CERN-TH/2001-009
NYU-TH/01/01/01
January 19, 2001

On the two-loop electroweak amplitude of the muon decay[1]

P.A. Grassi[a] and T. Hurth[b]

(a) Physics Department, New York University ,
4 Washington Place, New York, NY, 10003, USA.

(b) Theory Division, CERN,
CH-1211, Geneva 23, Switzerland.

We present an analysis of the two-loop amplitude of the muon decay in the Standard Model (SM) using algebraic renormalization techniques. In addition, we discuss a manifestly BRST invariant IR regulator for the photon within the SM.

[1]To appear in the Proceedings of the 5th International Symposium on Radiative Corrections (RADCOR-2000) Carmel CA, USA, 11–15 September, 2000. Talk by P. A. G..

1 Introduction

In perturbative multi-loop calculations, the subtraction of UV-divergences in quantum field theory generally leads to Green functions which fail to respect the symmetries of the theory. With the exception of the well-known γ_5 problem, the method of dimensional regularization is compatible with the gauge symmetry but it breaks the supersymmetry. In these cases a practical method is needed to restore the symmetry identities of the gauge symmetry or of the supersymmetry. Here the method of algebraic renormalization [1] supplies a complete solution. However, this method has rarely been used in practical calculations although it has been applied intensively in order to demonstrate the renormalizability of various models.

In a recent papers [2, 3], we reviewed the method of algebraic renormalization from a practical point of view and proposed an algebraic method combining the advantages of the background field method (BFM) and the simplification of (intermediate) Taylor subtractions. The method is independent of the regularization scheme; since the local breaking terms are under control, one can use the most convenient regularization scheme in a specific application. After a straightforward analysis of the corresponding (Ward-Takahashi Identities) WTIs and (Slavnov-Taylor Identities) STIs, the spurious anomalies introduced by a non-invariant regularization scheme were shown to reduce to a few universal breaking terms which depend only on finite Green's functions. The method was already applied to several phenomenologically relevant examples in the SM, such as the two-loop contributions to the processes $H \to \gamma\gamma$, to $B \to X_s\gamma$ and to the three-gauge boson vertices [2, 4, 3].

Because of the experimental precision of standard model observables at LEP (CERN, Geneva), at SLC (SLAC, Stanford) and at TEVATRON (FERMILAB), calculations of quantum corrections on the two-loop level are necessary; and the γ_5 play a critical role here. The purpose of this note is to offer a theoretical analysis of the electroweak two-loop contribution to the muon decay using our algebraic method, and to show its efficiency.

Since a detailed self-contained discussion of the fundamental symmetry constraints for the SM, of the algebraic renormalization procedure in the BFM, and of our specific subtraction method can be found in [2, 3], we restrict ourselves here to reviewing briefly the basic steps of our method (Sec. 2). Then we discuss the muon-decay amplitude, in particular the gauge-invariant subset of two-loop diagrams which is sensitive to the γ_5 problem (Sec. 3). It is well-known that there is a physical infrared divergence present in the muon-decay amplitude. For this purpose, we propose an IR regulator that is manifestly compatible with all the symmetries of the SM (Sec. 4).

2 General Strategy

In the following, we briefly review the main steps elaborated in [2, 3] to renormalize a gauge model with a non-invariant regularization technique. The BFM turns out to be very important for our purposes and, therefore, we quantize the SM in the 't Hooft background gauge [5, 6, 7].

The use of a non-invariant regularization scheme induces breaking terms into the STIs

$$\mathcal{S}\left(\Gamma^{(n)}\right) = \hbar^n \Delta^{(n),S} + \mathcal{O}(\hbar^{n+1})\,, \tag{1}$$

which implement the Becchi-Rouet-Stora-Tyutin (BRST) symmetry, and into the WTIs

$$\mathcal{W}_{(\lambda)}\left(\Gamma^{(n)}\right) = \hbar^n \Delta^{(n),W} + \mathcal{O}(\hbar^{n+1})\,, \tag{2}$$

which implement the background gauge invariance of the SM. The definition of $\mathcal{S}$ and $\mathcal{W}_{(\lambda)}$ is given in [2].

The local breaking terms are denoted by $\Delta^{(n),S}$ and $\Delta^{(n),W}$. Note that the locality is a consequence of the Quantum Action Principle. Here and in the following, $\Gamma^{(n)}$ denotes the n-loop order regularized and (minimally) subtracted one-particle-irreducible (1PI) function. $\Gamma^{(n)}$ includes the renormalization of all subdivergences. The STIs and the WTIs are not able to fix the Green functions completely. Indeed it is possible to add invariant local terms to the action changing the normalization conditions of the Green functions. A complete analysis on the normalization conditions for the SM can, for instance, be found in [7].

Acting on the broken WTIs (2) with the Taylor operator $(1-T^\delta)$ one gets

$$(1-T^\delta)\mathcal{W}_{(\lambda)}\left(\Gamma^{(n)}\right) \;=\; 0\,, \tag{3}$$

where δ has to be chosen in such a way that $(1-T^\delta)\Delta^{(n),S/W} = 0$. After commuting the Taylor operator $(1-T^\delta)$ with $\mathcal{W}_{(\lambda)}$, we obtain

$$\mathcal{W}_{(\lambda)}\left[(1-T^{\delta'})\Gamma^{(n)}\right] \;=\; \left[T^\delta \mathcal{W}_{(\lambda)} - \mathcal{W}_{(\lambda)} T^{\delta'}\right]\Gamma^{(n)} \;\equiv\; \hbar^n \Psi^{(n),W}(\lambda)\,, \tag{4}$$

where δ' is the naive power-counting degree of $\Gamma^{(n)}$. In general, one has $\delta \geq \delta'$, hence the commutation of the Taylor operator with $\mathcal{W}_{(\lambda)}$ leads to over-subtractions of $\Gamma^{(n)}$ and, thus, to the new breaking terms $\Psi^{(n),W}(\lambda)$ occurring on the r.h.s. of Eq. (4) (for more details see [2, 3]). The breaking terms $\Psi^{(n),S}(\lambda)$ for the STIs are defined in the same way. Therefore, the application of the Taylor subtraction on Eqs. (1) and (2) transforms them into

$$\mathcal{S}\left(\hat{\Gamma}^{(n)}\right) = \hbar^n \Psi^{(n),S} + \mathcal{O}(\hbar^{n+1}) \quad \text{and} \quad \mathcal{W}_{(\lambda)}\left(\hat{\Gamma}^{(n)}\right) = \hbar^n \Psi^{(n),S} + \mathcal{O}(\hbar^{n+1})\,, \tag{5}$$

where $\hat{\Gamma}^{(n)} = (1-T^\delta)\Gamma^{(n)}$.

The breaking terms $\Psi^{(n),S}$ and $\Psi^{(n),W}$ can be expressed in terms of a linear combination of ultra-violet (UV) finite Green functions. Here we assumed that up to the $(n-1)$-loop order the Green functions are already correctly renormalized. The main difference between $\Psi^{(n),S}$ and $\Psi^{(n),W}$ is due to the linearity of the corresponding operators $\mathcal{S}$ and $\mathcal{W}_{(\lambda)}$. In the former case one has to consider non-linear terms arising from lower loop orders. On the contrary, the linearity of the WTIs enormously simplifies the evaluation of the breaking terms and of counterterms.

Finally, we introduce

$$\mathrm{I\!\Gamma}^{(n)} = \hat{\Gamma}^{(n)} - \Xi^{(n)} = (1 - T^\delta)\Gamma^{(n)} - \Xi^{(n)} , \tag{6}$$

where $\Xi^{(n)}$ is chosen in such a way that the following identities are fulfilled:

$$\mathcal{S}\left(\mathrm{I\!\Gamma}^{(n)}\right) = 0 , \qquad \mathcal{W}_{(\lambda)}\left(\mathrm{I\!\Gamma}^{(n)}\right) = 0 . \tag{7}$$

In general, it is quite simple to compute the counterterm, $\Gamma^{C.T.} = T^\delta \Gamma^{(n)} + \Xi^{(n)}$, as it can be expressed in terms of Green functions expanded around zero external momenta.

As already mentioned above, there is still the freedom to add invariant counterterms. In other words, we have the freedom to impose normalization conditions that lead in addition to Eqs. (7) to the following equation being fulfilled:

$$\mathcal{N}_i\left(\mathrm{I\!\Gamma}^{(n)}\right) = 0 , \tag{8}$$

where the index i runs over all independent parameters of the SM. As the Green function $\Gamma^{(n)}$ also has to fulfill this condition we have for the counterterm

$$\mathcal{N}_i\left(T^\delta \Gamma^{(n)} + \Xi^{(n)}\right) = 0 , \tag{9}$$

which is a local equation. This means that, whenever the effort to impose the normalization conditions is done the changes due to the subtraction are only a local changes which can be easily compensated. For clarity let us consider an example: for the condition on the W boson mass we could choose $\mathcal{N}_1(\Gamma^{(n)}_{\hat{W}^+\hat{W}^-}) = \Gamma^{(n),T}_{\hat{W}^+\hat{W}^-}(p^*) = 0$ where the superscript T stands for the transverse part and $\mathrm{Re}(p^*) = M_W$. Notice that the imposition of normalization conditions is a very important ingredient for the computation in order to compare with other schemes and in order to simplify the breaking terms themselves.

The procedure described so far is heavily based on the Taylor operator T^δ. In the presence of massless particles this may introduce infra-red (IR) divergences. In [4] we presented a modified procedure which resolves this spurious IR problem generally.

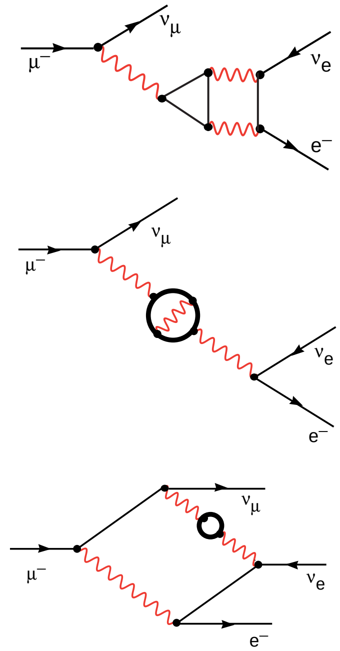

Figure 1: Example of $O(N_f\alpha^2)$ contributions to the muon decay amplitude with a two-loop three-point, two-loop two-point function and with box contribution

3 Muon Decay Amplitude

3.1 General settings

- *Muon decay amplitude*

  We want to focus on the $O(N_f\alpha^2)$ contributions including the two-loop three-point functions $\Gamma_{\hat{W}^+_\mu \overline{\nu} e}(p_\nu, p_e)$ * with a $\hat{W}$ and an electron (muon) and electron-(muon-) neutrino. This subgroup of $O(N_f\alpha^2)$ contributions to the muon decay amplitude is the most delicate one regarding the γ_5-problem. An example is given in Fig. 1.

  There are further contributions at the two-loop level such as the diagrams including the two-loop-two-point function $\Gamma_{\hat{W}^+_\mu \hat{W}^-_\nu}(p)$ (Fig. 1). These have already been discussed within our approach in [4], but there are no problems with γ_5

*All momenta are considered as incoming. In the Green functions $\Gamma_{\phi_1 \ldots \phi_n}$ they are assigned to the corresponding fields starting from the right. The momentum of the most left field is determined via momentum conservation. $\hat{W}$ denotes the background field corresponding to the quantum field W.

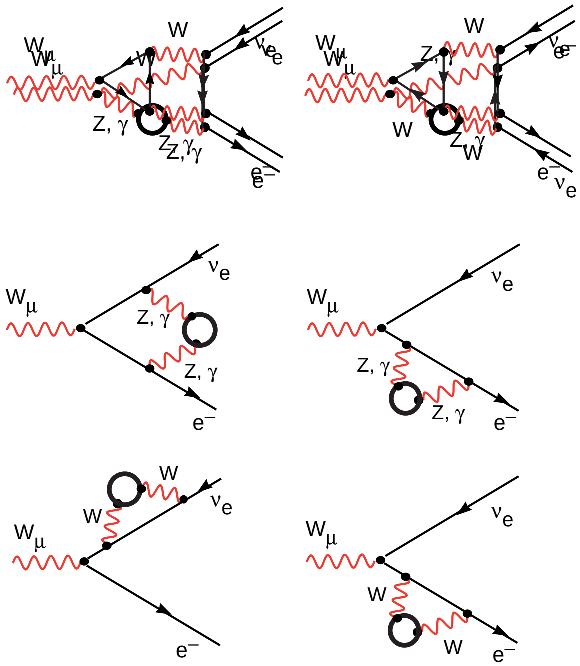

Figure 2: Gauge-invariant subset of $O(N_f\alpha^2)$ contributions to the muon decay including vertex and external leg corrections.

there. Moreover, there are two-loop box diagrams with a gauge-boson self-energy inside (as shown in the last diagram of Fig. 1).

Thus, let us focus on contributions like the one shown in the first picture of Fig. 1. We have to consider the complete gauge-invariant subset of two-loop contributions to the three-point function $\Gamma_{\hat{W}^+_\mu \overline{\nu} e}(p_\nu, p_e)$. Actually, there are various types of diagrams shown in Fig. 2. Here we note that only the first two diagrams in Fig. 2 change when switching from a conventional gauge to the 't Hooft-background gauge.

- *IR problems*

  Among the $O(N_f\alpha^2)$ gauge-invariant subset of diagrams (see Fig. (2)), one has to consider those with a virtual photon. Those diagrams are potentially IR divergent and only the $O(N_f\alpha^2)$ contributions to the physical amplitude, after the inclusion of the Bremsstrahlung radiation of the external electron and muon, turn out to be finite.

  Historically [8], the virtual photon contributions, namely the pure QED corrections, are analysed separately from the electroweak corrections by using a suitable decomposition of massless propagators. In that spirit, the pure QED corrections have been computed in [9, 10, 11] and the remaining complete electroweak corrections are IR finite. In paper [12] the $O(N_f \alpha^2)$ corrections are computed in the MS scheme and the massless quark approximation has been used. In [13, 14] the exact fermionic contributions, in on-shell scheme, are taken into account. However, since we are not interested in the explicit evaluation of the muon amplitude, but only to present a procedure to handle the γ_5 problem in the present process, we will not disentangle the QED corrections in our considerations.

As a consequence, we have to keep the possible IR divergences under control, namely we have to be sure that all the steps of the computation are IR regulated. The situation is worsened by the fact that, according to our procedure, the Taylor expansion in Eqs. (4) is performed at zero momentum. For those purposes, we propose a BRST invariant IR regularization for the photon within the SM (see Sec. 4). This method regulates both physical and spurious IR divergences simultaneously.

An alternative approach to IR problems is the following: regarding γ_5, the delicate diagram is shown in the first picture of Fig. (1) which belong to the pure vertex corrections $\Gamma_{\hat{W}^+_\mu \overline{\nu} e}(p_\nu, p_e)$. Luckily, this vertex Green function is the simplest one – compared to box and external-leg corrections – from the IR perspective. This is because its physical IR singularities are induced only by the on-shell wave-function renormalization.

Therefore, we can also avoid the physical IR problems by choosing an off-shell renormalization procedure. There are two ways of doing this: *i)* either one imposes an on-shell renormalization for the neutrino and, as a consequence of WTIs, an off-shell renormalization prescription of the electron is automatically provided (see next section), *ii)* or one can also choose a MS wave-function renormalization for external fermions which is infrared finite and compatible with the background gauge invariance.

Finally, to handle the spurious IR problems generated by means of the Taylor subtraction, a modification of the procedure is discussed in [4]. Here, the modified breaking terms $\Psi^{(2),W}$, occurring in the WTI for the Green function $\Gamma_{\hat{W}^+_\mu \overline{\nu} e}(p_\nu, p_e)$, is written explicitly.

- *Kinematic approximations*

  The specific kinematic situation allows for some simplifications: in $\Gamma_{\hat{W}^+_\mu \overline{\nu} e}(p_\nu, p_e)$ the W is off-shell, while both the electron and the electron-neutrino are on-shell. We can make the approximation $p_W^2 = 0$ and neglect m_{muon}/m_W-terms, because the muon is almost at rest. All momenta squared are zero, and hence

$$p_W = p_\nu = p_e = 0\,. \tag{10}$$

In the following we will derive all symmetry constraints for the general kinematic case and then specify to the zero-momentum setting (10).

- *Subdivergences*

  Considering only the $O(N_f\alpha^2)$ contributions to the two-loop three-point vertex function $\Gamma^{(2)}_{\hat{W}^+_\mu \overline{\nu}e}(p_\nu, p_e)$, we have to take into account three kinds of one-loop subdivergences: the three-gauge boson vertices with one background and two quantum fields (they have been largely discussed in [4] within the BFM framework), the quantum gauge boson self-energies (they have been analysed in [2, 3] without and with the BFM; in particular, in [3] the conversion from background field amplitudes to quantum ones is completely exploited) and the one-loop $\Gamma^{(1)}_{\hat{W}^+\overline{\nu}e}(p_\nu, p_e)$ amplitude together with their corresponding scalar vertices where the gauge boson is replaced by the Goldstone boson (notice that this amplitude appears also as subdivergence for two-loop three-gauge boson vertices and it is extensively discussed in [4]).

  Moreover, the renormalization of one-loop amplitudes can be quite easily handled within different regularization techniques, therefore we can assume, for the time being, that the one-loop Green functions already satisfy the WTIs (or the STIs) and fulfill certain normalization conditions. Nevertheless, to apply our procedure, we have to compute the one-loop counterterms which must be inserted in one-loop graphs. By using the notation of the introduction, these counterterms are given by

$$\begin{aligned}
\mathrm{I\!\Gamma}^{(n)} &= \hat{\Gamma}^{(n)} - \Xi^{(n)} = \Gamma^{(n)} - \left[T^\delta \Gamma^{(n)} + \ \Xi^{(n)}\right] \\
&= \Gamma^{(n)}_{\text{bare}} - \Gamma^{(n)}_{\text{UV}} - \left[T^\delta \Gamma^{(n)}_{\text{bare}} \ + T^\delta \Gamma^{(n)}_{\text{UV}} + \Xi^{(n)}\right].
\end{aligned} \tag{11}$$

  In the second line we have introduced the bare Green function $\Gamma^{(n)}_{\text{bare}}$ in addition. This quantity is defined by $\Gamma^{(n)} = \Gamma^{(n)}_{\text{bare}} - \Gamma^{(n)}_{\text{UV}}$ where $\Gamma^{(n)}_{\text{UV}}$ denotes the necessary UV counterterms computed in the specific regularization used in the calculation. Of course, the complete one-loop counterterms, namely $\mathrm{I\!\Gamma}^{(n)} - \Gamma^{(n)}_{\text{bare}}$, have to be taken into account at the two-loop level.

  For instance, in the case of charged gauge-boson self-energies, given $\mathrm{I\!\Gamma}^{(1)}_{W^+_\mu W^-_\nu}(p)$ (which satisfy the normalization conditions and the corresponding STIs) and given $\Gamma^{(1)}_{W^+_\mu W^-_\nu}(p)$, computed in the same regularization as will be used in the two-loop computation of $\Gamma^{(2)}_{\hat{W}^+_\mu \overline{\nu} e}(p_\nu, p_e)$, the counterterms are

$$\begin{aligned}
\Gamma^{(1),C.T.}_{W^+_\mu W^-_\nu}(p) &= T^2_p\left(\Gamma^{(1)}_{W^+_\mu W^-_\nu}(p)\right) + \Xi^{(1)}_{W^+_\mu W^-_\nu}(p)\,, \\
\Xi^{(1)}_{W^+_\mu W^-_\nu}(p) &= \xi^{(1)}_{W,1}\, p^2 g_{\mu\nu} + \xi^{(1)}_{W,2}\, p_\mu p_\nu + \xi^{(1)}_{M_W}\, g_{\mu\nu}\,,
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\xi^{(1)}_{W,1} &= \frac{1}{144}\left(5\,\partial^2_p\, \mathrm{I\!\Gamma}^{(1)}_{W^+_\mu W^-_\mu}(p)\Big|_{p=0} - 2\,\partial_{p^\mu}\partial_{p^\nu}\, \mathrm{I\!\Gamma}^{(1)}_{W^+_\mu W^-_\nu}(p)\Big|_{p=0}\right), \\
\xi^{(1)}_{W,2} &= \frac{1}{72}\left(-\partial^2_p\, \mathrm{I\!\Gamma}^{(1)}_{W^+_\mu W^-_\mu}(p)\Big|_{p=0} + 4\,\partial_{p^\mu}\partial_{p^\nu}\, \mathrm{I\!\Gamma}^{(1)}_{W^+_\mu W^-_\nu}(p)\Big|_{p=0}\right), \\
\xi^{(1)}_{M_W} &= \frac{1}{4}\, \mathrm{I\!\Gamma}^{(1)}_{W^+_\mu W^-_\mu}(p)\Big|_{p=0}\,.
\end{aligned} \tag{13}$$

In the same way, all the other possible one-loop divergences can be computed once the renormalized Green functions $\mathrm{I\!\Gamma}^{(1)}$ are known.

In some cases, the BFM does not achieve great advantages (for instance, in the cases of amplitudes with external fermions only) at the practical level and, on the other hand, it could be convenient to use the conventional gauge fixing. However, the Green functions computed with external background fields can be easily related to those with external quantum fields by using the extended versions of the BRST symmetry.

3.2 Two-loop vertex function

Working in the framework of the BFM, there is only one WTI for the vertex function $\Gamma^{(2)}_{\hat{W}^+_\mu \overline{\nu} e}(p_\nu, p_e)$ that has to be evaluated at two loops (cf. [2]):

$$\begin{aligned}
& i\,(p_\nu + p_e)_\rho\, \Gamma^{(2)}_{\hat{W}^+_\rho \overline{\nu} e}(p_\nu, p_e) + i\, M_W \Gamma^{(2)}_{\hat{G}^+ \overline{\nu} e}(p_\nu, p_e) \\
& \qquad + \frac{ie}{s_W\sqrt{2}}\left[\Gamma^{(2)}_{\overline{\nu}\nu}(-p_\nu)P_L - P_R \Gamma^{(2)}_{\overline{e} e}(p_e)\right] = \Delta^{(2),W}_{\lambda_+ \overline{\nu} e}(p_\nu, p_e)\,.
\end{aligned} \tag{14}$$

Here $\Delta^{(2),W}_{\lambda_+ \overline{\nu} e}(p_\nu, p_e)$ is a polynomial of the external momenta p_ν and p_e of maximum degree 1. We define the weak mixing angle through the on-shell relation $c_W = M_W/M_Z$ as we want to maintain the form of the WTIs to be the same to all orders. $P_{L/R} = (1 \mp \gamma_5)/2$ are the chiral projectors.

The breaking terms in (14) are generated by a non-invariant regularization procedure, for instance by using the 't Hooft-Veltman definition of γ_5 [15]. To remove them, according to the procedure described in [2, 4], we apply the Taylor operator $(1 - T^1_{p_\nu,p_e})$, obtaining

$$
\begin{aligned}
& i\,(p_\nu + p_e)_\rho \left[\left(1 - T^0_{p_\nu,p_e}\right)\Gamma^{(2)}_{\hat{W}^+_\rho \overline{p_\nu} p_e}(p_\nu, p_e)\right] + i\,M_W\left[\left(1 - T^0_{p_\nu,p_e}\right)\Gamma^{(2)}_{\hat{G}^+ \overline{\nu} e}(p_\nu, p_e)\right] \\
& \quad + \frac{ie}{s_W\sqrt{2}}\left\{\left[\left(1 - T^1_{p_\nu}\right)\Gamma^{(2)}_{\overline{\nu}\nu}(-p_\nu)\right]P_L - P_R\left[\left(1 - T^1_{p_e}\right)\Gamma^{(2)}_{\overline{e}e}(p_e)\right]\right\} = \Psi^{(2),W}_{\lambda_+ \overline{\nu} e}(p_\nu, p_e)\,.
\end{aligned}
\tag{15}
$$

where

$$
\Psi^{(2),W}_{\lambda_+ \overline{\nu} e}(p_\nu, p_e) = i\,M_W\left(p^\rho_\nu \partial_{p^\rho_\nu} + p^\rho_e \partial_{p^\rho_e}\right)\Gamma^{(2)}_{\hat{G}^+ \overline{\nu} p_e}(p_\nu, p_e)\Big|_{p_\nu = p_e = 0}\,, \tag{16}
$$

are finite and are generated by means of the over-subtraction.

Notice that the computation of $\Psi^{(2),W}_{\lambda_+ \overline{\nu} e}$ can be also performed without encountering any UV divergences. For that purpose, it is sufficient to implement the subtraction of UV subdivergences directly in such a way that all the diagrams are always finite. Technically, we suggest the use of Zimmermann's subtraction formula [16]. Since all the integrals involved can be performed analytically, $\Psi^{(2),W}_{\lambda_+ \overline{\nu} e}$ can be evaluated exactly. In this way, we use the Dirac algebra in 4 dimensions without any ambiguities.

Notice that the zero momentum subtraction in Eq. (15) removes exactly the contribution that we would like to evaluate to compute the muon decay amplitude in the approximation (10). However, this contribution can be computed as a counterterm. Notice in fact that, the local part of the muon decay amplitude is totally fixed when the normalization conditions for fermion two-point functions and the WTIs are used.

By using the parametrization

$$
\begin{aligned}
\Xi^{(2)}_{\overline{\psi}\psi}(p) &= \xi^{(2)}_{2,\psi}\,(\not{p} - m_\psi) + \xi^{(2)}_{\psi} m_\psi\,, \qquad \psi = \nu, e \\
\Xi^{(2)}_{\hat{W}^+_\mu \overline{\nu} e}(p_\nu, p_e) &= \xi^{(2)}_{L,\hat{W}^+ \overline{\nu} e}\,\gamma^\mu P_L + \xi^{(2)}_{R,\hat{W}^+ \overline{\nu} e}\,\gamma^\mu P_R\,,
\end{aligned}
\tag{17}
$$

for two- and three-point functions, and decomposing the breaking terms into scalar functions

$$
\Psi^{(2),W}_{\lambda_+ \overline{\nu} e}(p_\nu, p_e) = i\left(\psi^{(2)}_1\ \not{p}_\nu P_L + \psi^{(2)}_2\ \not{p}_\nu P_R + \psi^{(2)}_3\ \not{p}_e P_L + \psi^{(2)}_4\ \not{p}_e P_R\right)\,, \tag{18}
$$

we have the solution

$$
\begin{aligned}
\xi^{(2)}_{R,\hat{W}^+ \overline{\nu} e} &= \psi^{(2)}_2 = \psi^{(2)}_4\,, \\
\xi^{(2)}_{L,\hat{W}^+ \overline{\nu} e} &= \frac{e}{s_W\sqrt{2}}\,\xi^{(2)}_{2,\nu} + \psi^{(2)}_1\,, \\
\xi^{(2)}_{2,e} &= \xi^{(2)}_{2,\nu} + \frac{s_W\sqrt{2}}{e}\left(\psi^{(2)}_1 - \psi^{(2)}_3\right)\,,
\end{aligned}
\tag{19}
$$

for the coefficients. Notice that the equality $\psi_2^{(2)} = \psi_4^{(2)}$ follows from the consistency conditions (see, for example, the discussion in [2]) and it provides a check of the computation of the breaking terms. In addition, from Eqs. (19), it emerges that $\xi_{2,e}^{(2)} = \xi_{2,\nu}^{(2)}$ in the case of invariant regularization techniques, namely when $\psi_i^{(2)} = 0, \ \forall i$. This means that we are not allowed to impose any arbitrary normalization conditions for fermion residues. If the neutrino is renormalized in such a way that its residues is equal to 1, the electron residue will be clearly different from 1. In this way, we have a partial on-shell scheme, we maintain the background gauge symmetry and we can avoid the physical IR divergences for the vertex amplitude.

The final result, namely diagram computation plus counterterms, can be written in the following way

$$\begin{aligned}\mathrm{I\Gamma}^{(2)}_{\hat{W}^+_\mu \overline{\nu} e}(p_\nu, p_e) &= \Gamma^{(2)}_{\hat{W}^+_\mu \overline{\nu} e}(p_\nu, p_e) \\ &\quad - \left[T^0_{p_\nu, p_e} \Gamma^{(2)}_{\hat{W}^+_\mu \overline{\nu} e}(p_\nu, p_e) + \xi^{(2)}_{L,\hat{W}^+\overline{\nu} e} \gamma^\mu P_L + \xi^{(2)}_{R,\hat{W}^+\overline{\nu} e} \gamma^\mu P_R \right] . \end{aligned} \tag{20}$$

The parameters $\xi_\nu^{(2)}$ and $\xi_e^{(2)}$ are used to impose mass renormalization conditions on the fermion self-energies.

From this we learn that the symmetric amplitude $\mathrm{I\Gamma}^{(2)}_{\hat{W}^+_\mu \overline{\nu} e}$ at zero momentum is just given by the universal counterterm:

$$\mathrm{I\Gamma}^{(2)}_{\hat{W}^+_\mu \overline{\nu} e}(p_\nu = 0, p_e = 0) = -\xi^{(2)}_{L,\hat{W}^+\overline{\nu} e} \gamma^\mu P_L - \xi^{(2)}_{R,\hat{W}^+\overline{\nu} e} \gamma^\mu P_R \, . \tag{21}$$

The proposed procedure allows for an efficient computation of the amplitude $\mathrm{I\Gamma}^{(2)}_{\hat{W}^+_\mu \overline{\nu} e}$ at two-loop order avoiding the γ_5 problem. In the literature, different techniques with different prescription of γ_5 have been used to evaluate the Feynman diagrams of $\mathrm{I\Gamma}^{(2)}_{\hat{W}^+_\mu \overline{\nu} e}$, however we believe that a rigorous check of these result is desirable.

4 Massive U(1) BRST symmetry within the SM

It is a well-known problem that in the computation of Green's function there are IR divergences due to the vanishing photon mass. In this brief section, we will describe how to perform a regularization of the photonic IR divergences in a way that is consistent with the BRST symmetry of the SM and, thus, preserves the unitarity of the model (see also [18]). The choice of such a regulator is motivated essentially by the fact that it regulates both the physical and spurious IR divergences.

We will refer to the Stueckelberg method (see [19, 20] and references therein). This method gives rise to unsolvable renormalization problems in the case of Yang-Mills fields, which only can only be resolved through the Higgs mechanism. However, in the abelian case it provides a manifestly BRST (and the background gauge) invariant model of massive QED.

For pedagogical purposes, we present a short digression regarding the Stueckelberg formalism in QED. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4g^2}{F_{\mu\nu}}^2 + \frac{m^2}{2g^2}(A_\mu - \frac{1}{m}\partial_\mu\varphi)^2 + s(\overline{c}\mathcal{F}) + \mathcal{L}_{\rm matter}\,,$$
$$\mathcal{L}_{\rm matter} = \overline{\Psi}(i\,\partial\!\!\!/ - M)\Psi + \overline{\Psi}\gamma^\mu(A_\mu - \frac{1}{m}\partial_\mu\varphi)\Psi\,, \tag{22}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The apparent non-renormalizable derivative coupling $\overline{\Psi}\gamma^\mu\partial_\mu\varphi\Psi$ can be absorbed by a field redefinition

$$\psi = e^{\frac{i}{m}\varphi}\Psi, \qquad \mathcal{L}_{matter} = \overline{\psi}(\gamma^\mu(i\partial_\mu + A_\mu) - M)\psi. \tag{23}$$

Notice that in the non-abelian case a redefinition of fields of type (23) will generate new non-eliminable non-renormalizable terms [20].

The BRST transformations of the fundamental fields are given by

$$\begin{aligned} s\hat{A}_\mu &= 0\,, \quad & \hat{A}_\mu &= A_\mu - \frac{1}{m}\partial_\mu\varphi\,,\\ s\varphi &= m\,c\,, \quad & sc &= 0\,,\\ s\overline{c} &= b\,, \quad & s\,b &= 0\,. \end{aligned} \tag{24}$$

Thus, the BRST multiplets consist of two trivial pairs (φ, c), $(\overline{c}, b)$ and one singlet $\hat{A}_\mu = A_\mu - \frac{1}{m}\partial_\mu\varphi$. This means that the physical spectrum will be independent of (φ, c) and $(\overline{c}, b)$.

In the gauge-fixing-ghost term $s(\overline{c}\mathcal{F})$, $\mathcal{F}$ is a real bosonic function of all the fields and their derivatives. For example, the 't Hooft-Feynman gauge fixing

$$\mathcal{F} = \frac{1}{g^2}(\frac{1}{2}b - \partial^\mu A_\mu - m\varphi)\,, \tag{25}$$

leads directly to noninteracting ghosts $\overline{c}$ and c. Moreover, the $\hat{A} - \varphi$ sector does not contain higher derivatives or dipoles. The field strengths $F_{\mu\nu}$ calculated from A and from $\hat{A}$ coincide. The field b is auxiliary with the algebraic equation of motion $b = \partial^\mu A_\mu + m\varphi$ Thus, the $\mathcal{L}$ defines a massive abelian gauge field coupled to matter fields.

To extend the Stueckelberg formalism to the SM quantized in the background gauge, the scalar field φ transforms under the BRST symmetry[†] and under the background gauge transformations in the following way

$$s\,\varphi = \mu\,c\,, \qquad \delta_{(\lambda)}\varphi = \mu\,\lambda\,, \tag{26}$$

[†]Notation, definitions, and quantum numbers can be found in [2].

where λ is the infinitesimal parameter of the background gauge transformations. μ is the IR regulator and c is the $U(1)$ ghost. The latter can be written in terms of the combination $c = c_W c_A + s_W c_Z$, where c_A and c_Z are the photon ghost and the ghost associated with the Z boson, respectively. It follows that a term like

$$\Gamma^{\rm Stu} = \int d^4x \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \mu\partial_\mu\varphi B^\mu + \frac{\mu^2}{2} B^\mu B_\mu \right) , \tag{27}$$

is BRST and background gauge invariant for all the values of the parameter μ and the last term provides a mass term for the B_μ fields. Other φ-dependent invariant terms can be constructed, however it is easy to show that all of them, but (27), can be reabsorbed by a simple redefinition of the field B_μ (cf. [20]). In addition, in order to deal with diagonal two-point functions the 't Hooft gauge fixing

$$\begin{aligned} \Gamma^{\rm Stu,g.f.} &= s \int d^4x \, \overline{c} \left(\partial^\mu B_\mu - \rho\mu \, \varphi + \xi_0 b \right) \\ &= \int d^4x \left[b \left(\partial^\mu B_\mu - \rho\mu \, \varphi + \xi_0 b \right) - \overline{c} \, \partial^2 c + \rho\mu^2 \overline{c} \, c \right] , \end{aligned} \tag{28}$$

is used. Here ξ_0 is the conventional gauge fixing parameter of the $U(1)$ sector and ρ is the 't Hooft parameter. With this gauge fixing, it is easy to see that the gauge field B_μ, the scalar field φ and the ghosts $\overline{c}, c$ (with masses $\mu^2, \rho^2\mu^2/\xi_0$ and $\rho\mu^2$) form a quartet which ensures the unitarity of the model. Notice that the BRST variation of φ says that this field corresponds to a would-be-Goldstone boson, and the spontaneous symmetry breaking mechanism – in the abelian case – can be implemented without the Higgs counterpart.

In the SM framework, the field B_μ does not coincide with the physical photon field, but the mixing with the third component of $SU(2)$ gauge boson triplet W^3_μ has to be considered. We have to cancel this term by modifying the gauge fixing function $\mathcal{F}_B$ (cf. [2], Eq. (A.3)) for the abelian field in the following way

$$\mathcal{F}_B = \partial^\mu B_\mu + \rho_0 (\hat{\Phi} + v)^i t^0_{ij} \, (\Phi + v)^j + \frac{\xi_0}{2} \, b \longrightarrow \mathcal{F}_B - \rho\mu\varphi \, , \tag{29}$$

where ρ is the t'Hooft parameter for the φ field. By eliminating the Lagrange multiplier b, we have the gauge fixing terms

$$\begin{aligned} \mathcal{L}^{\rm g.f.} &= -\frac{1}{2\xi_0} \left[\partial^\mu B_\mu - \rho\mu\varphi + \rho_0 (\hat{\Phi} + v)^i t^0_{ij} \, (\Phi + v)^j \right]^2 \\ &= -\frac{1}{2\xi_0} (\partial^\mu B_\mu)^2 - \frac{\rho^2\mu^2}{2\xi_0} \varphi^2 - \frac{\rho_0 g'^2 v^2}{2\xi_0} G^2 + \frac{\rho\rho_0\mu v g'}{2\xi_0} \varphi \, G \\ &\quad + \frac{\rho\mu}{\xi_0} \partial^\mu B_\mu \, \varphi - \frac{\rho_0 v g'}{\xi_0} \partial^\mu B_\mu G \\ &\quad - \frac{\rho v g'}{2\xi_0} \left(\partial^\mu B_\mu - \mu\rho\varphi \right) \left(H \, \hat{G} - \hat{H} \, G \right) - \frac{\rho^2 g'^2 v^2}{2\xi_0} \left(H \, \hat{G} - \hat{H} \, G \right)^2 , \end{aligned} \tag{30}$$

where g' is the $U(1)$ gauge coupling, v is the vacuum expectation value, G and H are the Goldstone boson and the Higgs field, respectively, while $\hat{G}$ and $\hat{H}$ are their background partners. The first line contains the contribution to the quadratic part of the action, this shows that also the masses of the Goldstone boson G are modified by the introduction of the Stueckelberg field φ. The mixed terms $\varphi\partial^\mu B_\mu$ and $G\partial^\mu B_\mu$ are cancelled (in the restricted 't Hooft gauge) by the mixing terms coming from the covariant derivatives of the kinetic terms (i.e. from Eq. (27)). Finally the last terms describe the interactions between φ and the other fields. As can be noticed all the interaction terms depend on the background fields. Therefore, the Stueckelberg φ field can be generated only if the fields $\hat{G}$ or $\hat{H}$ appear as external vertices of the amplitude or due to the mixing with the neutral Goldstone boson. This is the only difference between the Stueckelberg formalism and its application to the SM quantized in the background gauge.

ACKNOWLEDGEMENTS

We thank P. Gambino, W. Hollik, G. Weiglein, A. Ferroglia and M. Pernici for useful comments and discussions. We also warmly thank our collaborator Matthias Steinhauser. The work of P.A.G. is supported by NSF grants no. PHY-9722083 and PHY-0070787.

References

[1] O. Piguet and S.P. Sorella, *Algebraic Renormalization* Lecture Notes in Physics Monographs, Springer-Verlag Berlin Heidelberg, 1995 and references therein.

[2] P.A. Grassi, T. Hurth, and M. Steinhauser, [hep-ph/9907426] to appear in Annals of Physics (NY).

[3] P.A. Grassi, T. Hurth, and M. Steinhauser, Rep. No. CERN-TH/2001-002, NYU-TH/00/09/9.

[4] P.A. Grassi, T. Hurth, and M. Steinhauser, [hep-ph/0011067] to appear in JHEP.

[5] G. 't Hooft, Nucl. Phys. **B33**, 436 (1971); H. Kluberg-Stern and J. Zuber, Phys. Rev. D **12**, 467 (1975); Phys. Rev. D **12**, 482 (1975); L.F. Abbott, Nucl. Phys. **B185**, 189 (1981); S. Ichinose and M. Omote, Nucl. Phys. **B203**, 221 (1982); D.M. Capper and A. MacLean, Nucl. Phys. **B203**, 413 (1982); D.G. Boulware, Phys. Rev. D **12**, 389 (1981).

[6] A. Denner, G. Weiglein, and S. Dittmaier, Phys. Lett. **B333**, 420 (1994); Nucl. Phys. **B440**, 95 (1995).

[7] P.A. Grassi, Nucl. Phys. **B462**, 524 (1996); Nucl. Phys. **B537**, 527(1999); Nucl. Phys. **B560**, 499 (1999).

[8] A. Sirlin, Phys. Rev. D **22**, 971 (1980). A. Sirlin, Phys. Rev. D **29**, 89 (1984).

[9] T. van Ritbergen and R. G. Stuart, Phys. Rev. Lett. **82**, 488 (1999) [hep-ph/9808283]. T. van Ritbergen and R. G. Stuart, Nucl. Phys. **B564**, 343 (2000) [hep-ph/9904240].

[10] M. Steinhauser and T. Seidensticker, Phys. Lett. **B467**, 271 (1999) [hep-ph/9909436].

[11] A. Ferroglia, G. Ossola, and A. Sirlin, Nucl. Phys. **B560**, 23 (1999) [hep-ph/9905442].

[12] P. Malde and R. G. Stuart, Nucl. Phys. **B552**, 41 (1999) [hep-ph/9903403].

[13] A. Freitas, W. Hollik, W. Walter, and G. Weiglein, Phys. Lett. **B495**, 338 (2000) [hep-ph/0007091].

[14] A. Freitas, S. Heinemeyer, W. Hollik, W. Walter, and G. Weiglein, Nucl. Phys. Proc. Suppl. **89**, 82 (2000) [hep-ph/0007129].

[15] G. 't Hooft and M. Veltman, Nucl. Phys. **B44**, 189 (1972); P. Breitenlohner and D. Maison, Comm. Math. Phys. **52**, 11, 39, 55 (1977).

[16] W. Zimmermann, *Local Operator Products and Renormalization in Quantum Field Theory* in 1970 Brandeis University Summer Institute Lectures, Cambridge, Mass. M.I.T. Press.

[17] C. Becchi, A. Rouet, and R. Stora, Comm. Math. Phys. **42**, 127(1975); Ann. of Phys. (NY) **98**, 287(1976); I.V. Tyutin, Lebedev Institute preprint N39 (1975); see also C. Becchi, *Lectures on Renormalization of Gauge Theories* in Relativity, group and topology II, Les Houches 1983.

[18] P.A. Grassi, Ph. D. Thesis, University of Milano, Italy, 1997.

[19] E.C.G. Stueckelberg, Helv. Phys. Acta **30**, 209 (1957).

[20] N. Dragon, T. Hurth, and P. van Nieuwenhuizen, Nucl. Phys. Proc. Suppl. **B56**, 318 (1997) [hep-th/9703017]; T. Hurth, Helv. Phys. Acta **70**, 406 (1997).