United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency

THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

RECURSIVE SUBNORMAL COMPLETION
AND THE TRUNCATED MOMENT PROBLEM

R. Ben Taher[1]
Département de Mathématiques, Faculté des Sciences,
B.P. 4010, Beni M'hamed, Méknés - Morocco,

M. Rachidi[2] and E.H. Zerouali[3]
Faculté des Sciences, Université Mohammed V,
B.P. 1014, Rabat, Morocco
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

The aim of this paper is to study properties of sequences that are recursively defined by a linear equation and their applications to the truncated moment problem in connection with the problem of subnormal completion of the truncated weighted shifts. Special cases are considered and some classical results due to Stampfli, Curto and Fialkow are recovered using elementary techniques.

MIRAMARE – TRIESTE

August 2000

[1]E-mail: bentaher@fsmek.ac.ma
[2]E-mail: rachidi@fsr.ac.ma
[3]Regular Associate of the Abdus Salam ICTP. E-mail: zerouali@fsr.ac.ma

1   Introduction.

Let $a_0, a_1, \cdots, a_{r-1}$ $(r \geq 2)$ be some fixed real numbers with $a_{r-1} \neq 0$ and consider the sequence $V = \{V_n\}_{n \geq 0}$ defined by the following linear recurrence relation of order r,

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \cdots + a_{r-1} V_{n-r+1}, \quad \text{for all} \ \ n \geq r - 1, \tag{1}$$

where $V_0, V_1, \cdots, V_{r-1}$ are specified by the initial conditions. Such sequences are largely studied in the literature, generally called $r-$generalized Fibonacci sequences (see [6] and [8] for example). We shall refer to them in the sequel as *sequences (1)*.

Let $\{V_n\}_{0 \leq n \leq p}$, where $p \leq +\infty$, be a sequence of real numbers and K be a compact subset of $\mathbb{R}$. The K-moment problem associated to $\{V_n\}_{0 \leq n \leq p}$ consists of finding a positive Borel measure μ such that,

$$V_n = \int_K t^n d\mu(t), \quad \text{for all} \ \ n \ \ (0 \leq n \leq p) \ \ \text{and} \ \ Supp(\mu) \subset K, \tag{2}$$

where $Supp(\mu)$ is the support of μ. A measure μ satisfying (2) is called a *generating measure* of the sequence $\{V_n\}_{0 \leq n \leq p}$. A positive generating measure is called *representing measure* (see [3] and [4]). For $p = +\infty$ the problem (2) is called the full K-moment problem (see [1], for example). When $p < +\infty$ the problem (2) is called the truncated K-moment problem (see [3], [4] and [7]).

Consider the classical separable Hilbert space $\mathcal{H} = l^2(\mathbf{Z}_+)$ and $\{e_n\}_{n \geq 0}$ its orthonormal basis. Let $\alpha = \{\alpha_n\}_{n \geq 0}$ be a bounded sequence of nonnegative real numbers. Let W_α be the bounded operator of $\mathcal{H} = l^2(\mathbf{Z}_+)$ defined by $W_\alpha e_n = \alpha_n e_{n+1}$. Set $\{V_n\}_{n \geq 0}$, where $V_n = \alpha_{n-1}^2 \cdots \alpha_1^2 \alpha_0^2$ $(n \geq 1)$ and $V_0 > 0$, the sequence of moments associated to W_α. By Berger's theorem, W_α is a subnormal operator if and only if there exists a nonnegative Borelean measure μ, that is called a representing measure of $\{V_n\}_{n \geq 0}$, with $supp(\mu) \subset [0, \|W_\alpha\|^2]$ and such that

$$V_n = \int_0^{\|W_\alpha\|^2} t^n d\mu(t), \quad \text{for all} \ \ n \geq 0.$$

where $\|W_\alpha\| = \sup_{n \geq 0} \alpha_n$. Hence, the moment problem and subnormal weighted shifts are closely related. A weighted shift W_α such that $\{V_n\}_{n \geq 0}$ is a sequence (1) is called a recursively generated weighted shift. It is known that every weighted shift is norm-limit of recursively generated weighted shifts. For further information we refer to [2], [4], [5] and [7] for example.

We characterize sequences (1) that are moment sequences in terms of their minimal polynomial and give some generalisations of results due to R. Curto and L. Fialkow. In the last section, we apply our technique to recover easily some results due to stampfli.

2   Sequences (1) and Moment Problem.

2.1   Minimal polynomial.

Let $V = \{V_n\}_{n \geq 0}$ be a sequence (1). The polynomial $P(x) = x^r - a_0 x^{r-1} - \cdots - a_{r-2} x - a_{r-1}$ is called a characteristic polynomial associated to $\{V_n\}_{n \geq 0}$. A sequence (1) can be defined in

various ways using different characteristic polynomials as shown in the following example.

Set $V_n = n + 1$, then $\{V_n\}_{n \geq 0}$ satisfies the following recursive relations,

$$V_{n+1} = V_n + V_{n-1} - V_{n-2},$$

for $V_0 = 1$, $V_1 = 2$, $V_2 = 3$ and

$$V_{n+1} = V_n + \frac{1}{2}(V_{n-1} - V_{n-2} + V_{n-3} - V_{n-4}),$$

for $V_0 = 1$, $V_1 = 2$, $V_2 = 3$, $V_3 = 4$, $V_4 = 5$.

Hence $P_1(X) = X^3 - X^2 - X + 1$ and $P_2(X) = X^5 - X^4 - \frac{1}{2}X^3 + \frac{1}{2}X^2 - \frac{1}{2}X + \frac{1}{2}$ are characteristic polynomials for the sequence $\{V_n\}_{n \geq 0}$.

Let $\mathcal{P}_V$ be the set of characteristic polynomial associated to the sequence $V = \{V_n\}_{n \geq 0}$.

Proposition 2.1 *For every sequence $V = \{V_n\}_{n \geq 0}$ given by (1), there exists a unique characteristic polynomial $P_V \in \mathcal{P}_V$ with minimal degree. Moreover, every $P \in \mathcal{P}_V$ is a multiple of P_V.*

Proof. It is suffice to prove the second assertion of our proposition. Set $P(X) = \prod_{i=1}^{r}(X - \lambda_i)^{n_i}$ for $P \in \mathcal{P}_V$. By the Binet formula we get

$$V_n = \sum_{i=1}^{r_V} \sum_{j=1}^{k_i} c_{i,j} n^j \lambda_i^n \quad (c_{i,k_i} \neq 0).$$

The polynomial $P_V(X) = \prod_{i=1}^{r_V}(X - \lambda_i)^{k_i}$ is a characteristic polynomial for $V = \{V_n\}_{n \geq 0}$. On the other hand λ_i is a zero of order at least k_i for every characteristic polynomial for $\{V_n\}_{n \geq 0}$. Hence P_V provides a positive answer to the proposition.$\square$

Definition 2.2 *Let $V = \{V_n\}_{n \geq 0}$ be a sequence (1). The polynomial $P_V(X) = X^r - a_o X^{r-1} - a_1 X^{r-2} - ... - a_{r-1}$ is called the minimal polynomial associated to $\{V_n\}_{n \geq 0}$. The sequence $\{V_n\}_{n \geq 0}$ is said to be of order r.*

Remark 2.3 *1. The minimal polynomial obtained from the Binet formula of sequence (1) coincides with the generating function of [3], which is obtained by computing the rank of the Hankel matrix associated with $\{V_n\}_{n \geq 0}$. Polynomials are the basic tools for constructing the r-atomic representing measure for the sequence $\{V_n\}_{n \geq 0}$.*

2. As shown in proposition 2.1, the minimal polynomial is a factor of any characteristic polynomials that depend only on the initial conditions. Hence it can be computed by 'testing' equation (1) for a finite number of characteristic polynomial's factors.

3

2.2   Generating measure and minimal polynomial.

Proposition 2.4 *Let $V = \{V_n\}_{n \geq 0}$ be a sequence (1). Then $\{V_n\}_{n \geq 0}$ admits a generating measure μ if and only if P_V has distinct real roots. Moreover, $Supp(\mu) = Z(P_V)$.*

Proof. Set $P_V(X) = \prod_{i=0}^{r-1}(X - \lambda_i)$ with $\lambda_0 < ... < \lambda_{r-1}$, then the Binet formula implies that

$$V_n = \rho_0 \lambda_0^n + \cdots + \rho_{r-1} \lambda_{r-1}^n, \quad \text{for any} \quad n \geq 0, \tag{3}$$

where $\rho_0, \cdots, \rho_{r-1}$ are nonzero real numbers derived from the following system of r linear equations $\rho_0 \lambda_0^n + \cdots + \rho_{r-1} \lambda_{r-1}^n = V_n$, $n = 0, 1, \cdots, r-1$ (see [6] for example). Consider the Borelean $r-$atomic measure μ given by $\mu = \sum_{j=0}^{r-1} \rho_j \delta_{\lambda_j}$, on the interval $[\lambda_0, \lambda_{r-1}]$. From expression (3) we derive that

$$V_n = \int_{\lambda_0}^{\lambda_{r-1}} t^n d\mu(t), \quad \text{for any} \quad n \geq 0.$$

Conversely, suppose that μ is a generating measure associated to $\{V_n\}_{n \geq 0}$ and $P \in \mathcal{P}_V$. Easy computations give $P.\mu = 0$, which assert that $Supp(\mu) \subset Z(P) = \{\lambda_0, ..., \lambda_n\}$. Set $\mu = \sum_{j=0}^{r-1} \rho_j \delta_{\lambda_j}$, where ρ_j are nonzero real numbers. Hence $V_n = \rho_0 \lambda_0^n + \cdots + \rho_{r-1} \lambda_{r-1}^n$ $(\rho_i \neq 0)$. We obtain P_V by setting $P_V(X) = \prod_{i=0}^{r-1}(X - \lambda_i)$, which is the minimal polynomial of $\{V_n\}_{n \geq 0}$. $\square$

Remark 2.5 *Proposition 2.4 is valid without assuming that the polynomial P_V has real roots, however in the subnormal completion problem this will be the case.*

In the following we consider sequences (1) that are given by P_V with distinct roots. Let μ be a nonnegative measure, solution of the moment problem associated to the initial conditions $\{V_n\}_{0 \leq n \leq r-1}$ of the sequence (1). Then ¿from properties of the truncated moment problem we derive that the Hankel matrix $H(r) = (V_{i+j})_{0 \leq i,j \leq r}$ is nonnegative (see [2], [3], [7] for example).

On the other hand, suppose that the Hankel matrix $H(r) = (V_{i+j})_{0 \leq i,j \leq r}$ is nonnegative. Expression (3) means that μ is an interpolating $r-$atomic measure of the sequence $\gamma = (\gamma_{i,j})_{0 \leq i,j \leq r}$, where $\gamma_{i,j} = V_{i+j}$. Thus Propositions 3.8, 3.17 of [3] and Proposition 3 of [7] allow us to derive that μ is a nonnegative measure. Hence, we have the following result,

Theorem 2.6 *Let $V = \{V_n\}_{n \geq 0}$ be a sequence (1) and $P_V(X) = \prod_{i=0}^{r-1}(X - \lambda_i)$ be its minimal polynomial, with $\lambda_0 < ... < \lambda_{r-1}$. Then, there exists a nonnegative representing measure μ for $\{V_n\}_{n \geq 0}$ if and only if the Hankel matrix $H(r) = (V_{i+j})_{0 \leq i,j \leq r}$ is nonnegative.*

3   Application to subnormal completion of weighted truncated Shifs.

We interpret Theorem 2.6 in terms of subnormality of W_α that is close to the positivity of Hankel matrices (see [2], [4], [5] and [7] for example). We derive the following property,

Theorem 3.1 *Let* $V = \{V_n\}_{n \geq 0}$ *be a positive sequence (1) and* $P_V(X) = \prod_{i=0}^{r-1}(X - \lambda_i)$ *be its minimal polynomial with* $\lambda_0 < ... < \lambda_{r-1}$. *Let* $\alpha = (\alpha_n)_{n \geq 0}$ *be the sequence defined by* $\alpha_n^2 = \frac{V_{n+1}}{V_n}$. *Then the operator* W_α *is subnormal if and only if the Hankel matrix* $H(r) = (V_{i+j})_{0 \leq i,j \leq r}$ *is nonnegative.*

Proof. From Theorem 2.6 we derive that the support of the positive measure μ is a subset of the interval $[\lambda_0, \lambda_{r-1}]$.□

Following the same proof of Theorem 2.6 and using Proposition 3.8 of [3] and Proposition 3 of [7], we can derive that $H(r-1) \geq 0$ if and only if W_α is subnormal operator. Hence, from Theorem 3.2 of [2] W_α is subnormal if and only if $H(k) \geq 0$ for any $k \geq 0$. Thus, we have the following proposition,

Proposition 3.2 *Let* $V = \{V_n\}_{n \geq 0}$ *be a positive sequence (1). Suppose that* $P_V(X) = \prod_{i=0}^{r-1}(X - \lambda_i)$ *is its minimal polynomial. Suppose that* $0 < \lambda_0 < \cdots < \lambda_{r-1}$, *then the following are equivalent,*

(i) $H(r-1) \geq 0$;

(ii) $H(r) \geq 0$;

(iii) $H(k) \geq 0$ *for any* $k \geq 0$.

Proposition 3.2 allows us to derive the following relation between subnormality and k-hyponormality in the case of sequences (1),

Proposition 3.3 *Let* $\{V_n\}_{n \geq 0}$ *be a positive sequence (1). Suppose that the characteristic roots* $\lambda_0, \cdots, \lambda_{r-1}$ *are simple and positive. Let* $\alpha = (\alpha_n)_{n \geq 0}$ *be the sequence defined by* $\alpha_n^2 = \frac{V_{n+1}}{V_n}$. *Then the following are equivalent,*

(i) W_α *is subnormal;*

(ii) W_α *is k-hyponormal;*

(iii) There exists $k_0 \geq r - 1$ *such that* W_α *is* k_0*-hyponormal.*

Proof. Implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ follow from [2]. Let us establish $(iii) \Rightarrow (i)$. Suppose that there exists $k_0 \geq r - 1$ such that $H(k_0) \geq 0$. Then, Proposition 3.2 implies that $H(r-1) \geq 0$. Hence, W_α is subnormal. □

4 Special Cases.

4.1 case n = 2 and n = 3.

Let $\{V_n\}_{n \geq 0}$ be a sequence (1) such that $V_{n+1} = a_0 V_n + a_1 V_{n-1}$ and $V_0 = 1, V_1$ are given. The characteristic polynomial is $P(X) = X^2 - a_0 X - a_1$. Suppose that $\{V_n\}_{n \geq 0}$ is a moment sequence associated to a measure μ. A simple use of the Cauchy-Schwartz inequality gives $V_2 = a_0 V_1 + a_1 \geq V_1^2$, hence $P(V_1) \leq 0$. On the other hand, if $P(V_1) \leq 0$ we get $V_1 = \rho_1 \lambda_1 + \rho_2 \lambda_2$, where

$\lambda_1 = \frac{a_0 - \sqrt{a_0^2 + 4a_1}}{2}$, $\lambda_2 = \frac{a_0 + \sqrt{a_0^2 + 4a_1}}{2}$ are the distinct real roots of P and ρ_1, ρ_2 are nonnegative real numbers such that $\rho_1 + \rho_2 = 1$. Thus $\{V_n\}_{n \geq 0}$ is a moment sequence associated to the nonnegative measure $\mu = \rho_1 \delta_{\lambda_1} + \rho_2 \delta_{\lambda_2}$. Hence,

Proposition 4.1 *A sequence $\{V_n\}_{n \geq 0}$ such that $V_0 = 1, V_1$ are given and $V_{n+1} = a_0 V_n + a_1 V_{n-1}$ is a moment sequence if and only if $P(V_1) \leq 0$.*

Remark 4.2 *1. For given $V_1 \geq 0$, we have $P(V_1) \leq 0$ if and only if $\frac{a_0 - \sqrt{a_0^2 + 4a_1}}{2} \leq V_1 \leq \frac{a_0 + \sqrt{a_0^2 + 4a_1}}{2}$, that is easy to obtain.*
2. By writing $\gamma_1^2 = \frac{V_2}{V_1}$ we get $P(V_1) \leq 0$ if and only if $\gamma_0 \leq \gamma_1$. Hence for $0 < \gamma_0 < \gamma_1$, we can construct γ_n $(\gamma_n^2 = \frac{V_{n+1}}{V_n})$ such that W_α is subnormal and the result of Stampfli is recovered (see [9]).
3. Consider $V_{n+1} = a_0 V_n + a_1 V_{n-1}$ for $n = 1, 2$. Thus $a_0 = \frac{V_2 V_1 - V_3}{V_1^2 - V_2}$ and $a_1 = \frac{V_3^2 - V_2}{V_1^2 - V_2}$. As $V_1 = \gamma_0^2, V_2 = \gamma_0^2 \gamma_1^2$ and $V_3 = \gamma_0^2 \gamma_1^2 \gamma_2^2$ we get $a_0 = \gamma_1^2 \frac{\gamma_2^2 - \gamma_0^2}{\gamma_1^2 - \gamma_0^2}$ and $a_1 = -(\gamma_0^2 \gamma_1^2) \frac{\gamma_2^2 - \gamma_1^2}{\gamma_1^2 - \gamma_0^2}$, as it was shown in [2].

In the following proposition, we characterize sequences (1) of order $r = 3$ that give moment sequences. Given $V_0 = 1, V_1$ and V_2 nonnegative real numbers, under which conditions a sequence (1) such that $V_{n+1} = a_0 V_n + a_1 V_{n-1} + a_2 V_{n-2}$ $(n \geq 3)$ is a moment sequence. Let $P(X) = X^3 - a_0 X^2 - a_1 X - a_2$ be its minimal characteristic polynomial and suppose that μ is a generating nonnegative measure, $\mu = \rho_0 \delta_{\lambda_0} + \rho_1 \delta_{\lambda_1} + \rho_2 \delta_{\lambda_2}$, where $\{\lambda_0, \lambda_1, \lambda_2\} = Z(P)$ $(\lambda_0 < \lambda_1 < \lambda_2)$ and ρ_0, ρ_1, ρ_2 are solutions of the following system of equations,

$$\begin{cases} \rho_0 + \rho_1 + \rho_2 & = 1, \\ \rho_0 \lambda_0 + \rho_1 \lambda_1 + \rho_2 \lambda_2 & = V_1, \\ \rho_0 \lambda_0^2 + \rho_1 \lambda_1^2 + \rho_2 \lambda_2^2 & = V_2. \end{cases}$$

Then ρ_0, ρ_1 and ρ_2 are given by $\rho_0 = \frac{V_2 - V_1(\lambda_1 + \lambda_2) + \lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_0)(\lambda_1 - \lambda_0)}$, $\rho_1 = \frac{-V_2 + V_1(\lambda_0 + \lambda_2) - \lambda_0 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_0)(\lambda_1 - \lambda_0)}$ and $\rho_2 = \frac{V_2 - V_1(\lambda_0 + \lambda_1) + \lambda_0 \lambda_1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_0)(\lambda_1 - \lambda_0)}$. Hence, we have the following result,

Proposition 4.3 *Given $\{V_n\}_{n \geq 0}$ a sequence (1) with initial condition $V_o = 1, V_1, V_2$ and minimal polynomial $P(X) = (X - \lambda_0)(X - \lambda_1)(X - \lambda_2)$. Then $\{V_n\}_{n \geq 0}$ is a moment sequence for a nonnegative measure if and only if*
$$\begin{cases} V_2 - V_1(\lambda_1 + \lambda_2) + \lambda_1 \lambda_2 & \geq 0, \\ -V_2 + V_1(\lambda_0 + \lambda_2) - \lambda_0 \lambda_2 & \geq 0, \\ V_2 - V_1(\lambda_0 + \lambda_1) + \lambda_0 \lambda_1 & \geq 0. \end{cases}$$

4.2 case n = 4.

Given $0 \leq \gamma_0 \leq \gamma_1 \leq ... \leq \gamma_n$. A subnormal completion weighted shift W_α of $(\gamma_0, \gamma_1, ..., \gamma_n)$ is a weighted shift W_α, where $\alpha = (\alpha_0, ..., \alpha_n....)$, so that $\alpha_i = \gamma_i$ $(0 \leq i \leq n)$. Let $\mathbf{W}(\gamma_0, ..., \gamma_n)$ be the set of all subnormal completion weighted shifts of $(\gamma_0, ..., \gamma_n)$ and

$$, (\gamma_0, ..., \gamma_n) = \inf\{\alpha_{n+1}, W_\alpha \in \mathbf{W}(\gamma_0, ..., \gamma_n)\}.$$

For $0 \leq \gamma_0 \leq \gamma_1$ a direct computation, using the norm-density of recursive weighted shifts, gives , $(\gamma_0, \gamma_1) = \gamma_1$. We derive that every sequence $\{\gamma_0, \gamma_1, \gamma_2\}$, where $0 \leq \gamma_0 \leq \gamma_1 \leq \gamma_2$, admits a subnormal completion weighted shift associated to a sequence (1) of order r.

As shown in [9], a sequence $\gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \gamma_3$ may have no subnormal completion weighted shift. We characterize those sequences that admit a subnormal completion weighted shift by computing , $(\gamma_0, \gamma_1, \gamma_2)$.

Proposition 4.4 *Given* $0 \leq \gamma_0 \leq \gamma_1 \leq \gamma_2$ *then*

$$, (\gamma_0, \gamma_1, \gamma_2) = \frac{\gamma_1^2(\gamma_1^2\gamma_0^2 + \gamma_2^4 - 2\gamma_0^2\gamma_2^2)}{\gamma_2^2(\gamma_1^2 - \gamma_0^2)}.$$

In particular, the sequence $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ *admits a subnormal completion weighted shift if and only if*

$$\gamma_3^2 \geq \frac{\gamma_1^2(\gamma_1^2\gamma_0^2 + \gamma_2^4 - 2\gamma_0^2\gamma_2^2)}{\gamma_2^2(\gamma_1^2 - \gamma_0^2)}.$$

Proof. By Proposition 3.2 $\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ admits a subnormal completion weighted shift if and only if

$$\begin{pmatrix} 1 & V_1 & V_2 \\ V_1 & V_2 & V_3 \\ V_2 & V_3 & V_4 \end{pmatrix} \geq 0.$$

Hence for x, y and z real numbers we get

$$\begin{pmatrix} 1 & V_1 & V_2 \\ V_1 & V_2 & V_3 \\ V_2 & V_3 & V_4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \geq 0,$$

which leads to

$$x^2 + V_2 y^2 + V_4 z^2 + 2V_1 xy + 2V_2 xz + 2V_3 yz \geq 0,$$

for any x, y, z. By setting $x' = x, y' = \sqrt{V_2}y, z' = \sqrt{V_4}z, a = \frac{V_1}{\sqrt{V_2}}, b = \frac{V_2}{\sqrt{V_4}}$ and $c = \frac{V_3}{\sqrt{V_2 V_4}}$ we obtain that $(x')^2 + (y')^2 + (z')^2 + 2ax'y' + 2bx'z' + 2cy'z' \geq 0$, for any x', y', z'. Hence

$$(x' + ay' + bz')^2 + (1 - a^2)(y')^2 + (1 - b^2)(z')^2 + 2(c - ab)y'z' \geq 0,$$

for any x', y', z' and so $(c-ab)^2 - (1-a^2)(1-b^2) \leq 0$. That is equivalent to $a^2 + b^2 + c^2 - 2abc - 1 \leq 0$. In terms of V_1, V_2, V_3 and V_4 we obtain that $V_4 \geq \frac{V_2^3 + V_3^2 - 2V_1V_2V_3}{V_2 - V_1^2}$, and by using our weight, we get $\gamma_3^2 \geq \frac{\gamma_1^2(\gamma_1^2\gamma_0^2 + \gamma_2^4 - 2\gamma_0^2\gamma_2^2)}{\gamma_2^2(\gamma_1^2 - \gamma_0^2)}$. $\square$

For $\gamma_0^2 = 1, \gamma_1^2 = 2$ and $\gamma_2^2 = 3$ we obtain that $\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ admits a subnormal completion weighted shifts if and only if $\gamma_3^2 \geq \frac{10}{3}$ as given in the example of [9].

In [9], it is also shown that for a given moment sequence $\{\gamma_n\}_{n \geq 0}$ we have $\gamma_n = \gamma_m$ whenever there exists $n_0 \geq 1$ such that $\gamma_{n_0} = \gamma_{n_0+1}$. This result can be obtained by using Proposition 4.5 in an inductive proof, however sequences (1) allow us to give it in a simple way.

Proposition 4.5 *Let* $\{V_n\}_{n \geq 0}$ *be a moment sequence and suppose that there exists* $n_0 \geq 2$ *such that* $\gamma_{n_0} = \gamma_{n_0+1}$, *then* $\gamma_n = \gamma_m$ *for any* $n, m \geq 1$.

Proof. Let $\{V_n\}_{n\geq 0}$ be a moment sequence and $\{V_n\}_{0\leq n\leq p}$ be a truncated subsequence for an arbitrary $p \geq n_0 + 1$. By theorem 2.6 $\{V_n\}_{0\leq n\leq p}$ can be extended in a recursive moment sequence (see also [3]), we can assume that $\{V_n\}_{n\geq 0}$ is a sequence (1). Let $\mu = \sum_{0\leq i\leq r-1} \rho_i \delta_{\lambda_i}$ be a representing measure. Then $\frac{V n_{0+1}}{V n_0} = \frac{V n_{0+2}}{V n_{0+1}}$ implies that,

$$\big(\sum_{0\leq i\leq r-1} \rho_i (\lambda_i)^{n_{0+1}} \big)^2 = \big(\sum_{0\leq i\leq r-1} \rho_i (\lambda_i)^{n_0} \big)\big(\sum_{0\leq i\leq r-1} \rho_i (\lambda_i)^{n_{0+2}} \big),$$

or

$$\sum_{0\leq i<j\leq r-1} \rho_i \rho_j (\lambda_i \lambda_j)^{n_{0+1}} (\lambda_i - \lambda_j)^2 = 0,$$

which gives $\lambda_i = \lambda_j$ whenever $\lambda_i.\lambda_j \neq 0$. Thus $V_n = \rho\lambda_1^n$ for $n \geq 1$, where $\rho = \sum_{0\leq i\leq r-1} \rho_i$. Finally $(\gamma_n)^2 = \lambda_1$ for all $n \geq 2$. As p is arbitrary we obtain the proposition.□

Acknowledgments.

References

[1] N.I. AKHIEZER, *The classical moment problem*, (Hafner Publ. Co., New York, 1965).

[2] R.E. CURTO, 'Existence of a rich collection of positively quadratically hyponormal weighted shifts', *Proceeding of the CIMR-99, Rabat* - April 1999 (To appear).

[3] R.E. CURTO and L. FIALKOW, *Solution of the truncated complex moment problem for flat data*, Memoire of the Amer. Math. Soc. Vol. 119, No. 568 (1996).

[4] R.E. CURTO and L. FIALKOW, 'Recursively generated weighted shifts and the subnormal completion problem', *Integr. Equat. and Oper. Theory* Vol. 17 (1993) 202-246.

[5] R.E. CURTO and L. FIALKOW, 'Recursively generated weighted shifts and the subnormal completion problem II', *Integr. Equat. and Oper. Theory* Vol. 18 (1994) 369-426.

[6] F. DUBEAU, W. MOTTA, M. RACHIDI and O. SAEKI, 'On Weighted r-generalized Fibonacci sequences', *Fibonacci Quarterly* 35 (1997) 102-110.

[7] L. A. FIALKOW, 'Positivity, extensions and the truncated moment problem', *Multi-variable operator theory, Contemporary Mathematics* 185, Amer. Math. Soc., Providence, RI, (1995), pp.135-150.

[8] E. P. MILES, 'Generalized Fibonacci numbers and associated matrices', *Amer. Math. Monthly* 67 (1960) 745-752.

[9] J. STAMPFLI, 'Which weighted shifts are subnormal', *Pacific Journal of Math.* 17 (1966) 367-379