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Progress on two-loop non-propagator integrals

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At variance with fully inclusive quantities, which have been computed already at the two- or three-loop level, most exclusive observables are still known only at one loop, as further progress was hampered up to very recently by the greater computational problems encountered in the study of multi-leg amplitudes beyond one loop. We discuss the progress made lately in the evaluation of two-loop multi-leg integrals, with particular emphasis on two-loop four-point functions.

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1 Introduction

Precision applications of particle physics phenomenology often demand theoretical predictions at the next-to-next-to-leading order in perturbation theory. Corrections at this order are known for many inclusive observables, such as total cross sections or sum rules, which correspond from a technical point of view to propagator-type Feynman amplitudes. For $2 \to 2$ scattering and $1 \to 3$ decay processes, the calculation of next-to-next-to-leading order corrections is a yet outstanding task. One of the major ingredients for these calculations are the two-loop virtual corrections to the corresponding four-point Feynman amplitudes. Depending on the process under consideration, these calculations require two-loop four-point functions with massless internal propagators and all legs on-shell (high energy limit of Bhabha scattering, hadronic two-jet production) or one leg off-shell (three-jet production and event shapes in electron–positron annihilation, two-plus-one-jet production in deep inelastic scattering, hadronic vector-boson-plus-jet production).

During the past two years, many new results on two-loop four-point functions became available, thus enabling the first calculations of two-loop virtual corrections to $2 \to 2$ scattering processes. A variety of newly developed techniques made this progress possible. In this talk, we describe these new techniques and their applications, and we summarise recent results. In an outlook, we discuss the remaining steps to be taken towards the completion of next-to-next-to-leading order calculations of $2 \to 2$ scattering and $1 \to 3$ decay processes.

2 New technical developments

Using dimensional regularization [1,2] with $d = 4 - 2\epsilon$ dimensions as regulator for ultraviolet and infrared divergences, the integrals appearing in the calculation of two-loop corrections take the generic form

$$I(p_1, \ldots, p_n) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{D_1^{m_1} \ldots D_t^{m_t}} S_1^{n_1} \ldots S_q^{n_q} \,, \tag{1}$$

where the D_i are massless scalar propagators, depending on k, l and the external momenta $p_1, \ldots, p_n$ while S_i are scalar products of a loop momentum with an external momentum or of the two loop momenta. The topology (interconnection of propagators and external momenta) of the integral is uniquely determined by specifying the set $(D_1, \ldots, D_t)$ of t different propagators in the graph. The integral itself is then specified by the powers m_i of all propagators and by the selection $(S_1, \ldots, S_q)$ of scalar products and their powers $(n_1, \ldots, n_q)$ (all the m_i are positive integers greater or equal to 1, while the n_i are greater or equal to 0). Integrals of the same topology with the same dimension $r = \sum_i m_i$ of the denominator and same total number

$s = \sum_i n_i$ of scalar products are denoted as a class of integrals $I_{t,r,s}$. The integration measure and scalar products appearing the above expression are in Minkowskian space, with the usual causal prescription for all propagators. The loop integrations are carried out for arbitrary space-time dimension d, which acts as a regulator for divergences appearing due to the ultraviolet or infrared behaviour of the integrand. For each topology appearing in the calculation, a sizable number of different scalar integrals has to be computed.

Recent progress in the computation of two-loop corrections to four-point amplitudes was based on three technical developments: an efficient procedure to reduce the large number of different scalar integrals to a very limited number of so-called master integrals, new techniques for the computation of these master integrals, and a new class of functions (harmonic polylogarithms), which can be extended to suit the needs of a particular calculation. We discuss these developments in the following.

2.1 Reduction to master integrals

The number $N(I_{t,r,s})$ of the integrals grows quickly as r, s increase, but the integrals are related among each other by various identities. One class of identities follows from the fact that the integral over the total derivative with respect to any loop momentum vanishes in dimensional regularization

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\partial}{\partial k^\mu} J(k, \ldots) = 0, \tag{2}$$

where J is any combination of propagators, scalar products and loop momentum vectors. J can be a vector or tensor of any rank. The resulting identities [2,3] are called integration-by-parts (IBP) identities.

In addition to the IBP identities, one can also exploit the fact that all integrals under consideration are Lorentz scalars (or, perhaps more precisely, "d-rotational" scalars) , which are invariant under a Lorentz (or d-rotational) transformation of the external momenta [4]. These Lorentz invariance (LI) identities are obtained from:

$$\left(p_1^\nu \frac{\partial}{\partial p_{1\mu}} - p_1^\mu \frac{\partial}{\partial p_{1\nu}} + \ldots + p_n^\nu \frac{\partial}{\partial p_{n\mu}} - p_n^\mu \frac{\partial}{\partial p_{n\nu}} \right) I(p_1, \ldots, p_n) = 0 \; . \tag{3}$$

In the case of two-loop four-point functions, one has a total of 13 equations (10 IBP + 3 LI) for each integrand corresponding to an integral of class $I_{t,r,s}$, relating integrals of the same topology with up to $s+1$ scalar products and $r+1$ denominators, plus integrals of simpler topologies (*i.e.* with a smaller number of different denominators). The 13 identities obtained starting from an integral $I_{t,r,s}$ do contain integrals of the following types:

- $I_{t,r,s}$: the integral itself.

- $I_{t-1,r,s}$: simpler topology.
- $I_{t,r+1,s}, I_{t,r+1,s+1}$: same topology, more complicated than $I_{t,r,s}$.
- $I_{t,r-1,s}, I_{t,r-1,s-1}$: same topology, simpler than $I_{t,r,s}$.

Quite in general, single identities of the above kind can be used to obtain the reduction of $I_{t,r+1,s+1}$ or $I_{t,r+1,s}$ integrals in terms of $I_{t,r,s}$ and simpler integrals - rather than to get information on the $I_{t,r,s}$ themselves.

If one considers the set of all the identities obtained starting from the integrand of all the $N(I_{t,r,s})$ integrals of class $I_{t,r,s}$, one obtains $(N_{\rm IBP} + N_{\rm LI})N(I_{t,r,s})$ identities which contain $N(I_{t,r+1,s+1}) + N(I_{t,r+1,s})$ integrals of more complicated structure. It was first noticed by S. Laporta [5] that with increasing r and s the number of identities grows faster than the number of new unknown integrals. As a consequence, if for a given t-topology one considers the set of all the possible equations obtained by considering all the integrands up to certain values r^*, s^* of r, s, for large enough r^*, s^* the resulting system of equations, apparently overconstrained, can be used for expressing the more complicated integrals, with greater values of r, s in terms of simpler ones, with smaller values of r, s. An automatic procedure to perform this reduction by means of computer algebra using FORM [6] and MAPLE [7] is discussed in more detail in [4].

For any given four-point two-loop topology, this procedure can result either in a reduction towards a small number (typically one or two) of integrals of the topology under consideration and integrals of simpler topology (less different denominators), or even in a complete reduction of all integrals of the topology under consideration towards integrals with simpler topology. Left-over integrals of the topology under consideration are called irreducible master integrals or just master integrals.

2.2 Computation of master integrals

The IBP and LI identities allow to express integrals of the form (1) as a linear combination of a few master integrals, i.e. integrals which are not further reducible, but have to be computed by some different method.

For the case of massless two-loop four-point functions, several techniques have been proposed in the literature, such as for example the application of a Mellin–Barnes transformation to all propagators [8] or the negative dimension approach [9]. Both techniques rely on an explicit integration over the loop momenta, with differences mainly in the representation used for the propagators.

A method for the analytic computation of master integrals avoiding the explicit integration over the loop momenta is to derive differential equations in internal propagator masses or in external momenta for the master integral, and to solve these with appropriate boundary conditions. This method has first been suggested by Kotikov [10] to relate loop integrals with internal masses to massless loop integrals.

It has been elaborated in detail and generalized to differential equations in external momenta in [11]; first applications were presented in [12]. In the case of four-point functions with one external off-shell leg and no internal masses, one has three independent invariants, resulting in three differential equations.

The derivatives in the invariants $s_{ij} = (p_i + p_j)^2$ can be expressed by derivatives in the external momenta:

$$
\begin{aligned}
s_{12}\frac{\partial}{\partial s_{12}} &= \frac{1}{2}\left(+p_1^\mu\frac{\partial}{\partial p_1^\mu} + p_2^\mu\frac{\partial}{\partial p_2^\mu} - p_3^\mu\frac{\partial}{\partial p_3^\mu}\right) , \\
s_{13}\frac{\partial}{\partial s_{13}} &= \frac{1}{2}\left(+p_1^\mu\frac{\partial}{\partial p_1^\mu} - p_2^\mu\frac{\partial}{\partial p_2^\mu} + p_3^\mu\frac{\partial}{\partial p_3^\mu}\right) , \\
s_{23}\frac{\partial}{\partial s_{23}} &= \frac{1}{2}\left(-p_1^\mu\frac{\partial}{\partial p_1^\mu} + p_2^\mu\frac{\partial}{\partial p_2^\mu} + p_3^\mu\frac{\partial}{\partial p_3^\mu}\right) .
\end{aligned} \tag{4}
$$

It is evident that acting with the right hand sides of (4) on a master integral $I_{t,t,0}$ will, after interchange of derivative and integration, yield a combination of integrals of the same type as appearing in the IBP and LI identities for $I_{t,t,0}$, including integrals of type $I_{t,t+1,1}$ and $I_{t,t+1,0}$. Consequently, the scalar derivatives (on left hand side of (4)) of $I_{t,t,0}$ can be expressed by a linear combination of integrals up to $I_{t,t+1,1}$ and $I_{t,t+1,0}$. These can all be reduced (for topologies containing only one master integral) to $I_{t,t,0}$ and to integrals of simpler topology by applying the IBP and LI identities. As a result, we obtain for the master integral $I_{t,t,0}$ an inhomogeneous linear first order differential equation in each invariant. For topologies with more than one master integral, one finds a coupled system of first order differential equations. The inhomogeneous term in these differential equations contains only topologies simpler than $I_{t,t,0}$, which are considered to be known if working in a bottom-up approach.

The master integral $I_{t,t,0}$ is obtained by matching the general solution of its differential equation to an appropriate boundary condition. Quite in general, finding a boundary condition is a simpler problem than evaluating the whole integral, since it depends on a smaller number of kinematical variables. In some cases, the boundary condition can even be determined from the differential equation itself.

To solve the differential equations for two-loop four-point functions with one off-shell leg [4,13], we express the system of differential equations for any master integral in the variables $s_{123} = s_{12} + s_{13} + s_{23}$, $y = s_{13}/s_{123}$ and $z = s_{23}/s_{123}$. We obtain a homogeneous equation in s_{123}, and inhomogeneous equations in y and z. Since s_{123} is the only quantity carrying a mass dimension, the corresponding differential equation is nothing but the rescaling relation obtained by investigating the behaviour of the master integral under a rescaling of all external momenta by a constant factor. The master integral can be determined by solving one of the inhomogeneous equations, the second equation can then serve as a check on the result.

In the y differential equation for the master integral under consideration, the coefficient of the homogeneous term as well as the full inhomogeneous term (coefficients and subtopologies) are then expanded as a series in ϵ. From the leading coefficient of the homogeneous term, one can determine a rational prefactor $\mathcal{R}$ for the master integral. Rescaling the master integral by this prefactor, one obtains a differential equation in which the coefficient of the homogeneous term is of $\mathcal{O}(\epsilon)$. This equation can then be solved order by order in ϵ by direct integration. The remaining constants of integration, which correspond to the boundary condition of the equation, are subsequently determined by using the fact that the master integral is regular in the whole kinematic plane with the exception of a few (at most three) branch cuts.

For each master integral, we obtain a result of the form

$$\sum_i \mathcal{R}_i(y,z;s_{123},\epsilon)\mathcal{H}_i(y,z;\epsilon) \ , \tag{5}$$

where the prefactor $\mathcal{R}_i(y,z;s_{123},\epsilon)$ is a rational function of y and z, which is multiplied with an overall normalization factor to account for the correct dimension in s_{123}, while $\mathcal{H}_i(y,z;\epsilon)$ is a Laurent series in ϵ. The coefficients of its ϵ-expansion are then written as the sum of two-dimensional harmonic polylogarithms up to a weight determined by the order of the series:

$$\mathcal{H}_i(y,z;\epsilon) = \frac{\epsilon^p}{\epsilon^4}\sum_{n=0}^{4}\epsilon^n\left[T_n(z) + \sum_{j=1}^{n}\sum_{\vec{m}_j\in V_j(z)} T_{n,\vec{m}_j}(z)H(\vec{m}_j;y)\right] \ , \tag{6}$$

where the $H(\vec{m}_j;y)$ are two-dimensional harmonic polylogarithms (2dHPL), which were introduced in [13] and $T_n(z)$, $T_{n,\vec{m}_j}(z)$ are z-dependent coefficients.

2.3 Harmonic polylogarithms

Harmonic polylogarithms (HPL) were introduced in [14] as an extension of the generalized polylogarithms of Nielsen [15,16]. They are constructed in such a way that they form a closed, linearly independent set under a certain class of integrations. We observe that the class of allowed integrations on this set can be extended *à la carte* by enlarging the definition of harmonic polylogarithms in order to suit the needs of a particular calculation. We made use of this feature by generalizing the one-dimensional HPL of [14] to two-dimensional harmonic polylogarithms (2dHPL), which appear in the solution of the differential equations for the three-scale master integrals discussed in[13]. We briefly recall the HPL formalism [14]:

1. The one-dimensional HPL $H(\vec{m}_w;x)$ is described by a w-dimensional vector $\vec{m}_w$ of parameters and by its argument x. w is called the weight of H.

2. The HPL of parameters $(+1,0,-1)$ form a closed set under the class of integrations

$$\int_0^x \mathrm{d}x'\left(\frac{1}{x'},\frac{1}{1-x'},\frac{1}{1+x'}\right)H(\vec{b};x') \ . \tag{7}$$

3. The HPL fulfil an algebra, such that a product of two HPL (with weights w_1 and w_2) of the same argument x is a combination of HPL of argument x with weight $w = w_1 + w_2$.

4. The HPL fulfil integration-by-parts identities.

5. The HPL are linearly independent.

The generalization from one-dimensional to two-dimensional HPL starts from (7), which defines the class of integrations under which the HPL form a closed set. By inspection of the various inhomogeneous terms of the y differential equations for the three-scale master integrals discussed in this paper, we find that, besides the denominators $1/y$ and $1/(1-y)$, also $1/(1-y-z)$ and $1/(y+z)$ appear. It is therefore appropriate to introduce an extension of the HPL, which forms a closed set under the class of integrations

$$\int_0^y \mathrm{d}y' \left(\frac{1}{y'}, \frac{1}{1-y'}, \frac{1}{1-y'-z}, \frac{1}{y'+z} \right) H(\vec{b}; y') \ . \tag{8}$$

Allowing $(z, 1-z)$ as components of the vector $\vec{m}_w$ of parameters does then define the extended set of HPL, which we call two-dimensional harmonic polylogarithms (2dHPL). They retain all properties of the HPL, in particular the algebra and the linear independence.

Two-dimensional harmonic polylogarithms can be expressed in terms of Nielsen's generalized polylogarithms up to weight 3, which is the maximum weight appearing in the divergent terms of two-loop four-point functions with one leg off-shell. These relations are tabulated in [13]. At weight 4, only some special cases relate to generalized polylogarithms.

3 Summary of recent results

For two-loop four-point functions with massless internal propagators and all legs on-shell, which are relevant for example in the next-to-next-to-leading order calculation of two-jet production at hadron colliders, all master integrals have been calculated over the past two years. The calculations were performed using the Mellin–Barnes method [8] and the differential equation technique [17]. The resulting master integrals can be expressed in terms of Nielsen's generalized polylogarithms. Very recently, these master integrals were already applied in the calculation of two-loop virtual corrections to Bhabha scattering [18] in the limit of vanishing electron mass and to quark–quark scattering [19].

In [13], we have used the differential equation approach to compute all master integrals for two-loop four-point functions with one off-shell leg. Earlier partial results on these functions were obtained in [9,20], and a purely numerical approach to these functions was presented in [21]. Our results [13] for these master integrals are in terms of two-dimensional harmonic polylogarithms. All 2dHPL appearing in the divergent parts of the master integrals can be expressed in terms of Nielsen's generalized polylogarithms of suitable non-simple arguments, while the 2dHPL appearing in the finite parts are one-dimensional integrals over generalized polylogarithms. An efficient numerical implementation of these functions is currently being worked out. Our results correspond to the kinematical situation of a $1 \to 3$ decay, their analytic continuation into the region of $2 \to 2$ scattering processes requires the analytic continuation of the 2dHPL, which is outlined in [13].

These four-point two-loop master integrals with one leg off-shell are a crucial ingredient to the virtual next-to-next-to-leading order corrections to processes such as three-jet production in electron–positron annihilation, two-plus-one-jet production in deep inelastic scattering and vector-boson-plus-jet production at hadron colliders.

4 Outlook

Owing to numerous technical developments in the past two years, virtual two-loop corrections to four-point amplitudes are now becoming available for a variety of phenomenologically relevant processes. One must however keep in mind that these corrections form only one part of a full next-to-next-to-leading order calculation, which also has to include the one-loop corrections to processes with one soft or collinear real parton [22] as well as tree-level processes with two soft or collinear partons [23]. Only after summing all these contributions (and including terms from the renormalization of parton distributions for processes with partons in the initial state), the divergent terms cancel among one another. The remaining finite terms have to be combined into a numerical programme implementing the experimental definition of jet observables and event-shape variables. A first calculation involving the above features was presented for case of photon-plus-one-jet final states in electron–positron annihilation in [23], thus demonstrating the feasibility of this type of calculations.

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