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FULL AND TRUNCATED MOMENT PROBLEMS IN $\mathbb{R}^2$ AND RECURSIVE RELATIONS

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Abstract

This paper concerns the full and truncated moment problems in dimension 2. The connection between these problems are considered using some linear recurrence relations. These relations allow us to have a bridge between Curto-Fialkow's method and Cassier-Vasilescu's method.

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1 Introduction.

Let $a_0, a_1, \cdots, a_{r-1}$ and $b_0, b_1, \cdots, b_{s-1}$ ($r,\ s \geq 2$) be some real numbers with $a_{r-1}b_{s-1} \neq 0$, and $\{\omega_{i,j}\}_{0\leq i\leq r-1,\, 0\leq j\leq s-1}$ be a sequence of real numbers.

Let $\{W_{n,m}\}_{n,\, m\geq 0}$ be the sequence defined by $W_{i,\, j} = \omega_{i,j}$ for $0 \leq i \leq r-1$, $0 \leq j \leq s-1$, and the following linear recurrence relations

$$W_{n+1,m} = \sum_{i=0}^{r-1} a_i W_{n-i,m} \text{ and } W_{n,m+1} = \sum_{j=0}^{s-1} b_j W_{n,m-j}, \text{ for } n \geq r-1, \ m \geq s-1. \quad (1)$$

In the sequel we shall refer to these sequences as *sequences (1)*.

Let $\Gamma_{p,q} = \{\gamma_{n,m}\}_{0\leq n\leq p,\ 0\leq m\leq q}$, where $p, q \leq +\infty$, be a sequence of real numbers and K be a compact subset of $I\!R^2$. The K-moment problem associated to $\Gamma_{p,q}$ consists of finding a positive Borel measure μ such that

$$\gamma_{n,m} = \int_K x^n y^m d\mu(t), \text{ for } 0 \leq n \leq p, \ 0 \leq m \leq q \text{ and } Supp(\mu) \subset K, \quad (2)$$

where $Supp(\mu)$ is the support of μ. A positive measure satisfying (2) is called a *representing measure* of $\Gamma_{p,q} = \{\gamma_{n,m}\}_{0\leq n\leq p,\ 0\leq m\leq q}$ on K. For $p = q = +\infty$ the problem (2) is called the *full K-moment problem.* When $p, q < +\infty$ the problem (2) is called the *truncated K-moment problem.*

There is a large amount of literature on the full K-moment problem studied by various methods and technics (see [1], [3], [4], [5], [12] and [14], for example). In dimension $n \geq 2$ the full K-moment problem has been solved for K compact with nonempty interior (see [4] and [5]), and for K semi-algebraic compact set (see [12] and [14]).

The truncated K-moment problem is studied by Curto-Fialkow for $K \subset \mathbb{C}$, using the positive matrix approach, and the subcase $K \subset I\!R$ is considered (see [6] and [11] for example). The Curto-Fialkow's method is motivated, because the classical full K-moment problem argumentations are obstructed. In [14] (see Question 3.9) the problem of the truncated moment sequences and its connection with subnormality of commuting multi-operator arose.

The connection between the full and truncated K-moment problems has been studied in [9], [10] for $K = [a, b] \subset I\!R$.

In [2], we consider the truncated moment problem in the one dimensional case and its connection with the subnormal completion problem. In this paper we investigate the closed relation between the full and the truncated K-moment problem for $K \subset I\!R^2$. More precisely, the linear recurrence relations (1) allow us to solve the truncated moment problem (2) for a sequence $,_{r,s} = \{\omega_{n,m}\}_{0\leq n\leq r-1,\, 0\leq m\leq s-1}$ in the case when $,_{r,s}$ is a set of initial values of a sequence (1).

This paper is organized as follows. In Section 2 we consider the relation between sequences (1), linear forms and properties of the representing measures. Section 3 is devoted to the existence of solutions of the K-moment problem (2) for sequences (1), using Cassier-Vasilescu's method. Finally, in Section 4 we investigate the connection of our method with Curto-Fialkow schemes and we give an explicit example.

2 Moment problem for sequences (1).

2.1 Sequences (1) and linear forms.

Let $\{W_{n,m}\}_{n,\, m\geq 0}$ be a sequence (1). A direct computation shows that, for all $n \geq r-1$ and $m \geq s-1$, we have

$$\begin{aligned} W_{n+1,m+1} &= \textstyle\sum_{i=0}^{r-1} a_i W_{n-i,m+1} \\ &= \textstyle\sum_{j=0}^{r-1} b_j W_{n+1,m-j} \\ &= \textstyle\sum_{0\leq i\leq r-1, 0\leq j\leq s-1} a_i b_j W_{n-i,m-j}. \end{aligned} \tag{3}$$

Equation (3) gives the compatibility condition of the two relations of (1). Hence the sequence $\{W_{n,m}\}_{m,\, m\geq 0}$ is well defined.

Consider the linear form $L : I\!R[X,Y] \rightarrow I\!R$ given by

$$L(X^n Y^m) = W_{n,m} \;, \quad \text{for all} \quad n,\, m \geq 0. \tag{4}$$

From (1) and (3) we derive that, for all $n \geq 0$, $m \geq 0$ and $k \geq 0$, we have

$$L(X^n Y^k P_1(X)) = 0 \quad \text{and} \quad L(X^k Y^m P_2(Y)) = 0, \tag{5}$$

where $P_1(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ and $P_2(Y) = Y^s - b_0 Y^{s-1} - \cdots - b_{s-1}$. Conversely, suppose that the linear form (4) satisfies (5). Then the sequence $\{W_{n,m}\}_{n,\, m\geq 0}$ defined by $L(X^n Y^m) = W_{n,m}$ is a sequence (1), associated to the two polynomials $P_1(X)$ and $P_2(Y)$.

2.2 Representing measure.

Let $,_{p,q} = \{\gamma_{n,m}\}_{0\leq n\leq p,\, 0\leq m\leq q}$ be a sequence of real numbers. A generating measure μ associated to $,_{p,q}$ is a real Borel measure satisfying:

$$\gamma_{n,m} = \int_K x^n y^m d\mu(x,y) \;, \text{ for } 0 \leq n \leq p,\, 0 \leq m \leq q \text{ and } \; Supp(\mu) \subset K, \tag{6}$$

Let $\{W_{n,m}\}_{n\geq 0,\, m\geq 0}$ be a sequence (1) associated to polynomials $P_1(X)$ and $P_2(Y)$. Let $\{\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{r-1}\}$ and $\{\beta_0 \leq \beta_1 \leq \cdots \leq \beta_{r-1}\}$ be the two sets of characteristic roots of $P_1(X)$ and $P_2(Y)$ respectively.

It is obvious from (1) that for any m_0 (respectively n_0) the sequence $\{W_{n,m_0}\}_{n\geq 0}$ (respectively $\{W_{n_0,m}\}_{m\geq 0}$) is a recursive sequence. Hence, in the case of moment sequences, $\{W_{n,m_0}\}_{n\geq 0}$ (respectively $\{W_{n_0,m}\}_{m\geq 0}$) is associated with a minimal polynomial P_{m_0} (respectively Q_{n_0}) with distinct roots, where P_{m_0} is a divisor of P_1 (respectively Q_{n_0} is a divisor of P_2) (see [2]). Hence we can suppose without loss of generality that $P_1(X)$ and $P_2(Y)$ have distinct roots. We have the following.

Proposition 1 *Let $\{W_{n,m}\}_{1\leq n\leq r-1, 1\leq m\leq s-1}$ be a sequence (1). Suppose that the two polynomials $P_1(X) = X^r - a_0X^{r-1} - \cdots - a_{r-1}$ and $P_2(Y) = Y^s - b_0Y^{s-1} - \cdots - b_{s-1}$ have distinct real roots $\{\lambda_0 < \lambda_1 < \cdots < \lambda_{r-1}\}$ and $\{\beta_0 < \beta_1 < \cdots < \beta_{r-1}\}$ respectively. Then $\{W_{n,m}\}_{1\leq n\leq r-1, 1\leq m\leq s-1}$ admits a generating atomic measure.*

Proof : Consider the atomic measure given by

$$\mu = \sum_{\substack{0\leq i\leq r-1\\ 0\leq j\leq s-1}} \rho_{i,j}\delta_{(\lambda_i,\beta_j)}.$$

Then μ is a generating measure associated to $\{W_{n,m}\}_{n,m\geq 0}$ if, and only if the sequence $\{\rho_{i,j}\}_{1\leq 0\leq r-1, 0\leq j\leq s-1}$ satisfies the following linear system of $r.s$ equations

$$\sum_{\substack{0\leq i\leq r-1\\ 0\leq j\leq s-1}} \rho_{i,j}\lambda_i^n\beta_j^m = W_{n,m} \quad \text{for} \quad 1\leq n\leq r-1,\ 1\leq m\leq s-1.$$

As the determinant of the preceding system of equations is nonzero (namely its absolute value is $\prod_{i<j}(\lambda_i - \lambda_j)^r(\beta_i - \beta_j)^s$), we derive the existence of the atomic measure μ. □

In the proof of Proposition 1, if $p = r-1 < q = s-1$ for example, we can complete our system of equations recursively to get q^2 equations (explicit computations in the case where $r = s = 2$ are given in section 4).

Let μ be a generating measure of a sequence $,_{p,q} = \{\gamma_{n,m}\}_{0\leq n\leq p,\, 0\leq m\leq q}$ on K. If $\mu \geq 0$ we say that μ is a *representing measure* of $,_{p,q} = \{\gamma_{n,m}\}_{0\leq n\leq p,\, 0\leq m\leq q}$ on K.

In the sequel we consider that $K = [\lambda_0, \lambda_{r-1}] \times [\beta_0, \beta_{s-1}]$. Then $K = \{(x,y) \in I\!R^2; Q_j(x,y) \geq 0,\ j = 1,\ 2,\ 3,\ 4\}$, where $Q_1(x,y) = \lambda_{r-1} - x$, $Q_2(x,y) = \beta_{s-1} - y$, $Q_3(x,y) = x - \lambda_0$ and $Q_4(x,y) = y - \beta_0$. Thus K is a semi-algebraic compact subset of $I\!R^2$ (see [5] and [14] for example).

The linear recurrence relations (1) allow us to establish the following reduction property.

Lemma 1 *Let $\{W_{n,m}\}_{n,m\geq 0}$ be a sequence (1) and μ be a positive measure supported by K. The following are equivalent.*
(i) μ is a representing measure of the sequence $\{W_{n,m}\}_{n,m\geq 0}$.
(ii) μ is a representing measure of the truncated sequence $\{W_{n,m}\}_{0\leq n\leq 2r, 0\leq m\leq 2s}$.

Let μ be a discrete positive measure on $\mathbb{R}^2$ with $supp(\mu) \subset K$ given by

$$\mu = \sum_{0\leq i\leq r-1, 0\leq j\leq s-1} a_{i\,j}\delta_{(x_i,y_j)},$$

where $a_{i\,j} \in \mathbb{R}$ and $\delta_{(a,b)}$ is the Dirac measure at (a, b). The moment sequence $\{\alpha_{n,\,m}\}_{n,m\geq 0}$ associated to μ on K is

$$\alpha_{n,\,m} = \int_K x^n y^m d\mu(x, y) = \sum_{0\leq i\leq r-1, 0\leq j\leq s-1} a_{i\,j} x_i^n y_j^m.$$

We have

$$\alpha_{n+1,m} = \sum_{i=0}^{r-1} c_i\alpha_{n-i,m}, \text{ for } n \geq r-1 \text{ and } \alpha_{n,m+1} = \sum_{j=0}^{r-1} d_j\alpha_{n,m-j}, \text{ for } m \geq s-1,$$

where the coefficients c_i $(0 \leq i \leq r-1)$ and d_j $(0 \leq j \leq s-1)$ are given as follows $Q_1(X) = \Pi_{j=0}^{r-1}(X - x_j) = X^r - c_0X^{r-1} - \cdots - c_{r-1}$ and $Q_2(Y) = \Pi_{j=0}^{s-1}(Y - y_j) = Y^s - d_0Y^{s-1} - \cdots - d_{s-1}$. Hence $\{\alpha_{n\,m}\}_{n,m\geq 0}$ is a sequence (1), whose initial values are $\{\alpha_{n\,m}\}_{0\leq n\leq r-1, 0\leq m\leq s-1}$.

Let $\{W_{n,\,m}\}_{n,m\geq 0}$ be a sequence (1). Suppose that $\{W_{n,\,m}\}_{0\leq n\leq 2r, 0\leq m\leq 2s}$ has a representing measure μ on $K \subset \mathbb{R}^2$. From Lemma 1 we derive that $W_{n,\,m} = \int_K x^n y^m d\mu(x, y)$ for all $n \geq 0$ and $m \geq 0$. The relation (1) implies that $\int_K R(x, y)P_1(x)d\mu(x, y) = \int_K S(x, y)P_2(y)d\mu(x, y) = 0$ for all R, S in $\mathbb{R}[X, Y]$. Thus $P_1(X)\mu = P_2(Y)\mu = 0$, which implies that $supp(\mu) \subset \{(x, y) \in \mathbb{R}^2; P_1(x) = 0\} \cap \{(x, y) \in \mathbb{R}^2; P_2(y) = 0\}$. Hence $supp(\mu) \subset \{\lambda_0 < \lambda_1 < \cdots < \lambda_{r-1}\} \times \{\beta_0 < \beta_1 < \cdots < \beta_{r-1}\}$ and we have $\mu = \sum_{0\leq i\leq r-1, 0\leq j\leq s-1} a_{i\,j}\delta_{(\lambda_i,\beta_j)}$. Thus we have the following property.

Proposition 2 *Let $\{W_{n,\,m}\}_{n,m\geq 0}$ be a sequence (1). Then the following are equivalent.*
(i) There exists a representing measure μ of $\{W_{n,\,m}\}_{n,m\geq 0}$ on K.
(ii) There exists a representing measure μ of $\{W_{n,\,m}\}_{0\leq n\leq 2r, 0\leq m\leq 2s}$ on K.
(iii) There exists μ a representing measure of $\{W_{n,\,m}\}_{0\leq n\leq 2r, 0\leq m\leq 2s}$ on K with a finite support.
(iv) There exists μ a representing measure of $\{W_{n,\,m}\}_{0\leq n\leq r-1, 0\leq m\leq s-1}$ on K with $supp(\mu) \subset Z(P_1) \times Z(P_2) = \{\lambda_0 < \lambda_1 < \cdots < \lambda_{r-1}\} \times \{\beta_0 < \beta_1 < \cdots < \beta_{r-1}\}$.

3 Existence of solutions.

3.1 Reduction properties for sequences (1).

In [4] and [5] Cassier gives some criteriums on the existence of the solution of the full K-moment problem (2) in dimension n, where K is a semi-algebraic compact set of $\mathbb{R}^n$. Schmudgen had studied the K-moment problem for semi-algebraic sets (see [12]). In [14] Vasilescu had considered the moment problem for multi-sequences on some explicit test set and applied this to establish the connection between the moment problem and subnormality.

Consider the following notations from [4], let $A(K)$ be the set of affine forms on $\mathbb{R}^2$ which can be identified with $\mathbb{R}_1[X,Y]$ and set

- $A_+(K) = \{T \in A(K); T \geq 0 \ \text{ on } \ K\}$
- $G(K) = \{T \in A_+(K);\ T \neq 0$ and generate extremal gernerating in $A_+(K)\}$
- $G_1(K) = \{T \in G(K); \|T\| = \sup_{(x,y)\in K} |T(x,y)| = 1\}$
- $\Delta(K) = \{T = \Pi_{i=1}^p T_i;\ p \geq 1,\ T_i \in G_1(K)\} \cup \{1\}$

For $r,\ s \geq 2$, we consider the following $\mathbb{R}$-vector space

$$\mathbb{R}_{r-1,\,s-1} = \{T \in \mathbb{R}[X,Y];\ deg_X T \leq r-1 \ \text{ and } deg_Y(T) \leq s-1\}$$

where deg_X (respectively deg_Y) is the degree in the variable X (respectively Y).

Let $P_1(X) = X^r - a_0X^{r-1} - \cdots - a_{r-1}$ and $P_2(Y) = Y^s - b_0Y^{s-1} - \cdots - b_{s-1}$. Note that for all $T(X,Y) \in \mathbb{R}[X,Y]$ there exist $Q_1(X,Y)$, $Q_2(X,Y)$ and $R(X,Y)$ with $deg_X R(X,Y) \leq r-1$ and $deg_Y R(X,Y) \leq s-1$ such that

$$T(X,Y) = Q_1(X,Y)P_1(X) + Q_2(X,Y)P_2(Y) + R(X,Y). \tag{7}$$

Set

$$\Delta_{r,\,s}(K) = \{H := T - Q_1P_1 - Q_2P_2 \,/\, T \in \Delta(K) \text{ and } Q_1, Q_2 \in \mathbb{R}[X,Y]\} \cap \mathbb{R}_{r-1,\,s-1}[X,Y]$$

Thus we derive from (4)-(5) that we have $L(T(X,Y)) = L(R(X,Y))$, where $R(X,Y)$ is given by (7). Hence, using equation (5), we have the following property.

Proposition 3 *Let $\{W_{n,m}\}_{n,m\geq 0}$ be a sequence (1) and L be the associated linear form defined by (4). Then we have $L(T) \geq 0$ for all $T \in \Delta(K)$ if and only if $L(R) \geq 0$ for all $R \in \Delta_{r,s}(K)$.*

Let $P(K)$ be the convex set of linear forms L on $\mathbb{R}[X,Y]$ such that $L(1) = 1$ and $L(T) \geq 0$ for all $T \in \Delta(K)$. Let $Q \in \mathbb{R}[X,Y]$, then we have $S = (\|Q\|1 \pm Q)|_K \geq 0$, where $\|Q\| = \sup_{(x,y)\in K} |Q(x,y)|$. This implies that $L(S) = \|Q\| \pm L(Q) \geq 0$, thus $|L(Q)| \leq \|Q\|$. Using Hahn-Banach Theorem and Proposition 2 we get the following reduction Lemma.

Lemma 2 *Let $\{W_{n,m}\}_{n,m\geq 0}$ be a sequence (1) and L be the associated linear form defined by (4). Suppose that $L \in P(K)$. Then the following are equivalent.*
(i) There exists a probability measure μ on K such that $L(T) = \int T d\mu$ for all $T \in \mathbb{R}[X,Y]$.
(ii) There exists a probability measure μ on K such that $L(T) = \int T d\mu$ for all $R \in \mathbb{R}_{r,s}[X,Y]$.

Thus we have the following result.

Proposition 4 *Let $\{W_{n,m}\}_{n,m\geq 0}$ be a sequence (1) and μ be a positive Borel measure on K. Then the following are equivalent.*
(i) μ is a representing measure of $\{W_{n,m}\}_{n,m\geq 0}$ on K.
(ii) $L(T) = \int_K T(x,y)d\mu(x,y) \geq 0$ for all $T \in \Delta(K)$.
(iii) $L(R) = \int_K R(x,y)d\mu(x,y) \geq 0$ for all $R \in \Delta_{r,s}(K)$.

3.2 Links with positive matrices.

Let , $= \{\alpha_{i\,j}\}_{i,j\geq 0}$ be a sequence of real numbers. To any polynomial $T(X,Y) = \sum_{0\leq i\leq k, 0\leq j\leq p} a_{ij}X^iY^j$ of $I\!R[X,Y]$, we associate the following infinite matrix introduced by Cassier (see [4] and [5]).

$$M_T(,) = [m_{(i_1,j_1),(i_2,j_2)}] \quad \text{where} \quad m_{(i_1,j_1),(i_2,j_2)} = \sum_{(k_1,k_2)} a_{k_1,k_2}\alpha_{i_1+i_2+k_1,j_1+j_2+k_2}.$$

Using Lemma 2 (of reduction) we obtain the following properties.

Proposition 5 *Let $\{W_{n,\,m}\}_{n,m\geq 0}$ be a sequence (1) and K be a compact subset of $I\!R^2$. The following are equivalent.*
(i) $\{W_{n,\,m}\}_{n,m\geq 0}$ is a moment sequence of a Borelean positive measure on K.
(ii) $\{W_{n,\,m}\}_{0\leq n\leq 2r, 0\leq m\leq 2s}$ is a moment sequence of a Borelean positive measure μ of finite support.
(iii) The matrix $m_1(\{W_{n,\,m}\}_{0\leq n\leq k, 0\leq m\leq p}) = [m_{i,\,j}]$ is positive for any k, p, where

$$m_{i,\,j} = (w_{i_1+i_2,j_1+j_2})_{0\leq i_1,i_2\leq k, 0\leq j_1,j_2\leq p} \quad i = (i_1,j_1), j = (i_2,j_2).$$

(iv) The matrix $m_1(\{W_{n,\,m}\}_{0\leq n\leq r-1, 0\leq m\leq s-1}) = [m_{i,\,j}]$ is positive, where

$$m_{i,\,j} = (w_{i_1+i_2,j_1+j_2})_{0\leq i_1,i_2\leq r-1, 0\leq j_1,j_2\leq s-1}.$$

Proof.

- The equivalence $(i) \Longleftrightarrow (ii)$ is due to Lemma 2 (of reduction).
- For $(i) \Longleftrightarrow (iii)$ see [4] or [5].
- $(ii) \Longleftrightarrow (iv)$.

  - $(ii) \Longrightarrow (iv)$ is obtained from a direct computation by considering the linear positive form $L(f) = \int_K f(x,y)d\mu(x,y)$.

  - $(iv) \Longrightarrow (ii)$. The first relation of (1) means that for any fixed m the sequence $\{W_{n,\,m}\}_{n\geq 0}$ is a linear recursive sequence of order r. Hence there exist r real numbers $C_{j,m}$ $(0 \leq j \leq r-1)$ such that

$$W_{n,m} = C_{0,m}\lambda_0^n + C_{1,m}\lambda_1^n + \cdots + C_{r-1,m}\lambda_{r-1}^n,$$

for any $n \geq 0$ (see [8] for example). The second relation of (1) implies that for any fixed j the sequence $\{C_{j,m}\}_{m\geq 0}$ is a recursive sequence of order s. Thus $C_{j,m} = d_{j,0}\beta_0^m + d_{j,1}\beta_1^m + \cdots + d_{j,s-1}\beta_{s-1}^m$, where $d_{j,0}$, $d_{j,1}$,...,$d_{j,s-1}$ are constant real numbers. Hence we derive that for any n, $m \geq 0$ we have $W_{n,m} = \sum_{0\leq i\leq r-1, 0\leq j\leq s-1} a_{i,j}\lambda_i^n\beta_j^m$. This implies that

$$W_{n,m} = L(X^nY^m) = \int_K x^n y^m d\mu(x,y), \tag{8}$$

where $\mu = \sum_{0\leq i\leq r-1,0\leq j\leq s-1} a_{i,j}\delta_{(\lambda_i,\beta_j)}$ and $K = [\lambda_0, \lambda_{r-1}] \times [\beta_0, \beta_{s-1}]$. To prove that μ is positive it suffice to have $a_{i,j} \geq 0$ in the expression (8). Consider $f_{i,j} = \frac{\prod_{(k,p)\neq(i,j)} (X-\lambda_k)(Y-\beta_p)}{\prod_{(k,p)\neq(i,j)} (\lambda_i-\lambda_k)(\beta_j-\beta_p)} \in \mathbb{R}[X,Y]$. Hence $f_{i,j}(\lambda_i, \beta_j) = 1$ and $f_{i,j}(\lambda_k, \beta_p) = 0$ for any $(k,p) \neq (i,j)$ and we have

$$L(f_{i,j}^2) = \int_K f_{i,j}^2 d\mu = a_{i,j} =< Mf_{i,j}, f_{i,j} >\geq 0.$$

Hence μ is a Borelean positive measure of finite support.□

3.3 Weakly multiplicative case.

Let $\{W_{n,m}\}_{n,m\geq 0}$ be a sequence (1) and consider the linear form $L : \mathbb{R}[X,Y] \longrightarrow \mathbb{R}$ defined by $L(X^nY^m) = W_{n,m}$. We say that L is weakly multiplicative if $L(X^nY^m) = L(X^n)L(Y^m)$, for all $n, m \geq 0$. Thus we have $W_{0,0} = 1$ and $W_{n,m} = U_nV_m$ where $U_n = W_{n,0}$ and $V_m = W_{0,m}$. We can easily derive that $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{m\geq 0}$ are defined by classical linear recurrence relations of order r and s respectively.

Proposition 6 *Let $\{W_{n,m}\}_{n,m\geq 0}$ be a sequence (1) and K be a compact of $\mathbb{R}^2$. Suppose that $W_{n,m} = L(X^nY^m) = L(X^n)L(Y^m)$ for all n, $m \geq 0$. Then the following are equivalent.*
(i) $\{W_{n,m}\}_{n,m\geq 0}$ is a moment sequence of positive Borel measure μ with supp(μ) $\subset K$.
(ii) $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{m\geq 0}$, where $U_n = W_{n,0}$, $V_m = W_{0,m}$, are two sequences of moments of positive Borel measures ν_1, ν_2 on $\mathbb{R}$ (respectively).
In this case we have $\mu = \nu_1 \otimes \nu_2$.

Proof.

- $(i) \Longrightarrow (ii)$. We have $W_{n,m} = L(X^nY^m) = \int_K x^ny^m d\mu(x,y)$. We identify $\mathbb{R}[X]$ to a subspace of $\mathbb{R}[X,Y]$ and set $L_1 = L_{|\mathbb{R}[X]}$. Hence $U_n = L_1(X^n) = \int_K x^n d\mu(x,y)$, and for any $S \in \mathbb{R}[X]$ such that $S_{|K} \geq 0$ we have $L_1(S) \geq 0$. By a classical process of extension ([4], [5], [9] for example) we get from L_1 a positive Borel measure ν_1 on K_1; the projection of K in $\mathbb{R} \equiv \mathbb{R} \times \{0\}$, such that $U_n = \int_{K_1} x^n d\nu_1(x)$.

  Using the same argument we exhibit a positive Borel measure ν_2 on K_2; the projection of K in $\mathbb{R} \equiv \{0\} \otimes \mathbb{R}$, such that $U_n = \int_{K_1} x^n d\nu_1(x)$.

- $(ii) \Longrightarrow (i)$. Let ν_1, ν_2 be the representing measure of $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{m\geq 0}$ and $K_j = supp(\nu_j)$ $(j = 1,\, 2)$. Consider the positive Borelean measure $\mu = \nu_1 \times \nu_2$ on $\mathbb{R}^2$. Then we can easily verify that $\{W_{n,\,m}\}_{n,m\geq 0}$ is a moment sequence of μ on $K = K_1 \times K_2$. □

Let $\{W_{n,\,m}\}_{n,m\geq 0}$ be a sequence (1) such that $W_{n,m} = U_n V_m$. Consider the two Hankel matrices

$$H_U(r-1) = [U_{i+j}]_{0\leq i,j\leq r-1} \quad \text{and} \quad H_V(s-1) = [V_{i+j}]_{0\leq i,j\leq s-1}.$$

From [2] and Proposition 6 we derive the following,

Proposition 7 *Let $\{W_{n,\,m}\}_{n,m\geq 0}$ be a sequence (1) such that $W_{n,m} = U_n V_m$. Let μ be a positive Borel measure on a compact subset K of $\mathbb{R}^2$. Then the following are equivalent.*
(i) $\{W_{n,\,m}\}_{n,m\geq 0}$ is a moment sequence of μ.
(ii) $\{W_{n,\,m}\}_{0\leq n\leq 2r, 0\leq m\leq 2s}$ is a moment sequence of μ.
(iii) $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{m\geq 0}$ are two sequences of moments of positive Borel measures ν_1, ν_2 on $\mathbb{R}$ (respectively).
(iv) $H_U(r-1) \geq 0$ and $H_V(s-1) \geq 0$.

4 Concluding remarks and Example.

4.1 Application to Curto-Fialkow schemes.

Let $,_{2r} = \{W_{n,m}\}_{0\leq n+m\leq 2r}$ be a sequence of real numbers. Suppose that $,_{2r}$ is a sequence of moment of positive measure μ on $\mathbb{R}^2$. Then μ is a representing measure of $,_{2r}$ and

$$W_{j,k} = \int x^j y^k d\mu(x,y),$$

for $0 \leq j+k \leq 2r$. Let $\mathbb{C}[X,Y]_{2r}$ be the $\mathbb{C}$-vector space of polynomials in two variables X, Y of degree $\leq 2r$. Consider the bilinear form $\phi : \mathbb{C}[X,Y]_{2r} \to \mathbb{C}$ defined as follows

$$\phi(X^n Y^m) = W_{n,m},$$

for any n, m such that $0 \leq n+m \leq 2r$. Let $\{U_{n,m}\}_{0\leq n+m\leq 2r}$ be the sequence defined by

$$U_{n,m} = \phi([X-iY]^n [X+iY]^m),$$

where $i^2 = -1$ and $0 \leq n+m \leq 2r$. Using [6](see section 6.2.2) and also [13], we derive that $,_{2r} = \{W_{n,m}\}_{0\leq n+m\leq 2r}$ is a truncated moment sequence on $\mathbb{R}^2$ if and only if $\{U_{n,m}\}_{0\leq n+m\leq 2r}$ is a truncated moment sequence on $\mathbb{C}$.

Suppose that $\{U_{n,m}\}_{0\leq n+m\leq 2r}$ is a truncated moment sequence on $\mathbb{C}$. It is also established in [6] that $\{U_{n,m}\}_{0\leq n+m\leq 2r}$ had a representing r-atomic discrete measure ν on $\mathbb{C}$, where we have

$$\nu = \sum_{k=0}^{r-1} a_k \delta_{\alpha_k + i\beta_k}.$$

If we set $d\mu(x,y) = d\nu(x+iy, x-iy)$. We derive that μ is a representing measure of $\{W_{n,m}\}_{0\le n+m\le 2r}$ on $\mathbb{R}^2$ and

$$W_{n,m} = \int (\frac{z+\overline{z}}{2})^n (\frac{z-\overline{z}}{2i})^m d\nu(z,\overline{z}) = \int x^n y^m d\mu(x,y) = \sum_{k=0}^{r-1} a_k \alpha_k^n \beta_k^m.$$

Hence μ is a discrete representing measure of $\{W_{n,m}\}_{0\le n+m\le 2r}$ on $\mathbb{R}^2$. Thus we have the following property.

Proposition 8 *Let $\{W_{n,m}\}_{0\le n+m\le 2r}$ be a truncated moment sequence of positive measure μ on $\mathbb{R}^2$. Then there exists $\{V_{n,m}\}_{n,m\ge 0}$ a full moment sequence (1) on $\mathbb{R}^2$ such that $V_{n,m} = W_{n,m}$ for all n, m with $0 \le n+m \le 2r$.*

The following corollary shows the important role that sequences (1) play in the treatment of the truncated moment problem.

Corollary 1 *Let $,_{2r} = \{W_{n,m}\}_{0\le n+m\le 2r}$ be real numbers. The following are equivalent.*

1. *$,_{2r}$ admit a representing measure.*
2. *$,_{2r}$ admit a finitely atomic representing measure.*

4.2 Example.

We give here an example of truncated sequence in the case of $r = s = 2$ going through all computations for generating measure. Let $\{W_{0,0}(=1), W_{0,1}, W_{1,0}, W_{1,1}\}$ given real numbers and $\{W_{n,m}\}_{n\ge 0, m\ge 0}$ a sequence (1) associated to the polynomials $P_1(X) = (X-\lambda_0)(X-\lambda_1)$ and $P_2(Y) = (Y-\beta_0)(Y-\beta_1)$ with $\lambda_0 \ne \lambda_1$ and $\beta_0 \ne \beta_1$. A measure $\mu = C_{0,0}\delta_{\lambda_0,\beta_0} + C_{1,0}\delta_{\lambda_1,\beta_0} + C_{0,1}\delta_{\lambda_0,\beta_1} + C_{1,1}\delta_{\lambda_1,\beta_1}$ is a generating measure for $W_{n,m}$ if and only if $(C_{i,j})_{i\le 1, j\le 1}$ satisfy the following system of equations

$$\begin{cases} C_{0,0} + C_{1,0} + C_{0,1} + C_{1,1} & = & W_{0,0}(=1) \\ C_{0,0}\beta_0 + C_{1,0}\beta_0 + C_{0,1}\beta_1 + C_{1,1}\beta_1 & = & W_{0,1} \\ C_{0,0}\lambda_0 + C_{1,0}\lambda_1 + C_{0,1}\lambda_0 + C_{1,1}\lambda_1 & = & W_{1,0} \\ C_{0,0}\beta_0\lambda_0 + C_{1,0}\beta_0\lambda_1 + C_{0,1}\beta_1\lambda_0 + C_{1,1}\beta_1\lambda_1 & = & W_{1,1}. \end{cases}$$

The determinant of the preceding system of equations is $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \beta_0 & \beta_0 & \beta_1 & \beta_1 \\ \lambda_0 & \lambda_1 & \lambda_0 & \lambda_1 \\ \beta_0\lambda_0 & \beta_0\lambda_1 & \beta_1\lambda_0 & \beta_1\lambda_1 \end{vmatrix} = -((\lambda_1 - \lambda_0)(\beta_1-\beta_0))^2 \ne 0$, thus we get the existence of μ. Suppose that the solutions $C_{i,j}$ $(0 \le i,j \le 1)$ are nonnegative, then the measure μ is representing for $\{W_{n,m}\}_{n\ge 0, m\ge 0}$ on K.

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References

[1] N.I. Akhiezer, *The classical moment problem*, Hafner Publ.co.(1965), New York.

[2] R. Ben Taher, M. Rachidi and E.H. Zerouali, *Recursive subnormal completion and the truncated moment problem*, Bulletin of London Mathematical Society. (accepted).

[3] Ch. Berg and J.P.R. Christensen, *Density questions in the classical theory of moments*, Ann. Inst. Fourier,Grenoble, 31.3 (1981), 99-114.

[4] G. Cassier, *Problème des moments n-dimensionnel mesures quasi-spectrales et semi-groupes*, Thèse de troisieme cycle, Universite Claude Bernard-Lyon 1 (1983).

[5] G. Cassier, *Problème des moments sur un compact de* $\mathbb{R}^n$ *et décomposition des polynômes à plusieurs variables*, J. Funct. Anal. 58 (1984), 254-266.

[6] R.E. Curto and L. A. Fialkow, *Solution of the truncated complex moment problem for flat extension*, Mem. Amer. Math. Soc. 568, Vol. 119 (1996).

[7] R.E. Curto and L. A. Fialkow, *Flat extensions of positive moment matrices: Recursively generated relations*, Mem. Amer. Math. Soc. 648, Vol. 136 (1998).

[8] F. Dubeau, W. Motta, M. Rachidi and O. Saeki, *On weighted r-generalized Fibonacci sequence*, Fibonacci Quart. 35, (1997), 102-110.

[9] B. El Wahbi and M. Rachidi, *r-generalized Fibonacci sequences and Hausdorff moment problem*, Fibonacci Quart. (accepted).

[10] B. El Wahbi and M. Rachidi, *r-generalized Fibonacci sequences and Linear moment problem*, Fibonacci Quart. (accepted).

[11] L. Fialkow, *Positivity, extensions and the truncated complex moment problem*, Multivariable operator theory, Comtemporary Mathematics 185, Amer. Math. Soc., Providence, RI (1995), 133-150.

[12] K. Schmudgen, *The K-moment problem for semi-algebraic sets*, Math. Ann. 289 (1991), 203-205.

[13] J. Stochel and F.H. Szafraniec, *The complex moment problems and subnormality: A polar decomposition approach* , J. Funct. Anal. 159 (1998), 432-491.

[14] F.-H. Vasilescu, *Moment Problems for multi-sequences of operators*, J. Math. Anal. and Appli. 219 (1998), 246-259.