}{Q^2},\frac{k_2^2}{Q^2}\right)\frac{dk_1^2}{k_1^2}\frac{dk_2^2}{k_2^2} \\
&+ \bar{c}_2 \int \bar{a}(k_1^2)\bar{a}(k_2^2)\bar{a}(k_3^2)\,\phi_2\left(\frac{k_1^2}{Q^2},\frac{k_2^2}{Q^2},\frac{k_3^2}{Q^2}\right)\frac{dk_1^2}{k_1^2}\frac{dk_2^2}{k_2^2}\frac{dk_3^2}{k_3^2} + \cdots , \qquad (1)
\end{aligned}
$$

where ϕ_i are the momentum distribution functions (normalised to 1) and the $\bar{c}_i$ are the conformal coefficients in the skeleton scheme. For simplicity the above expression has been written including just one skeleton at each order but in general there can be several different skeletons which contribute at the same order. For comparison, the conformal theory gives $a_R(Q^2) = \bar{a} + \bar{c}_1\bar{a}^2 + \bar{c}_2\bar{a}^3 + \cdots$.

Another important property of the skeleton expansion is that each term in the expansion is renormalisation scheme and scale-invariant by itself. In addition the skeleton coupling is gauge-invariant. The skeleton expansion thus provides an alternative way of writing the perturbative series for an observable in which each term is given by one or several integrals over the running coupling. One complication of the skeleton expansion is that in general one needs a diagrammatic construction to identify the different skeletons. However, at low orders this requirement can be bypassed.

In QCD, the existence of an all-order skeleton expansion has so far not been proved. The basic complication arises from the gluon self-interactions and the related difference between gluon-propagator and charge renormalisation. Nevertheless it is reasonable to assume that something similar to the skeleton expansion in QED can also be constructed for QCD. In fact, the so called pinch technique [13] provides a realisation of the skeleton expansion in QCD at the one-loop level. As an example Fig. 2 illustrates how the three-gluon vertex is divided into a pinch part which contributes to the renormalisation of the effective propagator and a non-pinch part which contributes to renormalisation of the "external" vertex.

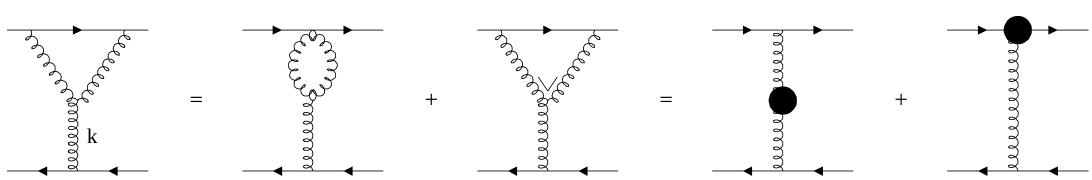

Figure 2: Illustration of the subdivision of the three-gluon vertex into a pinch part and a non-pinch part using the pinch technique

In this way the pinch technique arrives at the following QED-like Ward identities,

$$
\begin{aligned}
Z_1^{(PT)} &= Z_2^{(PT)} = 1 - \frac{1}{\varepsilon}\frac{C_F}{4}\bar{a} \\
Z_3^{(PT)} &= 1 + \frac{1}{\varepsilon}\left(\frac{11}{12}C_A - \frac{1}{3}T_F N_F\right)\bar{a} = 1 + \frac{1}{\varepsilon}\beta_0 \bar{a}
\end{aligned}
$$

such that all the one-loop running coupling effects are contained in the effective gluon propagator. The coupling defined by the pinch technique has a simple relation to the $\overline{\rm MS}$ scheme,

$$
\bar{a}(Q^2) = a_{\overline{\rm MS}}(\mu^2) + \left[-\beta_0\left(\log\frac{Q^2}{\mu^2} - \frac{5}{3}\right) + 1\right] a^2_{\overline{\rm MS}}(\mu^2) + \cdots .
$$

Recently there has been progress in extending the pinch-technique to two loops [14] and this may eventually lead to an extension of the skeleton expansion in QCD to two loops as well. Another possibility may be to use light-front quantization of QCD in light-cone gauge [15].

4 Identifying conformal coefficients

Given the advantages of the skeleton expansion compared to the standard perturbative expansion, it is instructive to consider the following simplified ansatz for QCD as a starting point for further investigations: assume there is only one skeleton coupling, that there is only one skeleton graph at each order in $\bar{a}$, and that the dependence on the number of light flavours (N_F) can be used to identify the non-conformal parts of the perturbative coefficients. Given these assumptions the first conformal coefficients in the skeleton expansion can be obtained from the perturbative ones in the following way [2].

The starting point is the skeleton expansion of an effective charge given by Eq. (1). Next the skeleton couplings $\bar{a}(k^2)$ under the integration sign can be expanded in the coupling $\bar{a}(Q^2)$ using the solution to the renormalisation group equation,

$$
\bar{a}(k^2) = \bar{a}(Q^2) + \beta_0 \log\left(\frac{Q^2}{k^2}\right)\bar{a}^2(Q^2) + \left[\beta_1 \log\left(\frac{Q^2}{k^2}\right) + \beta_0^2 \log^2\left(\frac{Q^2}{k^2}\right)\right]\bar{a}^3(Q^2) + \cdots .
$$

Inserting this into Eq. (1) then gives,

$$
a_R(Q^2) = \bar{a}(Q^2) + \left(\bar{c}_1 + \beta_0\phi_0^{(1)}\right)\bar{a}^2(Q^2) + \left(\bar{c}_2 + \bar{c}_1\beta_0\phi_1^{(1)} + \beta_1\phi_0^{(1)} + \beta_0^2\phi_0^{(2)}\right)\bar{a}^3(Q^2) + \cdots
$$

where $\phi_i^{(n)}$ are log-moments of the momentum distribution functions,

$$
\begin{aligned}
\phi_0^{(n)} &= \int \log^n\left(\frac{Q^2}{k^2}\right)\phi_0\left(\frac{k^2}{Q^2}\right)\frac{dk^2}{k^2} \\
\phi_1^{(1)} &= \int\left(\log\frac{Q^2}{k_1^2} + \log\frac{Q^2}{k_2^2}\right)\phi_1\left(\frac{k_1^2}{Q^2}, \frac{k_2^2}{Q^2}\right)\frac{dk_1^2}{k_1^2}\frac{dk_2^2}{k_2^2} .
\end{aligned}
$$

This can now be directly compared with the standard perturbative expansion,

$$a_R(Q^2) = \bar{a}(Q^2) + \bar{r}_1 \bar{a}^2(Q^2) + \bar{r}_2 \bar{a}^3(Q^2) + \cdots ,$$

which gives the relations

$$\begin{aligned} \bar{r}_1 &= \bar{c}_1 + \beta_0 \phi_0^{(1)} \\ \bar{r}_2 &= \bar{c}_2 + \bar{c}_1 \beta_0 \phi_1^{(1)} + \beta_1 \phi_0^{(1)} + \beta_0^2 \phi_0^{(2)} . \end{aligned}$$

Based on the N_F dependence of the perturbative coefficients $\bar{r}_i$ it is thus possible to identify in a unique way the conformal coefficients $\bar{c}_1$ and $\bar{c}_2$ as well as the log-moments $\phi_0^{(1)}$, $\phi_0^{(2)}$, and $\phi_1^{(1)}$. (This follows since the coefficients $\bar{r}_i$ are polynomials in N_F of order i.) In fact, given the assumptions made, it is possible to decompose the perturbative coefficients up to order $\bar{a}^4$ without any additional information. At higher orders the N_F dependence alone does not provide enough information even with the simplifying assumptions that have been made.

In general there are several ways in which the assumed ansatz can break down. Most notably, at higher orders there are skeletons which are N_F-dependent by themselves. In contrast to QED where the N_F-dependent skeletons (such as the light-by-light scattering diagrams) can be easily identified based on the dependence on the external charge there is in general no such simple identification possible in QCD. Another complication is that there may be more than one skeleton at each order. To resolve these two problems one will need an explicit diagrammatic construction of the skeleton expansion. It may also be the case that the skeleton expansion in QCD can only be systematically extended to all orders by having several skeleton couplings. However, even if some of the assumptions that have been made are wrong, it may still be true that the general properties of the ansatz are valid. This includes the property that running-coupling effects can be associated with different skeleton graphs in a renormalisation-group-invariant way, and that the skeleton coefficients are conformal. In practice there is usually no problem in identifying the skeleton structure at next-to-leading order but special care has to be taken as will be discussed below when the application of BLM scale-setting to the thrust-distribution in e^+e^--annihilation is re-examined.

5 Relation to the Banks-Zaks expansion

As already realised at the time of the discovery of asymptotic freedom, perturbative QCD has an perturbative infrared fixed-point [16] ($k^2 \to 0$),

$$\frac{da_{\text{FP}}(k^2)}{d\ln k^2} = -\beta_0 a_{\text{FP}}^2(k^2) - \beta_1 a_{\text{FP}}^3(k^2) + \cdots = 0$$

in the so called conformal window $8 < N_F < 16$ since for this range of N_F the first two terms in the β-function have opposite signs, $\beta_0 = \frac{11}{4} - \frac{1}{6}N_F > 0$ and $\beta_1 = \frac{51}{8} - \frac{19}{24}N_F < 0$.

If the coupling at the fixed point $a_{\rm FP}$ is small, such that perturbation theory is still applicable, then it can be written as a so called Banks-Zaks expansion [3] in the parameter $a_0 = -\beta_0 / \left.\beta_1\right|_{\beta_0=0} = \frac{16}{107}\beta_0$,

$$a_{\rm FP} = a_0 + v_1 a_0^2 + \cdots ,$$

where the coefficients v_i can be calculated from the higher order terms (β_2 etc.) in the β-function.

In the same way an arbitrary effective charge a_R can also be expanded in a_0. Starting from the ordinary perturbative expansion the coefficients r_i can be rewritten in terms of a_0 using the polynomial N_F-dependence,

$$a_R(Q^2) = a(Q^2) + (r_{1,0} + r_{1,1}a_0)a^2(Q^2) + (r_{2,0} + r_{2,1}a_0 + r_{2,2}a_0^2)a^3(Q^2) + \cdots .$$

From this it follows that it is also possible to get a relation between the fixed-point value of the effective charge $a_R^{\rm FP}$ and the coupling $a_{\rm FP}$. Taking the limit $Q^2 \to 0$ (assuming that this is well defined) and inserting $a_0 = a_{\rm FP} + u_1 a_{\rm FP}^2 + \cdots$ gives the fixed point relation,

$$a_R^{\rm FP} \;=\; a_{\rm FP} + r_{1,0}a_{\rm FP}^2 + (r_{2,0} + r_{1,1})a_{\rm FP}^3 + \cdots .$$

Comparison with the conformal coefficients obtained from the skeleton decomposition of the perturbative coefficients shows that, if $a_{\rm FP}$ is identified with the skeleton coupling then, they are indeed the same, *i.e.* $r_{1,0} = \bar{c}_1$ and $r_{2,0} + r_{1,1} = \bar{c}_2$ etc. Thus, the conformal coefficients in QCD can also obtained from the Banks-Zaks expansion by analytically continuing the number of light quark flavours into the conformal window and taking the infrared limit [2].

6 Connection to BLM scale-setting

Once the conformal coefficients have been identified one also has to evaluate the corresponding skeleton integrals. For the leading skeleton this can be done using the momentum distribution function calculated in the large β_0-approximation. At the same time the associated renormalon ambiguity indicates the form of the non-perturbative corrections in terms of power-corrections. The combination gives a framework for analysing the renormalon resummation and the non-perturbative corrections together [17,18]. An alternative is to approximate the skeleton integrals by using BLM scale-setting [4,5] as will be discussed below.

The starting point is the skeleton expansion of the effective charge in question where each integral is evaluated using the mean value theorem (MVT) in the following way,

$$\begin{aligned} a_R(Q^2) &= \int \bar{a}(\ell^2)\,\phi_0\left(\frac{\ell^2}{Q^2}\right)\frac{d\ell^2}{\ell^2} + \bar{c}_1 \int \bar{a}(\ell_1^2)\bar{a}(\ell_2^2)\,\phi_1\left(\frac{\ell_1^2}{Q^2},\frac{\ell_2^2}{Q^2}\right)\frac{d\ell_1^2}{\ell_1^2}\frac{d\ell_2^2}{\ell_2^2} \\ &+ \bar{c}_2 \int \bar{a}(\ell_1^2)\bar{a}(\ell_2^2)\bar{a}(\ell_3^2)\,\phi_2\left(\frac{\ell_1^2}{Q^2},\frac{\ell_2^2}{Q^2},\frac{\ell_3^2}{Q^2}\right)\frac{d\ell_1^2}{\ell_1^2}\frac{d\ell_2^2}{\ell_2^2}\frac{d\ell_3^2}{\ell_3^2} + \cdots \\ \text{(MVT)} &\equiv \bar{a}(k_0^2) + \bar{c}_1\bar{a}^2(k_1^2) + \bar{c}_2\bar{a}^3(k_2^2) + \cdots \end{aligned}$$

The "BLM" scales k_0, k_1, k_2, etc. are uniquely determined by requiring a one-to-one correspondence between the skeleton integrals and the terms in the "BLM" series [2]. In other words k_0 depends only on ϕ_0, k_1 on ϕ_1, and so on. Thus there is no ambiguity in determining the scales as is the case for commensurate scale relations [6,7]. Expanding the couplings $\bar{a}(k^2)$ in terms of $\bar{a}(Q^2)$ under the integration sign the "BLM" scales are obtained as a perturbative series in the skeleton coupling with the coefficients given in terms of the moments of the distribution functions,

$$\begin{aligned} \ln\frac{Q^2}{k_0^2} &= \underset{\text{mean}}{\phi_0^{(1)}} + \underset{\text{variance}}{\left[\phi_0^{(2)} - \left(\phi_0^{(1)}\right)^2\right]}\beta_0\bar{a}(k_0^2) + \cdots\,, \\ \ln\frac{Q^2}{k_1^2} &= \frac{1}{2}\phi_1^{(1)} + \cdots\,. \end{aligned}$$

It is important to realize that this provides a systematic improvement of the original BLM-scale, $k_{0,\mathrm{BLM}}^2 = Q^2\exp\left(-\phi_0^{(1)}\right)$. In the lowest order approximation the scale k_0 is simply given by the mean of the momentum distribution as indicated above. By going to higher orders one then takes into account the variance of the distribution and so on. This corresponds to performing the skeleton integral with successively improved approximations to ϕ_0.

Given the conformal expansions of two observables in the skeleton scheme it is also possible to eliminate the skeleton scheme and get a direct relation between the two observables – a so called commensurate scale relation (CSR). From renormalisation group transitivity it follows that the coefficients in the commensurate scale relation are also conformal and thus free of factorial growth due to renormalons. However, there is no clear interpretation of the scales that appear in the CSRs, and in addition there is no unique scale setting procedure as has been already mentioned.

7 Re-examining BLM scale-setting for thrust

The new insights gained from the relation between the skeleton expansion and BLM scale-setting makes it interesting to re-examine BLM scale-setting for event shape observables in e^+e^- annihilation [19]. In the following the thrust distribution will be considered as a concrete example but general criteria for the applicability of BLM scale-setting will also be given.

Thrust is an event shape observable defined by,

$$T = \max_{\vec{n}_T} \frac{\sum_i \vec{n}_T \cdot \vec{p}_i}{\sum_i |\vec{p}_i|}$$

where the sum runs over all particles in the final state. The thrust-axis $\vec{n}_T$ is varied until the maximal value for T is obtained. An event with two narrow back-to-back jets corresponds to $T = 1$ whereas the minimal thrust value $T = 0.5$ is obtained for an event with isotropic distribution of particles as illustrated in Fig. 3.

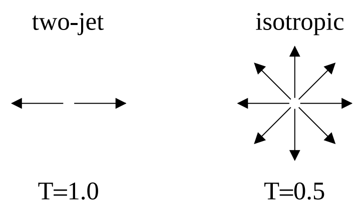

Figure 3: The values of thrust for an event with two narrow back-to-back jets (left) and an event with isotropic distribution of particles (right).

In the quark parton model the thrust distribution is a delta-function at $T = 1$. The leading order QCD-corrections have been calculated analytically [20] whereas the next-to-leading order QCD-corrections have only been calculated numerically [21,22]. From the definition of thrust one expects that at leading order there is only one skeleton which contributes and that all the N_F-dependence at next-to-leading order is from running coupling effects. Thus BLM scale-setting should be straight forward. There is however one possible complication, namely the non-inclusiveness of the definition. The easiest way to see this is that at leading order thrust can have values in the range $2/3 < T < 1$ whereas at next-to-leading order the range is given by $1/\sqrt{3} < T < 1$. Thus, if the next-to-leading order N_F-dependence is non-zero for $1/\sqrt{3} < T < 2/3$ then this cannot be attributed to the leading skeleton. However, as will be shown below the problems for the case of thrust are minimal.

At next-to-leading order the BLM series for the thrust-distribution can be written as,

$$\frac{1}{\sigma}\frac{d\sigma^{\rm BLM}}{dT}(s,T) = \delta(1-T) + R_0(T)\,\bar{a}\Big(k^2_{0,\rm BLM}(s,T)\Big) + \bar{R}_1(T)\,\bar{a}^2\Big(k^2_{0,\rm BLM}(s,T)\Big)\ , \quad (2)$$

where the skeleton coupling has been identified with the pinch technique coupling, $\bar{R}_1(T)$ is the conformal coefficient in the pinch scheme, and the BLM-scale $k^2_{0,\rm BLM}$ is used to approximate the unknown scale k_1^2 which should appear in the $\bar{R}_1(T)$-term. It is important to realize that the BLM-scale $k^2_{0,\mathrm{BLM}}(s,T)$ is a function of both kinematic variables, s and T. In addition the BLM scale is undefined for $T < 2/3$ where R_0 vanishes.

The expansion given above should be compared with the standard $\overline{\mathrm{MS}}$ expansion using $\mu^2 = s$,

$$\frac{1}{\sigma}\frac{d\sigma^{\overline{\mathrm{MS}}}}{dT}(s,T) = \delta(1-T) + R_0(T)a_{\overline{\mathrm{MS}}}(s) + R_{1,\overline{\mathrm{MS}}}(\mu^2 = s, N_F, T)a^2_{\overline{\mathrm{MS}}}(s)\,.$$

The leading order coefficient R_0 is scheme-invariant and thus the same in both expansions. However, the next-to-leading order coefficient R_1 is very different in the two cases as is illustrated in Fig. 4 which shows the conformal coefficient $\bar{R}_1$ compared to the standard $\overline{\mathrm{MS}}$ coefficient $R_{1,\overline{\mathrm{MS}}}(\mu^2 = s, N_F = 5)$ and the leading order coefficient R_0. The coefficients have been calculated numerically using the Beowulf program [22] which is shown as points in the figures. The lines are fits to this points taking into account the know logarithmic parts of the coefficients [25].

From the figure it is clear that the next-to-leading order coefficient is large compared to the leading order one in both cases. However, the conformal coefficient is more stable over a large range of T (when multiplied with $(1-T)$) except for $T \to 1$ where it becomes negative. This is the Sudakov region which can only be properly treated by resumming all singular terms in the Sudakov form-factor. Another important feature which is clear from the figure is that the non-conformal part of R_1 more or less vanishes for $T < 2/3$, which is a good indication that the N_F dependence can indeed be used to separate the conformal and non-conformal parts and that the problems with non-inclusiveness are only minor (see also [23,18,24]). This property is different for other event shape observables depending on how they are defined. For example, oblateness is defined as the difference between an observable that starts at order α_s and one that starts at order α_s^2. As a consequence there are N_F dependent contributions to the next-to-leading term which do not come from the leading skeleton. This could also explain why BLM scale-setting seems to fail for some event shape observables [19].

Fig. 4 also shows the resulting BLM-scale $k_{0,\mathrm{BLM}}(s,T)$ for the case $\sqrt{s} = M_Z$. From the figure it is clear that the scale vanishes as $T \to 1$ which is reasonable since the available phase-space for gluon emission vanishes in this limit. For comparison the figure also shows the approximation $k_{0,\mathrm{BLM}} \simeq 1.4(1-T)\sqrt{s}$ which gives an overall good description of the T-dependence. The scale can also be understood physically as the transverse momentum which approximately scales as $(1-T)\sqrt{s}$ for a three-jet configuration with one of the jets being much less energetic than the other two, *i.e.* in the soft limit. For $T \to 2/3$ the BLM-scale grows rapidly since the $R_0 \to 0$ but even at $T = 0.69$ (the point with the smallest T-value shown in the figure) the BLM-scale is still smaller than $\sqrt{s}$ which should be true in a physical scheme following from the mean value theorem.

For illustration, the fixed order BLM expression for the thrust distribution given by Eq. (2) has been fitted to data from the OPAL collaboration [26] at $\sqrt{s} = M_Z$ in the range $0.70 < T < 0.95$ using a two-loop running coupling. The result of the fit, which is shown together with the data in Fig. 5, corresponds to the value $\alpha_{\overline{\rm MS}}(M_Z^2) = 0.117$. (To translate the fit into a value for $\alpha_{\overline{\rm MS}}(M_Z^2)$ the commensurate scale relation, $\bar{a}(e^{5/3}M_Z^2) = a_{\overline{\rm MS}}(M_Z^2) + a^2_{\overline{\rm MS}}(M_Z^2)$, was used.) For comparison, using the fixed order $\overline{\rm MS}$ expression gives $\alpha_{\overline{\rm MS}}(M_Z^2) = 0.143$. This illustrates the importance of taking running coupling effects into account. However, it should be kept in mind that a complete analysis should also include the Sudakov form-factor and non-perturbative effects.

It is also possible to see the running of the coupling $\bar{\alpha}_s$ as a function of the BLM-scale $k_{0,\rm BLM}$ directly from the data. For each data point Eq. (2) is a simple second order equation which can be solved for $\bar{a} = \bar{\alpha}_s/\pi$. The resulting values of $\bar{\alpha}_s$ obtained in this way are shown in Fig. 5 as a function of the corresponding BLM-scales. (The figure only shows the points that where used in the fit. For larger values of T the next-to-leading order coefficient $\bar{R}_1$ is negative and for smaller values of T the next-to-leading order correction is larger than 100%.) Thus, even though the experiment is done at a fixed energy, it is still possible to observe the running of the coupling.

8 Conclusions

The standard perturbative expansion of observables in QCD is plagued by renormalisation scheme and scale ambiguities as well as higher order coefficients which grow factorially due to renormalons. In this talk I have presented an alternative approach which avoids, or at least minimizes, these problems by using conformal expansions, especially the skeleton expansion.

In contrast to the ordinary perturbative expansion the skeleton expansion is free of renormalisation scheme and scale ambiguities and the coefficients are free of factorial growth due to renormalons. Presently the pinch technique provides a realization of the skeleton expansion in QCD at next-to-leading order but it is not known whether an all-order expansion exists or not. Even so, the skeleton expansion has important phenomenological consequences.

The leading skeleton integral makes it possible to include non-perturbative effects in a consistent way which takes into account the arbitrariness of the definition of perturbation theory. The renormalon ambiguities which appear in the evaluation of the leading skeleton integral can be used to parametrize the non-perturbative contributions in the form of power-corrections.

By making a simple ansatz for the skeleton expansion in QCD the first steps in making a more systematic study of it properties have been taken [2]. One result of this study is that the conformal coefficients coincide with the ones obtained in case QCD has a perturbative infrared fixed-point (the Banks-Zaks expansion).

The skeleton integrals which appear in the skeleton expansion can also be approximated by the BLM-scale setting method and its generalisations. Requiring a one-to-one correspondence between the BLM-scales and the skeleton integrals gives a unique prescription for setting the scales [2] in contrast to the situation for commensurate scale relations. The connection between the skeleton expansion and BLM scale-setting also gives new criteria for the applicability of the latter.

Acknowledgments

I would like to thank Stan Brodsky and Einan Gardi for helpful comments on the manuscript. I would also like to thank the organizers for financial support and for organizing such an inspiring meeting.

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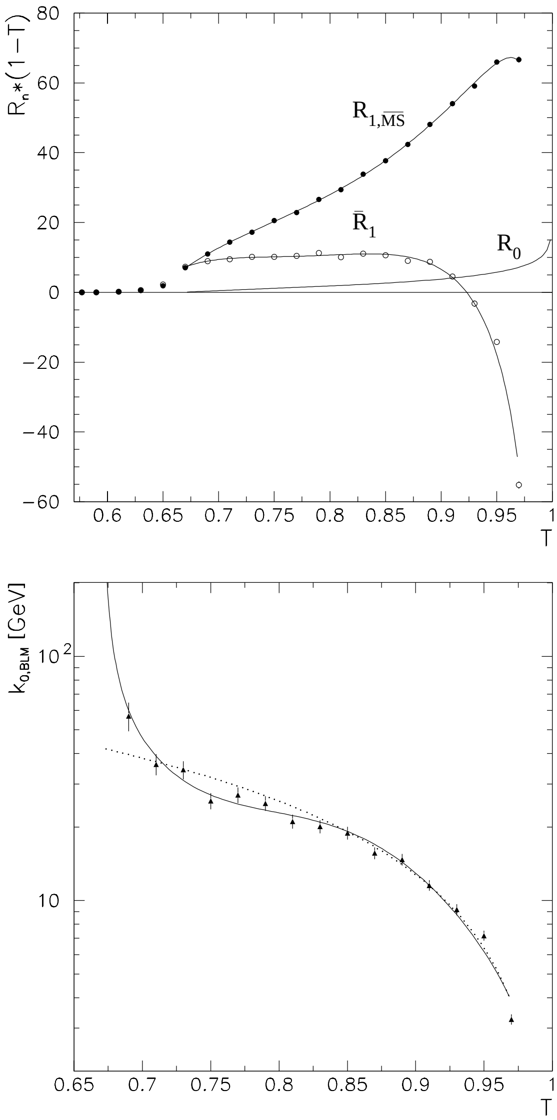

Figure 4: (a) The next-to-leading order conformal coefficient $\bar{R}_1$ compared to the standard $\overline{\rm MS}$ coefficient $R_{1,\overline{\rm MS}}(\mu^2 = s, N_F = 5)$ and the leading order (scheme-invariant) coefficient R_0. (b) The BLM-scale for $\sqrt{s} = M_Z$. For both figures the points show the numerical values that have been calculated and the lines are fits to these points taking into account the known logarithmic terms. In (b) the dotted line show the approximation $k_{0,\rm BLM} \simeq 1.4(1-T)\sqrt{s}$.

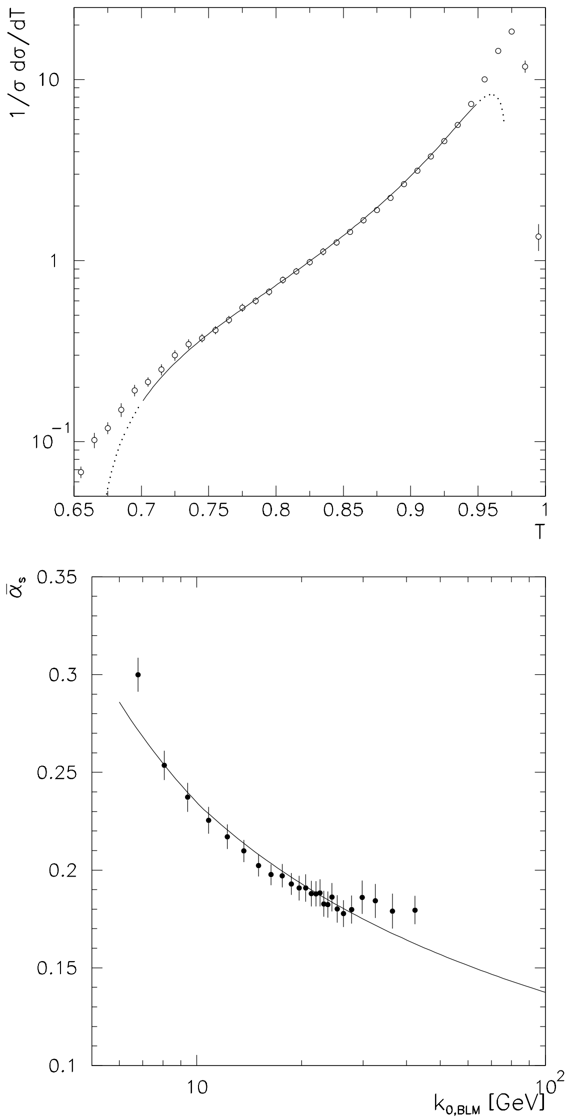

Figure 5: (a) Fit to OPAL data using the fixed order BLM series. The full line corresponds to the range fitted ($0.7 < T < 0.95$). (b) Value of running coupling $\bar{\alpha}_s$ extracted from OPAL data in the range $0.7 < T < 0.95$ at the corresponding BLM-scale.