Available at: http://www.ictp.trieste.it/~pub_off IC/98/49

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

A CLASSIFICATION OF TWO-DIMENSIONAL HYPOREDUCTIVE TRIPLE ALGEBRAS

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Abstract

Two-dimensional hyporeductive triple algebras (h.t.a) are investigated. Using the K. Yamaguti's approach for the classification of two-dimensional Lie triple systems (L.t.s), a classification of two-dimensional h.t.a is suggested.

MIRAMARE – TRIESTE
May 1998

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1 Introduction

Hyporeductive algebras were introduced (see [5],[6]) as an infinitesimal tool for the study of smooth hyporeductive loops which are a generalization both of smooth Bol loops and smooth reductive loops (i.e. smooth A-loops with monoalternative property [7]). By way of fact it is shown that the fundamental vector fields of any smooth hyporeductive loop constitute an algebra called a hyporeductive algebra of vector fields. Further (see [2],[3]) this notion has been extended to the one of abstract hyporeductive triple algebras (h.t.a for brevity) meaning a finite-dimensional linear space with two binary and one ternary operations satisfying some specific identities. It turns out that hyporeductive algebras generalize Bol algebras and Lie triple algebras (see [7],[9] about Bol and Lie triple algebras).
In this paper we consider 2-dimensional h.t.a and the prospecting of explicit expressions of operations for such algebras led us to their classification. In section 2 some specifications of hyporeductive algebras are given and the classification theorem is stated. Section 3 is devoted to its proof (this proof gives the classification process). As a consequence it is pointed out that there is no proper 2-dimensional Lie triple algebra. We conclude with some remarks in section 4.

2 Background and results.

Hyporeductive algebras were originally introduced ([5],[6]) as algebras of vector fields on a smooth finite-dimensional manifold, satisfying a specific condition. More exactly it was given the following
Definition 1 [5],[6]. *A linear space V of vector fields on an n-dimensional manifold with a singled out point e, satisfying*

$$[X,[Y,Z]] = [X, a(Y,Z)] + r(X;Y,Z) \tag{1}$$

is called a hyporeductive algebra of vector fields with determining operations a and r, if $dim\{X(e) : X \in V\} = n$.

Obviously $a(Y,Z)$ is a bilinear operation and $r(X;Y,Z)$ a trilinear one on V and $a(Y,Y) = 0$, $r(X;Y,Y) = 0$. The relation (1) we called (see [2],[3]) the *hyporeductive condition* for algebras of vector fields. Considering a hyporeductive algebra as a tangent algebra at the identity e of a smooth hyporeductive loop it is shown ([5]) that a hyporeductive algebra may be viewed as an algebra with two binary operations $a(X,Y)$, $T_e(X,Y) = [X,Y](e)$ and one ternary operation $r(Z;X,Y)$ and then working out the Jacobi identities in the corresponding enveloping Lie algebra, one can get the whole system of identities linking the operations a, T_e, r. A similar construction is carried out in [2],[3], where instead of $T_e(X,Y)$ the operation $b(X,Y) = [X,Y](e) - a(X,Y)$ is introduced (this is made in connection with a more suitable differential geometric interpretation of a hyporeductive algebra of vector fields and then the system of identities above mentioned constitutes the integratibility criteria of the structure equations of the affinely connected smooth manifold associated with a local smooth hyporeductive loop).
This led us to introduce the notion of (abstract) hyporeductive triple algebras (h.t.a):

Definition 2 [2],[3]. *Let $\mathcal{V}$ be a finite-dimensional linear space. Assume that on $\mathcal{V}$ are defined two binary anticommutative operations $(.), (*)$ and one ternary operation $< -; -, - >$ skew-symmetric with respect to the two last variables. We say that the algebra $(\mathcal{V}, ., *, < -; -, - >)$ is an abstract h.t.a if, for any $\xi, \eta, \zeta, \kappa, \chi, \theta$ in $\mathcal{V}$ the following identities hold:*

$$\sigma \ \{ \ \xi \, . \, (\eta \, . \, \zeta) - < \xi; \eta, \zeta > \ \} = 0 \tag{2}$$

$$\sigma \ \{ \ \zeta \ * \ (\xi \, . \, \eta) \ \} = 0 \tag{3}$$

$$\sigma \ \{ \ < \theta; \zeta, \ \xi \, . \, \eta > \ \} = 0 \tag{4}$$

$$\begin{aligned} \kappa. < \zeta; \xi, \eta > \quad & - \zeta. < \kappa; \xi, \eta > + < \zeta.\kappa; \xi, \eta > = \\ & < \xi * \eta; \zeta, \kappa > - < \zeta * \kappa; \xi, \eta > \\ & + \zeta * < \kappa; \xi, \eta > - \kappa * < \zeta; \xi, \eta > \\ & + (\xi * \eta) * (\zeta * \kappa) + (\xi * \eta).(\zeta * \kappa) \end{aligned} \tag{5}$$

$$\begin{aligned} \chi.(\kappa. < \zeta; \xi, \eta > \quad & - \zeta. < \kappa; \xi, \eta > + < \zeta.\kappa; \xi, \eta >) \\ & + << \chi; \xi, \eta >; \zeta, \kappa > - << \chi; \zeta, \kappa >; \xi, \eta > \\ & + < \chi; \zeta, < \kappa; \xi, \eta >> - < \chi; \kappa, < \zeta; \xi, \eta >> = 0 \end{aligned} \tag{6}$$

$$\chi \ * \ (\kappa. < \zeta; \xi, \eta > - \zeta. < \kappa; \xi, \eta > + < \zeta.\kappa; \xi, \eta >) = 0 \tag{7}$$

$$< \theta; \chi, \ \ \kappa. < \zeta; \xi, \eta > - \zeta. < \kappa; \xi, \eta > + < \zeta.\kappa; \xi, \eta >> = 0, \tag{8}$$

where σ denotes the cyclic sum with respect to ξ, η, ζ.

The study of h.t.a is more handy if they are given in terms of identities as in the definition above. For instance, we notice that if in (2)-(8) we set $\xi.\eta = 0$ for any ξ, η of $\mathcal{V}$ then we get the determining identities of Bol algebras. On the other hand, setting $\xi * \eta = 0$, we get Lie triple algebras (i.e. generalized Lie triple systems) and if, moreover, we put $\xi.\eta = 0$ then we obtain Lie triple systems (L.t.s).

The question naturally arises whether there exist proper abstract h.t.a. The answer to this problem is easier to seek among low-dimensional h.t.a because of the specific properties of operations $(.), (*)$ and $< -; -, - >$. Thus we are led to the study of two-dimensional h.t.a, that is to find the explicit expressions for their determining operations. Our investigations led us to the following classification theorem for such h.t.a.

Theorem. *There exist proper 2-dimensional h.t.a. Moreover any 2-dimensional h.t.a is isomorphic to one of the algebras described in the table below:*

type	$x_1 * x_2$	$x_1.x_2$	$< x_1; x_1, x_2 >$	$< x_2; x_1, x_2 >$	observations
I	$\delta x_1 + \epsilon x_2$	0	σx_2	ρx_1	
II	$\delta x_1 + \epsilon x_2$	0	$\lambda(x_1 + \sigma x_2)$	$\lambda(\rho x_1 - x_2)$	$\lambda \neq 0$
III	δx_1	$\alpha x_1 + \beta x_2$	0	$\rho x_1 + \sigma x_2$	$\alpha \neq 0, \beta \neq 0,$ $\delta \neq 0, \sigma \neq 0,$ $\alpha\sigma - \beta\rho \neq 0$
IV	δx_1	βx_2	0	$\rho x_1 + \sigma x_2$	$\beta \neq 0, \delta \neq 0,$ $\rho \neq 0, \sigma \neq 0$
V	$-\alpha x_1$	αx_1	0	$\lambda(\rho x_1 + x_2)$	$\alpha \neq 0, \lambda \neq 0$
VI	ϵx_2	$\alpha x_1 + \beta x_2$	$\rho x_1 + \sigma x_2$	0	$\alpha \neq 0, \beta \neq 0,$ $\epsilon \neq 0, \rho \neq 0$ $\alpha\sigma - \beta\rho \neq 0$
VII	ϵx_2	αx_1	$\rho x_1 + \sigma x_2$	0	$\alpha \neq 0, \epsilon \neq 0,$ $\rho \neq 0, \sigma \neq 0$
VIII	$-\beta x_2$	βx_2	$\lambda(x_1 + \sigma x_2)$	0	$\beta \neq 0, \lambda \neq 0$

3 Proof.

First we shall prove the following two lemmas.

Lemma 1. *If $\{x_1, x_2\}$ is a basis of a 2-dimensional h.t.a $\mathcal{V}$, then the determining identities (2)-(8) of abstract h.t.a has the following type:*

$$J(x_1, x_2) \quad - < x_1.x_2; x_1, x_2 > + x_1. < x_2; x_1, x_2 > - x_2. < x_1; x_1, x_2 >= 0 \tag{9}$$

$$x_i.J(x_1, x_2) \quad - < x_i; x_1, < x_2; x_1, x_2 >> + < x_i; x_2, < x_1; x_1, x_2 >>= 0 \tag{10}$$

$$x_i * J(x_1, x_2) = 0 \tag{11}$$

$$< x_j; x_i, J(x_1, x_2) = 0, \tag{12}$$

where $J(x_1, x_2) = x_1 * < x_2; x_1, x_2 > - x_2 * < x_1; x_1, x_2 >$ and $i, j = 1, 2$.

Proof. With respect to the basis $\{x_1, x_2\}$, (2),(3) and (4) are clearly satisfied identically. Next the left-hand side of (5) now reads $x_i. < x_j; x_1, x_2 > - x_j. < x_i; x_1, x_2 > + < x_j.x_i; x_1, x_2 >$ while the right-hand side reads $< x_1 * x_2; x_j, x_i > - < x_j * x_i; x_1, x_2 > + x_j * < x_i; x_1, x_2 > - x_i * < x_j; x_1, x_2 > + (x_1 * x_2) * (x_j * x_i) + (x_1 * x_2).(x_j * x_i)$, with $i, j = 1, 2$. Furthermore, appealing to the skew-symmetry of operations $(.), (*), (< -; -, - >)$ one observes that the identity (5) gets the form $x_1 * < x_2; x_1, x_2 > - x_2 * < x_1; x_1, x_2 >=< x_1.x_2; x_1, x_2 > - x_1. < x_2; x_1, x_2 > + x_2. < x_1; x_1, x_2 >$ so that we obtain (9).

In view of (9), the identities (7) and (8) are straightforward transformed into (11) and (12) respectively.

Finally, and again appealing to (9), we work out the identity (6) as follows: we replace $\xi, \eta, \zeta, \kappa, \chi$ by x_1, x_2, x_k, x_j, x_i respectively where $i, j, k = 1, 2$ and then keeping in mind (9), we see that (6) gets the form

$x_i.(x_1 * < x_2; x_1, x_2 > - x_2 * < x_1; x_1, x_2 >) + < x_i; x_2, < x_1; x_1, x_2 >> - < x_i; x_1,$ $< x_2; x_1, x_2 >> = 0$ that is, we get (10). $\diamondsuit$

Lemma 2. *If $\{x_1, x_2\}$ is a basis of a 2-dimensional h.t.a with zero binary operations, then $< x_1; x_1, x_2 >= \alpha x_1 + \beta x_2$, $< x_2; x_1, x_2 >= \gamma x_1 - \alpha x_2$, where α, β, γ are real numbers.*

Proof. If $x_1.x_2 = 0$ and $x_1 * x_2 = 0$ then (9)-(12) give

$$< x_i; x_1, < x_2; x_1, x_2 >> - < x_i; x_2, < x_1; x_1, x_2 >>= 0, \tag{13}$$

$i = 1, 2$. Now let $< x_s; x_1, x_2 >= r_s^1 x_1 + r_s^2 x_2$, $s = 1, 2$. Then, by the skew-symmetry, (13) reads $r_2^2 < x_i; x_1, x_2 > - r_1^1 < x_i; x_2, x_1 >= 0$ that is $(r_1^1 + r_2^2) < x_i; x_1, x_2 >= 0$. So that if $r_1^1 + r_2^2 \neq 0$, then $r_i^1 = r_i^2 = 0$ i.e. $r_1^1 = r_1^2 = 0$, $r_2^1 = r_2^2 = 0$ and this leads to a contradiction. $\diamondsuit$

Remark. Lemma 2 is actually another version of lemma 5.1 in [8]. This is expected since h.t.a with zero binary operations are precisely L.t.s. Therefore the Yamaguti's classification for 2-dimensional L.t.s (see also [4], p.312) is included in 2-dimensional h.t.a of types I and II.

We now set ourself about the proof of the theorem.

Proof of the theorem. Since $J(x_1, x_2)$ seems to be conclusive for the system (9)-(12), it would be rational to work out this system according to whether $J(x_1, x_2)$ is zero or not. Thus we shall consider the two cases $J(x_1, x_2) = 0$ and $J(x_1, x_2) \neq 0$.

A/ $J(x_1, x_2) = 0$.

If we set $< x_s; x_1, x_2 >= r_s^1 x_1 + r_s^2 x_2$, $s = 1, 2$ then we get $J(x_1, x_2) = (r_1^1 + r_2^2) x_1 * x_2$ and $J(x_1, x_2) = 0$ amounts to $r_1^1 + r_2^2 = 0$ or $x_1 * x_2 = 0$.

1/. Let $r_1^1 + r_2^2 = 0$ and $x_1 * x_2$ be any.

a). If $r_1^1 = r_2^2 = 0$, then $< x_1; x_1, x_2 >= \sigma x_2$ and $< x_2; x_1, x_2 >= \rho x_1$, where σ and ρ are some reals. Therefore from (9)-(12) we draw $< x_1.x_2; x_1, x_2 >= 0$ i.e. $x_1.x_2 = 0$. Hence we get the following first set of values for $< x_s; x_1, x_2 >, x_1 * x_2$ and $x_1.x_2$:

$$x_1 * x_2 = \delta x_1 + \epsilon x_2, \qquad < x_1; x_1, x_2 >= \sigma x_2,$$
$$< x_2; x_1, x_2 >= \rho x_1, \; x_1.x_2 = 0 \tag{14}$$

(with any reals $\delta, \epsilon, \sigma, \rho$).

b). If $r_1^1 = -r_2^2$ with $r_1^1 \neq 0, r_2^2 \neq 0$, then from (9) one infers $< x_1.x_2; x_1, x_2 >= 0$, whence $x_1.x_2 = 0$ so that lemma 2 gives the expressions for $< x_s; x_1, x_2 >$, $s = 1, 2$. Thus we have the following set of operations values:

$$x_1 * x_2 = \delta x_1 + \epsilon x_2, \qquad < x_1; x_1, x_2 >= \lambda(x_1 + \sigma x_2),$$
$$< x_2; x_1, x_2 >= \lambda(\rho x_1 - x_2), \; x_1.x_2 = 0 \tag{15}$$

(with any reals $\delta, \epsilon, \sigma, \rho$ and $\lambda \neq 0$).

2/. Now let $r_1^1 + r_2^2 \neq 0$ and $x_1 * x_2 = 0$.

Under such assumptions the system (9)-(12) leads to the following system :

$$< x_1.x_2; x_1, x_2 >= (r_1^1 + r_2^2) x_1.x_2 \tag{16}$$

$$< x_i; x_1, < x_2; x_1, x_2 >> - < x_i; x_2, < x_1; x_1, x_2 >>= 0. \tag{17}$$

Whence, from the proof of lemma 2 (since (17) is (13)), we get $< x_i; x_1, x_2 >= 0, i = 1, 2$ and then $x_1.x_2 = 0$. Thus we obtain the trivial algebra (i.e. the algebra with zero operations) that is included in the algebras of type (14).

B/. $J(x_1, x_2) \neq 0$.

Then $r_1^1 + r_2^2 \neq 0$ and $x_1 * x_2 \neq 0$. If we set $x_1 * x_2 = b_1 x_1 + b_2 x_2$ and $x_1.x_2 = a_1 x_1 + a_2 x_2$, one finds that the system (9)-(12) splits as follows:

$$(r_1^1 + r_2^2) x_1 * x_2 = (a_2 r_2^1 - a_1 r_2^2) x_1 + (a_1 r_1^2 - a_2 r_1^1) x_2 \tag{18}$$

$$\begin{aligned} (r_1^1 + r_2^2) x_1.(x_1 * x_2) &= < x_1; x_1, < x_2; x_1, x_2 >> - \\ &- < x_1; x_2, < x_1; x_1, x_2 >> \end{aligned} \tag{19}$$

$$x_1 * (x_1 * x_2) = 0 \tag{20}$$

$$< x_j; x_1, x_1 * x_2 >= 0, \tag{21}$$

(for $i = 1$), $j = 1, 2$, and

$$(r_1^1 + r_2^2) x_1 * x_2 = (a_2 r_2^1 - a_1 r_2^2) x_1 + (a_1 r_1^2 - a_2 r_1^1) x_2 \tag{22}$$

$$\begin{aligned} (r_1^1 + r_2^2) x_2.(x_1 * x_2) &= < x_2; x_1, < x_2; x_1, x_2 >> - \\ &- < x_2; x_2, < x_1; x_1, x_2 >> \end{aligned} \tag{23}$$

$$x_2 * (x_1 * x_2) = 0 \tag{24}$$

$$< x_j; x_2, x_1 * x_2 >= 0, \tag{25}$$

(for $i = 2$), $j = 1, 2$.

1/. Consider the system (18)-(21). From (20) and (18) we conclude that $a_1 r_1^2 - a_2 r_1^1 = 0$ and $a_2 r_2^1 - a_1 r_2^2 \neq 0$ (since $x_1 * x_2 \neq 0$). Thus we have

$$\begin{aligned} x_1 * x_2 = \delta x_1 (\delta \neq 0), \qquad & < x_1; x_1, x_2 >= 0 \text{ (from (19))}, \\ & < x_2; x_1, x_2 > \neq 0, \ x_1.x_2 \neq 0 \text{ (from (18))}. \end{aligned} \tag{26}$$

Next, discussing on the coefficients a_1, a_2, r_2^1, r_2^2 and their compatibility with respect to (18)-(21), (26), the set of values (26) splits as follows:

$$\begin{aligned} x_1 * x_2 = \delta x_1, \qquad & < x_1; x_1, x_2 >= 0, \\ & < x_2; x_1, x_2 >= \rho x_1 + \sigma x_2, \ x_1.x_2 = \beta x_2, \end{aligned} \tag{27}$$

(with $\delta \neq 0, \ \rho \neq 0, \ \sigma \neq 0, \ \beta \neq 0$),

$$\begin{aligned} x_1 * x_2 = -\alpha x_1, \qquad & < x_1; x_1, x_2 >= 0, \\ & < x_2; x_1, x_2 >= \lambda (\rho x_1 + x_2), \ x_1.x_2 = \alpha x_1, \end{aligned} \tag{28}$$

(with $\alpha\neq 0,\ \lambda\neq 0$),

$$x_1 * x_2 = \delta x_1, \qquad < x_1; x_1, x_2 >= 0,$$
$$< x_2; x_1, x_2 >= \rho x_1 + \sigma x_2,\ x_1.x_2 = \alpha x_1 + \beta x_2, \tag{29}$$

(with $\alpha\neq 0,\ \beta\neq 0\ \delta\neq 0,\ \sigma\neq 0$ and $\alpha\sigma - \beta\rho\neq 0$).
2/. Consider now the system (22)-(25). One may notice that this system is "symmetric" to the preceeding (18)-(21). Therefore, proceeding as in 1/, we get the following three sets of operations values:

$$x_1 * x_2 = \epsilon x_2, \qquad < x_1; x_1, x_2 >= \rho x_1 + \sigma x_2,$$
$$< x_2; x_1, x_2 >= 0,\ \ x_1.x_2 = \alpha x_1, \tag{30}$$

(with $\epsilon\neq 0,\ \rho\neq 0,\ \sigma\neq 0,\ \alpha\neq 0$),

$$x_1 * x_2 = -\beta x_2, \qquad < x_1; x_1, x_2 >= \lambda(x_1 + \sigma x_2),$$
$$< x_2; x_1, x_2 >= 0,\ \ x_1.x_2 = \beta x_2, \tag{31}$$

(with $\beta\neq 0,\ \lambda\neq 0$),

$$x_1 * x_2 = \epsilon x_2, \qquad < x_1; x_1, x_2 >= \rho x_1 + \sigma x_2,$$
$$< x_2; x_1, x_2 >= 0,\ \ x_1.x_2 = \alpha x_1 + \beta x_2, \tag{32}$$

(with $\alpha\neq 0,\ \beta\neq 0,\ \rho\neq 0,\ \epsilon\neq 0$ and $\alpha\sigma - \beta\rho\neq 0$).
Hence, gathering the set of values (14), (15), (27)-(29) and (30)-(32), we get the table of the theorem. This completes our proof. $\diamondsuit$

Corollary. *There is no proper 2-dimensional Lie triple algebra; more precisely the only 2-dimensional such algebras are L.t.s or the trivial algebra. Proof.* As noticed in section 2, any h.t.a $\{\mathcal{V}, *, ., < -; -, - >\}$ becomes a Lie triple algebra by setting $x * y = 0$ for all x, y in $\mathcal{V}$. So that 2-dimensional Lie triple algebras must be included in algebras of types I,II. But then $x.y = 0$, for all x, y in $\mathcal{V}$. This means that we get either the trivial algebra or L.t.s.$\diamondsuit$

4 Concluding remarks.

1. The algebras of types I,II are 2-dimensional Bol algebras (proper or not). Those of types III-VIII are proper 2-dimensional h.t.a, and so the class of such algebras is rather wide.
2. Although the survey may be conjectured tedious and lengthy, the classification of 3-dimensional h.t.a is similar to the 2-dimensional ones. For instance, one easily checks up that any 3-dimensional h.t.a is completely determined by the given of 15 different binary and ternary operations on basis elements with respect to operations $(*), (.)$ and $(< -; -, - >)$. Moreover, for dimension 3 there are nontrivial identities of types (2)-(4). This latter circumstance obviously makes the classification more fastidious. However it may be helpful to observe that such a classification must include examples of 3-dimensional Bol algebras that could be drawn from [1].

Acknowledgements.
This work was done within the framework of the Associateship Scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. Financial support from the Swedish International Development Cooperation Agency is acknowledged. The author would like to express his hearty gratitude to Professor Kiyosi Yamaguti.

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