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ON THE GROWTH OF THE LOCAL RESOLVENT

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Abstract

This paper is devoted to the study of growth of local resolvents. We give necessary conditions to obtain bounded local resolvents. The boundedness of derivatives of the local resolvent is studied in the case of reflexive Banach spaces and some results are given for admissible operators.

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1 Introduction

Let X be a complex Banach space and $\mathcal{L}(X)$ be the algebra of continuous linear operators on X. For $T \in \mathcal{L}(X)$, we denote by $\sigma(T)$ its spectrum and by $R_T : \lambda \in \mathbb{C} \setminus \sigma(T) \to R_T(\lambda) = (T-\lambda)^{-1} \in \mathcal{L}(X)$ its resolvent map. Let $x \in X$ be the analytic function $\lambda \to \tilde{x}(\lambda) := R_T(\lambda)x$ for $\lambda \notin \sigma(T)$ may have analytic extensions, solutions of the equation $(T-\lambda)f(\lambda) = x$. If for every $x \in X$ any two extensions of $R_T(\lambda)x$ agree on thier common domain, T is said to have the single valued extension property (that we denote $SVEP$) [2]. In this case, let $\rho(x,T)$ be the maximal domain of such extensions. The set $\sigma(x,T) = \mathbb{C} \setminus \rho(x,T)$ is called the local spectrum of T at x. It is obvious that T has the $SVEP$ if, and only if, the zero function is the only analytic function, on a given open set, that satisfies $(T-\lambda)f(\lambda) = 0$. By the Liouville theorem, it is clear that T has the $SVEP$ if, and only if, for any nonzero $x \in X$, we have $\sigma(x,T) \neq \emptyset$. Denote in the sequel for $A \subset \mathbb{C}$ the closure by $\overline{A}$ and by A^o the interior.
Recall that, for any arbitrary closed set F in the complex field, the spectral subspace associated to F is : $X_T(F) = \{x \in X, \sigma(x,T) \subset F\}$. The algebraic subspace $E_T(F)$ is the maximal element (if ordered by inclusion) of subspaces $Y \subset X$ which satisfy $(T-\lambda)Y = Y$ for all $\lambda \notin F$. It is obvious that $X_T(F) \subset E_T(F)$ (see also [3], [6] and [7]). By the open mapping theorem, we observe, for a closed set $F \subset \mathbb{C}$ that if $E_T(F)$ is closed, we have $E_T(F) = X_T(F)$ (see [4] for further information).

For $\lambda \in \rho(T)$, let $dist(\lambda, \sigma(T))$ be the distance of λ to $\sigma(T)$. As we have:

$$\|(T-\lambda)^{-1}\| \geq \frac{1}{dist(\lambda, \sigma(T))}$$

we see that the resolvent map is never bounded.
The aim of the paper [1] is the study of boundedness of the local resolvent. Examples of operators with bounded local resolvent (for some $x \in X$) and conditions for the existence of such elements are given. In [5] M.M. Neumann gives a large classe of operators with the same property. We pursue in this direction by studying properties of the local resolvent. We give in section 2 operators for which all the derivatives of the local resolvent are bounded and we show that if $\sigma(x,T)$ has an empty interior, the local resolvent is never uniformly continuous.
Section 3 is devoted to the case where X is a reflexive Banach space. We show that if $x \in X/E_T(\sigma(x,T)^o)$, then there exists an integer p such that the derivative of order p of the local resolvent at x is unbounded. Section 3 is devoted to admissible operators, we show that if $x \in X$ has bounded derivatives, then $\sigma(x,T) = \overline{\sigma(x,T)^o}$. In section 4 we introduce a class of spectral subspaces with growth properties, and we use them to obtain some results of [1].

2 On the boundedness of the local resolvent

Suppose that T has the $SVEP$ and let $\tilde{x}(\lambda)$ be the local resolvent, that is the maximal extension of $R_T(\lambda)x$. We give in the following examples of operators with bounded local resolvent.

Example 2.1 .

- *[1] Let* $C_0(\mathbb{Z}) = \{(x_n)_{n\in\mathbb{Z}} / \lim_{|n|\to+\infty} x_n = 0\}$ *equipped with the supermum norm and* $(e_n)_{n\in\mathbb{Z}}$ *its canonical basis. Consider the backward shift* T *(i.e.* $Te_n = e_{n-1}$*). The local resolvent at* e_0 *is:*

$$\tilde{e}_0(\lambda) = \begin{cases} \Sigma_1^\infty \lambda^{n+1} e_n, & |\lambda| < 1 \\ \Sigma_0^\infty - \frac{1}{\lambda^{n+1}} e_{-n}, & |\lambda| > 1 \end{cases}$$

  Hence $\|\tilde{e}_0(\lambda)\| \leq 1$.

- *[5] Let* $\Omega \subset \mathbb{C}$ *be a compact set and* $X = \mathcal{C}(\Omega)$ *be the Banach algebra of continuous functions. Let* $T \in \mathcal{L}(X)$ *given by* $T(f)(z) = zf(z)$ *and* $f_0(z) = dist(z, \mathbb{C} \setminus \Omega)$*. If* $\Omega^o \neq \emptyset$*, we have* $f_0 \neq 0$ *and*

$$\tilde{f}_0(\lambda)(z) = \frac{dist(z, \mathbb{C} \setminus \Omega)}{\lambda - z}$$

  so $\sigma(f_0, T) = \overline{\Omega^o}$ *and* $\|\tilde{f}_0(\lambda)\|_\infty \leq 1$.

- *Let* $H = L^2(\mathbb{D})$ *with* $\mathbb{D} = \{z \in \mathbb{C}/|z| \leq 1\}$ *and* M *be the shift operator on* H *given by:* $M(f)(z) = zf(z)$*. Let* $f \in H$*:* $f(z) = 1 - |z|$ *for all* $z \in \mathbb{D}$*. We have* $\sigma(f, M) = \mathbb{D}$ *and the local resolvent of* f *is* $\tilde{f}(z)(\lambda) = \frac{1-|z|}{z-\lambda}$ *for every* $|\lambda| > 1$ *. It is clear that* $\tilde{f}(z)$ *is bounded in* H*.*

We show in the following proposition, in connection with the examples above, that the local resolvent is never uniformly continuous, when the local spectrum has an empty interior.

Proposition 2.2 *Let* $x \in X$ *such that* $\sigma(x, T)$ *has an empty interior. If the local resolvent* $\tilde{x}$ *is uniformly continuous, then* $x = 0$*.*

Proof: Let $\lambda \in \sigma(x, T)$ and let $\lambda_n \in \rho(x, T)$ converging to λ. Then by uniform continuity $\tilde{x}(\lambda)$ converges and hence $\tilde{x}$ has a continuous extension through $\sigma(x, T)$. Such extension is analytic and we get by usual argument that $x = 0$.
We give a generalization of theorem 1 of [1].

Proposition 2.3 *Let* X *be a reflexive Banach space,* $x \in X$ *and* $T \in \mathcal{L}(X)$*.*
If $x \in X \setminus \bigcap_{\lambda \in \partial\sigma(x,T)} (\lambda - T) X_T(\sigma(x, T))$*, then* $\tilde{x}(\lambda)$ *is not bounded.*

Proof: Suppose that $\tilde{x}(\lambda)$ is bounded and let $\lambda \in \partial\sigma(x, T)$, there exists $\lambda_n \in \rho(x, T)$ that is converging to λ. As X is a reflexive Banach space, we can choose λ_n so that $\tilde{x}(\lambda_n)$ is a convergent sequence, let y be its limit, thus $(T - \lambda)y = x$. It is easy to check that $\sigma(x, T) = \sigma(y, T)$. Hence $x \in \bigcap_{\lambda \in \partial\sigma(x,T)} (\lambda - T) X_T(\sigma(x, T))$. Contradiction.

We derive

Corollary 2.4 *([1], Theorem 1) Let X be a reflexive Banach space , $x \in X$ and $T \in \mathcal{L}(X)$. Suppose that $x \in X \setminus \bigcap_{\lambda \in \mathbb{C}} (\lambda - T)X$ and $\sigma(x,T)$ has an empty interior, then the local resolvent at x is unbounded.*

In fact more is given:

Proposition 2.5 *Let X be a reflexive Banach space, $x \in X$ and $T \in \mathcal{L}(X)$.*
If $x \in X \setminus \bigcap_{\lambda \in \partial\sigma(x,T)} \bigcap_{n \geq 0} (\lambda - T)^n X_T(\sigma(x,T))$, then there exists some derivative of $\tilde{x}(\lambda)$ $(\lambda \in \rho(T,x))$ that is unbounded.

Proof: Suppose that all the derivatives of the local resolvent at x are bounded. Since the set $\{\tilde{x}^{'}(\lambda)\}$ is bounded, there exists a positive constant C such that:

$$\|\tilde{x}(\lambda) - \tilde{x}(\nu)\| \leq C\|\lambda - \nu\| \quad (\lambda, \nu \in \rho(x,T)).$$

Thus, for $\lambda_0 \in \partial\sigma(x,T)$, there exists $(\lambda_n) \subset \rho(x,T)$ that is converging to λ_0. As (λ_n) is a Cauchy sequence, $\tilde{x}(\lambda_n)$ is also a Cauchy sequence. Denote x_1 its limit, we have: $(T - \lambda_0)x_1 = x$ and $\sigma(x,T) = \sigma(x_1,T)$. Hence $(T - \lambda_0)(T - \lambda)\tilde{x}_1(\lambda) = (T - \lambda)x(\lambda)$ and by the $SVEP$ of T, we obtain $(T - \lambda_0)\tilde{x}_1(\lambda) = \tilde{x}(\lambda)$, and we have

$$\tilde{x}_1(\lambda) = \frac{\tilde{x}(\lambda) - x_1}{\lambda - \lambda_0} = \lim_{n \to +\infty} \frac{\tilde{x}(\lambda) - \tilde{x}(\lambda_n)}{\lambda - \lambda_n}$$

Using the preceding inequality, we obtain $\|\tilde{x}_1(\lambda)\| \leq C$. As the Banach space X is reflexive, we can choose λ_n so that $\tilde{x}_1(\lambda_n)$ is a convergent sequence, let x_2 be its limit. As before, we get $x_1 = (\lambda_o - T)x_2$ with $\sigma(x_2,T) = \sigma(x_1,T)$. By our assumption, there exists positive constants M_1 and M_2 such that:

$$\|\tilde{x}(\lambda) - \tilde{x}(\nu) - (\lambda - \nu)\tilde{x}^{'}(\nu)\| \leq M_1\|\lambda - \nu\|^2, \ \lambda, \nu \in \rho(x,T)$$

$$\|\tilde{x}^{'}(\lambda) - \tilde{x}^{'}(\nu)\| \leq M_2\|\lambda - \nu\|, \ \lambda, \nu \in \rho(x,T).$$

We derive that $\tilde{x}^{'}(\lambda_n)$ is also a Cauchy sequence (denote z its limit). By the preceding inequality, we obtain: $\|\tilde{x}_1(\lambda) - z\| \leq M_1\|\lambda - \lambda_0\|$. It is clear that: $z = x_2$ and $\|\tilde{x}_2(\lambda)\| \leq M_1$. Hence, we construct by induction a sequence (x_n) such that:

$$x = x_0, \ (\lambda_0 - T)x_n = x_{n+1}, \quad \text{and } \sigma(x_{n+1},T) = \sigma(x_n,T).$$

Thus

$$x \in \bigcap_{\lambda \in \partial\sigma(x,T)} \bigcap_{n \geq 0} (\lambda - T)^n X_T(\sigma(x,T))$$

which gives a contradiction.

Let A be a subset of $\mathbb{C}$ and $x \in X$. It is known that $x \in E_T(A)$ if, for every $\lambda \in \mathbb{C} \setminus A$, there exists (x_n) in X such that $(\lambda - T)x_{n+1} = x_n$ and $x = x_0$ [5].
Using this remark, and the preceding proof we obtain:

Proposition 2.6 *Let X be a reflexive Banach space and $T \in \mathcal{L}(X)$ with $SVEP$. If $x \in X \setminus E_T(\sigma(x,T)^o)$, then there exists some derivative of $\tilde{x}(\lambda)$ that is unbounded.*

3 Admissible operators

An operator $T \in \mathcal{L}(X)$ is said to be admissible if $E_T(F)$ is closed for every closed set $F \subset \mathbb{C}$. We have the following result:

Proposition 3.1 *Let X be a reflexive Banach space and $T \in \mathcal{L}(X)$ an admissible operator. Let $x \in X$. If all the derivatives of the local resolvent at x are bounded, then $\overline{\sigma(x,T)^o} = \sigma(x,T)$.*

Proof: Let $x \in \mathcal{L}(X)$ with bounded derivatives of its local resolvent and let $\lambda_0 \in \partial\sigma(x,T)$. By Proposition 2.5 there exists a sequence $(x_n) \in X$ such that $(\lambda_0 - T)x_{n+1} = x_n$ and $x_0 = x$. Hence $x \in E_T(\sigma(x,T)^o) \subset E_T(\overline{\sigma(x,T)^o})$. As T is admissible , we have $E_T(\overline{\sigma(x,T)^o}) = X_T(\overline{\sigma(x,T)^o})$. Thus $X_T(\sigma(x,T)) \subset X_T(\overline{\sigma(x,T)^o})$ and so $\sigma(x,T) \subseteq \overline{\sigma(x,T)^o}$. The reverse implication being obvious, we obtain the proposition.

We give in the following an example of an operator with bounded derivatives of the local resolvent.

Example 3.2 *Let Ω be a compact set with nonempty interior and $f \in L^2(\Omega)$ given by:*

$$f(z) = exp(-\frac{1}{dist(z, \mathbb{C} \setminus \Omega)}) \tag{1}$$

Then the local resolvent of f is

$$\tilde{f}(z)(\lambda) = \frac{exp(-\frac{1}{dist(z,\mathbb{C}\setminus\Omega)})}{\lambda - z}.$$

and we have

$$\tilde{f}^n(z)(\lambda) = \frac{(-1)^n(n-1)!exp(-\frac{1}{dist(z,\mathbb{C}\setminus\Omega)})}{(\lambda - z)^{n+1}} \quad \text{for all } n \geq 1.$$

Also, we obtain: $\sigma(f,T) = \overline{\Omega^o}$ and f satisfies the required properties of the preceding proposition.

For normal operators, we generalize ([1], Corollary 3):

Proposition 3.3 *Let T be a normal operator on a complex Hilbert space H. Then $\overline{\sigma(T)^o} = \sigma(T)$ if, and only if, there exists a vector $x \in H$ with bounded derivatives of its local resolvent are bounded and $\sigma(x,T) = \sigma(T)$.*

Proof: The converse implication follows ¿from the preceding proposition. To prove the direct implication, assume that $\overline{\sigma(T)^o} = \sigma(T)$.

If $U = (\overline{\sigma_p(T)})^o$ is nonempty, we consider a sequence (α_n) of distinct eigenvalues that is dense on U, and let (x_n) the normalized eigenvectors corresponding to (α_n). We define the numbers c_n by:

$$c_n = \frac{exp(-\frac{1}{dist(\alpha_n,\mathbb{C}\setminus U)})}{n},$$

and the vector

$$x = \sum_{n\geq 1} c_n x_n.$$

It is clear that $\overline{U} = \sigma(x,T)$, and the local resolvent is given by:

$$\tilde{x}_T(\lambda) = \sum_{n\geq 1} \frac{c_n x_n}{\lambda - \alpha_n}.$$

Since all the derivatives of the function

$$\lambda \in \rho(x,T) \to \frac{exp(-\frac{1}{dist(\alpha_n, \mathbb{C}\backslash U)})}{\lambda - \alpha_n}$$

are bounded, for every integer p, there exists a constant C_p such that:

$$\|\tilde{x}_T^p(\lambda)\|^2 \leq C_p \sum_{n\geq 1} \frac{1}{n^2}, \ (\lambda \in \rho(x,T)).$$

Hence all the derivatives of the local resolvent are bounded.
If U is empty, we may assume that T is a normal operator having non eigenvalues [see the proof of [1], Theorem 2]. Hence T is unitarily equivalent to the multiplication operator M_f on $L^2(\mu)$, given by

$$(M_f v)(t) := f(t)v(t), \ v \in L^2(\mu)$$

with f a bounded complex valued function on X. Boundedness properties are invariants under unitary transformations, we can assume without loss of generality that T is the multiplication operator M_f on $L^2(\mu)$. We have $\sigma(M_f) = \overline{f(X)}$ and $W := \sigma(M_f)^o$ is nonempty. Let $x \in L^2(\mu)$ given by :

$$x(t) = \begin{cases} exp(-\frac{1}{dist(f(t), \mathbb{C}\backslash W)}) & t \in f^{-1}(W) \\ 0 & \text{otherwise} \end{cases}$$

We have: $(\tilde{x}_{M_f}(\lambda))(t) = \frac{x(t)}{\lambda - f(t)}$ for $\lambda \in \rho(x, M_f)$, and $\sigma(M_f) = \overline{W}$. Since all the derivatives of the function

$$\lambda \in \rho(x,T) \to \frac{exp(-\frac{1}{dist(f(t), \mathbb{C}\backslash W)})}{\lambda - f(t)}$$

are bounded, there exists, for any $p > 0$, a positive constant C_p such that.

$$\|\tilde{x}_T^p(\lambda)\|^2 \leq C_p \int_{\overline{W}} \|y(t)\|^2 d\mu(z), \ \lambda \in \rho(x,T).$$

Hence

$$\|\tilde{x}_T^p(\lambda)\|^2 \leq C_p \|y\|^2.$$

We conclude that all the derivatives of the local resolvent at x are bounded on $\rho(x,T)$.

4 Spectral sets and the growth of the local resolvent

Let $F \subset \mathbb{C}$ be a closed set and $\phi : \mathbb{C} \setminus F \to X$ be an analytic function. Set

$$X_0^\phi(F) = \{x \in X_T(F) \cup X_T(\mathbb{C} \setminus F) \text{ such that } \lim_{dist(\lambda,F)\to 0} |\frac{\tilde{x}(\lambda)}{\phi(\lambda)}| = 0\}$$

We have

Proposition 4.1 *Let λ_0 be an isolated point in $\sigma(T)$, then $X_0^\phi(\{\lambda_0)\}$ is closed.*

Proof: Let λ_0 be an isolated point in $\sigma(T)$. There exists an open set V such that $V \cap \sigma(T) = \{\lambda_0\}$. Consider $x_n \in X_0^\phi(\{\lambda_0\})$ a convergent sequence with limit x and let $\lambda \in V \setminus \{\lambda_0\} \subset \rho(T)$, we have $\tilde{x}_n(\lambda) = (T-\lambda)^{-1}x_n$ and $g_n(\lambda) := \frac{(T-\lambda)^{-1}x_n}{\phi(\lambda)}$ is also a convergent sequence to $g(\lambda) := \frac{(T-\lambda)^{-1}x}{\phi(\lambda)}$ for all $\lambda \in V \setminus \{\lambda_0\} \subset \rho(T)$. As $g_n(\lambda) = \int_\gamma \frac{g_n(\mu)}{\mu-\lambda} d\mu$, where γ is a Jordan curve surrounding λ_0. If we denote $\alpha = dist(\gamma, V^c)$ the distance between γ and V^c, we obtain $\|g_n(\lambda - g(\lambda)\| \leq \frac{\varepsilon}{2\pi\alpha}$ and $\sup_{\lambda \in V \setminus \{\lambda_0\}} \|g_n(\lambda) - g(\lambda)\| \leq \frac{\varepsilon}{2\pi\alpha}$.

Using the fact that $\lim_{|\lambda-\lambda_0|\to 0} |\frac{\tilde{x}_n(\lambda)}{\phi(\lambda)}| = 0$ we get $x \in X_0^\phi(\lambda_0)$.

Proposition 4.2 *Let λ_0 be an isolated point in $\sigma(T)$ and ϕ_n a sequence of analytic functions on a neighborhood of λ_0. Suppose that $\cup_{n\geq 0} X_o^{\phi_n}(\{\lambda_0\}) = X$. Then there exists n_0 such that $X_0^{\phi_{n_0}}(\{\lambda_0\}) = X$.*

Proof: By proposition 4.1 $X_0^{\phi_n}(\{\lambda_o\})$ is closed for every $n \geq 0$ and we conclude by using the Baire category theorem.
We derive the following corollary:

Corollary 4.3 *Let $\lambda \in \sigma(T)$. If $\tilde{x}$ admits a pole in λ, for every $x \in X$, so is the case for R_T.*

Proof: By considering $x \in X$ such that $\sigma(x,T) = \sigma(T)$, we see that λ is isolated in $\sigma(T)$. Applying the proposition with $\phi_n(z) = \frac{1}{(z-\lambda)^n}$, we obtain the corollary.

Remark 4.4 *The Kaplansky theorem for locally algebraic operators can be derived by the preceding proposition in a classical way.*

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