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# A NOTE ON ZARISKI DENSE SUBGROUPS OF SEMISIMPLE ALGEBRAIC GROUPS WITH ISOMORPHIC $p$-ADIC CLOSURES

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**Abstract**

We prove under certain natural conditions the finiteness of the number of isomorphism classes of Zariski dense subgroups in semisimple groups with isomorphic $p$-adic closures.

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**Introduction.**

The present paper was inspired by Mazur [Ma], where he considered various types of *local-global principles* in number theory and also the problem, for a given number field $k$, to determine the *companions* of a given algebraic $k$-variety $V$ (i.e. those $k$-forms of $V$, locally everywhere $k_v$-isomorphic to $V$). He also conjectured that for projective smooth varieties $V$ over $k$, there are, up to $k$-isomorphism, only finite number of companions of $V$. For algebraic groups which are not necessarily linear, such a (well-known) question was answered in affirmative by Borel and Serre [BS]. We consider here an analog in the case of Zariski dense subgroups of semisimple groups. The following provides a connection with similar question. Let $k$ be a number field, $S$ a finite set of valuations of $k$, containing the set $\infty$ of archimedean ones. Let $\mathcal{O} = \mathcal{O}(S)$ be the ring of $S$-integers of $k$, $\mathbf{\Omega}$ be a fixed universal domain containing $k$. For a valuation $v$ of $k$ we denote by $k_v$ the $v$-adic completion of $k$, $\mathcal{O}_v$ = $v$-adic integers of $k_v$, $m_v$ = maximal ideal of $\mathcal{O}_v$, $\mathbf{A}$ = adèle ring of $k$. Algebraic groups under consideration are identified with their points over $\mathbf{\Omega}$. Assume that $G \subset \mathbf{G}(k)$, $\mathbf{G} \hookrightarrow \mathrm{GL}_n(\mathbf{\Omega})$, where $\mathbf{G}$ denotes the Zariski-closure of $G$ in $\mathrm{GL}_n$, $\mathbf{G}(\mathbf{B})$ will denote the $\mathbf{B}$-points of a linear algebraic group $\mathbf{G}$, with respect to the matrix realization of $\mathbf{G} \hookrightarrow \mathrm{GL}_n$ and for some ring $\mathbf{B}$. $Cl_v(G)$ denotes the ($v$-adic) closure of $G$ in $\mathbf{G}(k_v)$ with respect to the $v$-adic topology on $\mathbf{G}(k_v)$. So there attaches to a given $G$ a collection $(Cl_v(G))_v$ of $v$-adic closures of $G$, which measures how big $G$ is locally. One may ask the following natural question:

($*$) *To what extent the collection $(Cl_v(G))_v$ determines the group $G$ up to isomorphism ? Is the number of isomorphism classes finite ?*

We are most interested in the finiteness aspect of above question, i.e., given topological isomorphisms $Cl_v(G) \simeq Cl_v(G_i)$ for all $v$, where $i$ runs over a set of indices $I$, we ask whether the set of isomorphism classes of $\{G_i\}_i$ is finite.

These questions are closely related also to the congruence subgroup problem and strong approximation in simply connected algebraic groups in its wide sense.

It is our objective to establish the finiteness of the number of isomorphism classes in the case of semisimple groups (a partial answer to ($*$)).

In general, this is a difficult question and we will show the finiteness to hold under certain restrictions. The first restriction is to require the groups $G_i$ to be "big" in the sense below. For simplicity we restrict ourselves to the case $k = \mathbf{Q}$. Let $I$ be a set of indices. For each $i \in I$ let $G_i$ be a Zariski dense subgroup of simply connected absolutely almost simple $\mathbf{Q}$-group $\mathbf{G}_i \hookrightarrow \mathrm{GL}_{n_i}$, such that $G_i \subset \mathbf{G}_i(\mathbf{Z})$ and $G_i \not\simeq G_j$ if $i \neq j$. Assume that each $G_i$ satisfies the following condition

$$\cap_p(\mathbf{G}_i(\mathbf{Q}) \cap Cl_p(G_i))) = G_i. \qquad (B)$$

Here $Cl_p(\cdot)$ means taking the closure in the $p$-adic topology of $\mathbf{G}_i(\mathbf{Q}_p)$. This condition means that $G_i$ are "big" so that one can recover the group $G_i$ from local closures. A Zariski dense subgroup $G_i \subset \mathbf{G}_i$ satisfying this condition $(B)$ such that all closures $Cl_p(G_i)$ are open and compact subgroups of $\mathbf{G}_i(\mathbf{Q}_p)$ will be called *big*.

# 1 The Theorem

Our main result can be stated as follows.

**Theorem.** *With the above notation and convention, let $I$ be a set of indices and for $i \in I$, $G_i$ be a big subgroup of a simply connected absolutely almost simple $\mathbf{Q}$-group $\mathbf{G}_i$. Then the number of isomorphic classes of groups $G_i$ with isomorphic $p$-adic closures is finite.*

The proof of the theorem will be given in few steps.

We fix two groups $G$, $H$ from the set $\mathcal{B}(G) := \{G_i\}_{i\in I}$. $L(\mathbf{G})$ will denote the Lie algebra of a Lie (resp. $v$-adic or algebraic) group $\mathbf{G}$ and we fix once for all a matrix realization of $\mathbf{G}$ into $\mathrm{GL}_n(\mathbf{\Omega})$. $Ad(\mathbf{G})$ will denote the adjoint group of $\mathbf{G}$.

**1. Lemma.** *The set $\mathcal{B}(G)$ is a disjoint union of finitely many classes of groups $G_i$ with $\mathbf{Q}$-isomorphic Zariski closures.*

*Proof.* By our assumption, each $p$-adic closure $Cl_p(G_i)$ is an open and compact subgroup of $\mathbf{G}_i(\mathbf{Q}_p)$ and they are isomorphic to each other as topological groups. Denote by $f_p : Cl_p(G) \simeq Cl_p(H)$ the given topological isomorphism, where $G$ and $H$ are two fixed groups from $\{G_i\}_i$. By [Pin], Corollary 0.3, $f_p$ can be extended uniquely to a $\mathbf{Q}_p$-isomorphism $\bar{f}_p : \mathbf{G} \simeq \mathbf{H}$, so $\mathbf{G}$ and $\mathbf{H}$ are $\mathbf{Q}$-linear algebraic groups which are $\mathbf{Q}_p$-isomorphic for all $p$. By Borel - Serre [BS], Théorème 7.1, it follows that such groups lie in finitely many $\mathbf{Q}$-isomorphic classes. ∎

* * * * *

*From now on we assume that all groups $G_i$ have $\mathbf{Q}$-isomorphic Zariski closures.*

The following lemma shows the adèle nature of the family $(f_p)$.

**2. Lemma.** *With notation as in the proof of Lemma 1, for almost all $p$, $\bar{f}_p$ is a $\mathbf{Z}_p$-polynomial isomorphism with respect to the given matrix realization of the groups $\mathbf{G}$ and $\mathbf{H}$.*

*Proof.* Recall that we have fixed an embedding $G \subset \mathrm{GL}_n(\mathbf{Q})$. Since $f_p$ is a topological isomorphism, it is also an isomorphism of $p$-adic analytic Lie groups, thus it maps a open uniform powerful subgroup $S_G$ ($p$-saturable subgroup in terminology of [La], or standard subgroup in terminology of [Se]) of $Cl_p(G)$ onto a open uniform powerful subgroup $S_H$ of $Cl_p(H)$ (see [DDMS], Ch. 4, [Se], Ch. 4, [La] for more details). It follows from the definition of Lie algebras of analytic groups ([Se], Ch. 5, [DDMS], Sec. 8.2, 10.4) and the construction of standard subgroups that $L(S_G) \simeq L(S_H)$ as $\mathbf{Z}_p$-Lie algebras, i.e., with structural constants belonging to $\mathbf{Z}_p$ (loc. cit.) so $df_p$ must be a $\mathbf{Z}_p$-linear map with repsect to a given matrix realization (which is always fixed). Since $S_G$ is an open uniform subgroup of $Cl_p(G)$, its Lie algebra $L(S_G)$ is a $\mathbf{Z}_p$-lattice of $L(Cl_p(G))$, in particular, $L(Cl_p(G)) = L(S_G) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, and the same is true for $H$ instead of $G$. Therefore $df_p : L(Cl_p(G)) \simeq L(Cl_p(H))$ is defined over $\mathbf{Z}_p$, thus the same is true for isomorphism $L(\mathbf{G}) \simeq L(\mathbf{H})$, thus also for $d_p : Aut(L(\mathbf{G})) \simeq Aut(L(\mathbf{H}))$. Since the map $d_p$ is given by the following rule :

$$d_p : \phi \mapsto df_p \circ \phi \circ df_p^{-1},$$

it follows that $\bar{f}'_p : Ad(\mathbf{G}) \to Ad(\mathbf{H})$ will be a $\mathbf{Z}_p$-polynomial isomorphism. Since $f_p$ extends uniquely to $\mathbf{Q}_p$-isomorphism $\bar{f}_p : \mathbf{G} \to \mathbf{H}$ by [Pin], Corollary 0.3, the following diagram is commutative :

$$\begin{array}{ccc} \mathbf{G} & \xrightarrow{\bar{f}_p} & \mathbf{H} \\ \pi_1 \downarrow & & \pi_2 \downarrow \\ Ad(\mathbf{G}) & \xrightarrow{\bar{f}'_p} & Ad(\mathbf{H}) \end{array}$$

here $\pi_i$ denotes the corresponding isogeny. It follows that for those $p$ not lying in the set $T$ of primes dividing $m$, where $m = Card(\text{Ker }(\pi_1))$, $\bar{f}_p$ is also defined over $\mathbf{Z}_p$. Therefore $\bar{f}_p$ is defined over $\mathbf{Z}_p$ for all $p$ not belonging to $T$. ■

We need the following lemma in the sequel in order to realise Aut($\mathbf{G}$) as linear algebraic group over $\mathbf{Q}$.

**3. Lemma.** *With the above notation, let $f_1, \ldots, f_N$ be $\mathbf{Q}$-rational functions over $\mathbf{G}$ which are linearly independent over $\mathbf{Q}$. Then there exists $x_1, \ldots, x_N \in \mathbf{G}(\mathbf{Q})$ such that*

$$det(f_i(x_j))_{1 \leq i,j, \leq N} \in \mathbf{Q} \setminus \{0\}.$$

*Proof.* We prove by induction on $N$. The case $N = 1$ is trivial. Recall that $\mathbf{G}(\mathbf{Q})$ is Zariski dense in $\mathbf{G}$. Denote

$$f(x_1, ..., x_n) := det(f_i(x_j))_{1 \leq i,j, \leq N}.$$

Let $N > 1$ and assume that we have found $N \Leftrightarrow 1$ points $x_1, \ldots, x_{N-1}$ such that

$$c = det(f_i(x_j)_{1 \leq i,j \leq N-1}) \neq 0.$$

Consider the following $\mathbf{Q}$-rational function $g(z)$ on $\mathbf{G}$ defined as follows

$$g(z) := f(x_1, ..., x_{N-1}, z),$$

and expand the determinant $g(z)$ after the last row we have

$$g(z) = a_1 f_1(z) + \cdots + a_{N-1} f_{N-1}(z) + c f_N(z).$$

If for all $z \in \mathbf{G}(\mathbf{Q})$ we had $g(z) = 0$, then due to the Zariski density of $\mathbf{G}(\mathbf{Q})$ in $\mathbf{G}$, it would follow that $g(z) = 0$, hence $c = 0$ since $f_1, ..., f_N$ are $\mathbf{Q}$-linearly independent, which contradicts the choice of $c$. ∎

Denote by $M = \mathrm{Aut}(\mathbf{G})$ the group of rational automorphisms of $\mathbf{G}$. It is well-known that $M$ has a natural structure of linear $\mathbf{Q}$-algebraic group (see, e.g., [BS], [HM]) We need a specific realization of the group $M$, which plays a crucial role in our proof, as follows. Recall that $\mathbf{A}$ denotes the adèle ring of $\mathbf{Q}$.

**4. Proposition.** *With the above notation there is a realization of $M$ as a linear algebraic $\mathbf{Q}$-group such that for every $H \in \mathcal{B}(G)$ and for any $\mathbf{Q}$-isomorphism $g : \mathbf{H} \to \mathbf{G}$, the family $(g \circ f_p)$ ($p$ runs over all prime numbers) is belong to $M(\mathbf{A})$.*

*Proof.* First we fix a universal domain $\mathbf{\Omega}$. It follows from results of [HM] that $\mathbf{G}$ is a conservative $\mathbf{Q}$-group, i.e., the group $M$ acts locally finitely on the $\mathbf{Q}$-algebra $\mathbf{Q}[\mathbf{G}]$ of regular functions defined over $\mathbf{Q}$ on $\mathbf{G}$. As before, we fix an embedding $\mathbf{G} \hookrightarrow \mathrm{GL}_n(\mathbf{\Omega})$ and let $x_{ij} (1 \leq i, j \leq n)$ be the coordinate matrix functions on $\mathbf{G}$. Let $V$ be the smallest finite dimensional $\mathbf{Q}$-vector subspace of $\mathbf{Q}[\mathbf{G}]$ containing $x_{ij}, 1 \leq i, j \leq n$, which is $M$-invariant (i.e. $V$ is generated by $x_{ij}$ and their images under the action of $M$). Let $\{f_1, ..., f_N\}$ be $\mathbf{Q}$-regular functions over $\mathbf{G}$ which form a $\mathbf{Q}$-basis of $V$ containing all $x_{ij}$ (notice that all $x_{ij}$ are $\mathbf{Q}$-linearly independent). By multiplying $f_k$ with a suitable integer, we may assume that $f_k$ are all $\mathbf{Z}$-polynomial functions.

For $\phi \in M$ let the action of $\phi$ be given as follows

$$\phi : f_i \mapsto f_i \circ \phi = \Sigma_{1 \leq j \leq N} a_{ij}^{(\phi)} f_j,$$

where $a_{ij}^{(\phi)} \in \mathbf{\Omega}$ (=universal domain). Since the $\mathbf{Q}$-basis $\{f_1, ..., f_N\}$ contains all coordinate functions, it follows that the mapping

$$\Phi : \phi \mapsto (a_{ij}^{(\phi)})$$

is a faithful $\mathbf{Q}$-representation of $M$ into $\mathrm{GL}(V)$, where the latter is identified with $\mathrm{GL}_N(\mathbf{\Omega})$ by means of the basis $\{f_1, ..., f_N\}$. Further we will identify $M$ with a closed $\mathbf{Q}$-subgroup of $\mathrm{GL}_N(\mathbf{\Omega})$. Thus

$$\phi \in M(\mathbf{Z}_p) \Leftrightarrow a_{ij}^{(\phi)} \in \mathbf{Z}_p, \ \forall i, j.$$

Now let

$$\bar{f}_p : \mathbf{G} \simeq \mathbf{H}$$

be the isomorphism extending the isomorphism $f_p : Cl_p(G) \simeq Cl_p(H)$ (so $\bar{f}_p$ is defined over $\mathbf{Q}_p$) and let $g : \mathbf{H} \simeq \mathbf{G}$ be any $\mathbf{Q}$-isomorphism.

We now choose elements $x_1, ..., x_N$ as in Lemma 3. For the convenience, we denote $a_{ij} = a_{ij}^{(\bar{f}_p)}$, where $p$ is fixed. Then we have the following systems of equations

$$(A_1) \begin{cases} f_1(g \circ \bar{f}_p(x_1)) = a_{11} f_1(x_1) + \cdots + a_{1N} f_N(x_1) \\ \cdot \\ \cdot \\ \cdot \\ f_1(g \circ \bar{f}_p(x_N)) = a_{11} f_1(x_N) + \cdots + a_{1N} f_N(x_N) \end{cases}$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$(A_N) \begin{cases} f_N(g \circ \bar{f}_p(x_1)) = a_{N1} f_1(x_1) + \cdots + a_{NN} f_N(x_1) \\ \cdot \\ \cdot \\ \cdot \\ f_N(g \circ \bar{f}_p(x_N)) = a_{N1} f_1(x_N) + \cdots + a_{NN} f_N(x_N) \end{cases}$$

Denote

$$r = c/d := det(f_i(x_j))(1 \leq i, j \leq N),$$

where $c, d \in \mathbf{Z} \setminus \{0\}$. Since $x_i \in \mathbf{G}(\mathbf{Q})$ are finite in number, we may assume that $x_i \in \mathbf{G}(\mathbf{Z}[S_1^{-1}])$ for all $i$, where $\mathbf{Z}[S_1^{-1}]$ is the localization at a finite set $S_1$ of primes, which contains the set of primes dividing $c$. By Lemma 2, for certain finite set $S_2$ of primes the isomorphism $\bar{f}_p$ (see notation above) is defined over $\mathbf{Z}_p$ for $p \notin S_2$. For a finite set $S_3$ of primes, we see that $g$ is defined over $\mathbf{Z}_p$ for $p \notin S_3$. Let $S = S_1 \cup S_2 \cup S_3$. Then by solving the system $A_t$ above with respect to $a_{t1}, ..., a_{tN}$, we have

$$a_{ij} = (1/r) d_{ij}, \forall i, j,$$

where $a_{ij} \in \mathbf{Z}_p[S^{-1}]$. So for $p \notin S$ we have $g \circ f_p \in M(\mathbf{Z}_p)$ as required. ∎

Denote by

$$\mathcal{C}(G) = \{(f_p) \in \textstyle\prod_p M(\mathbf{Q}_p) : f_p(Cl_p(G)) = Cl_p(G), \forall p, \text{ and}$$

$$f_p \in M(\mathbf{Z}_p) \text{ for almost all } p\} \ .$$

It is clear that $\mathcal{C}(G)$ is an infinite subgroup of $M(\mathbf{A})$. Next we want to parametrize the set $\mathcal{B}(G)$ by assigning to each $H \in \mathcal{B}(G)$ a double coset class in $M(\mathbf{Q}) \backslash M(\mathbf{A}) / \mathcal{C}(G)$ defined as follows :

If $g : \mathbf{H} \simeq \mathbf{G}$ is a $\mathbf{Q}$-isomorphism, $\bar{f}_p : \mathbf{G} \simeq \mathbf{H}$ is the isomorphism extending $f_p : Cl_p(G) \simeq Cl_p(H)$ for all $p$, then we set

$$a(G, H) := M(\mathbf{Q})(g \circ \bar{f}_p)\mathcal{C}(G).$$

According to Proposition 4, $(g \circ \bar{f}_p) \in M(\mathbf{A})$ so $a(G, H)$ is an element of the set of double coset classes $M(\mathbf{Q})\backslash M(\mathbf{A})/\mathcal{C}(G)$.

**5. Proposition.** *The correspondence defined above is a well-defined map.*

*Proof.* First we have to show that the class $M(\mathbf{Q})(g \circ \bar{f}_p)\mathcal{C}(G)$ does not depend on the choice of $g$ and $(\bar{f}_p)$.

Let $g' : \mathbf{H} \simeq \mathbf{G}$ be another $\mathbf{Q}$-isomorphism, $f'_p : Cl_p(G) \simeq Cl_p(H)$ be an isomorphism with the extension $\bar{f}'_p : \mathbf{G} \to \mathbf{H}$ for all $p$. Then we have

$$(*) \qquad g \circ \bar{f}_p = (g \circ g'^{-1}) \circ (g' \circ \bar{f}'_p) \circ (\bar{f}'^{-1}_p \circ \bar{f}_p).$$

Since $g \circ g'^{-1}$ is a $\mathbf{Q}$-isomorphism of $\mathbf{G}$, $g \circ g'^{-1} \in M(\mathbf{Q})$. For all $p$ we have

$$(\bar{f}'^{-1}_p \circ \bar{f}_p)(Cl_p(G)) = Cl_p(G).$$

Hence for all $p$ we have $\bar{f}'^{-1}_p \circ \bar{f}_p \in M(\mathbf{Q}_p)$ and thus for almost all $p$, $\bar{f}'^{-1}_p \circ \bar{f}_p \in M(\mathbf{Z}_p)$, because $\bar{f}'_p$ and $\bar{f}_p$ are so. Hence $(\bar{f}'^{-1}_p \circ \bar{f}_p) \in \mathcal{C}(G)$. Thus

$$M(\mathbf{Q})(g \circ \bar{f}_p)\mathcal{C}(G) = M(\mathbf{Q})(g' \circ \bar{f}'_p)\mathcal{C}(G).$$

■

The injectivity of the map $H \mapsto a(G, H)$ now follows from the following

**6. Proposition.** *If $(G, H)$ and $(G, K)$ have the same double coset class then $H = K$.*

*Proof.* With notation as in the proof of Proposition 5, by the assumption we have for all primes $p$

$$f \circ \bar{f}_p = g_{\mathbf{Q}}(g \circ \bar{g}_p)h_p,$$

where $g_{\mathbf{Q}} \in M(\mathbf{Q})$ and $(h_p) \in \mathcal{C}(G)$. Denote $f' = g_{\mathbf{Q}}^{-1} \circ f$, $\bar{g}'_p = \bar{g}_p \circ h_p$. Then for all $p$ we have

$$f' \circ \bar{f}_p = g \circ \bar{g}'_p,$$

or

$$g^{-1} \circ f' = \bar{g}'_p \circ \bar{f}_p^{-1},$$

i.e., $g^{-1} \circ f'$ is a $\mathbf{Q}$-isomorphism $\mathbf{H} \simeq \mathbf{K}$, mapping $Cl_p(H)$ onto $Cl_p(K)$ for all primes $p$.

For $h \in H \subset \mathbf{H}(\mathbf{Q})$ we have $(g^{-1} \circ f')(h) \in \mathbf{K}(\mathbf{Q})$, and $(g^{-1} \circ f')(h) \in Cl_p(K)$ for all $p$. Thus

$$(g^{-1} \circ f')(h) \in \mathbf{K}(\mathbf{Q}) \cap (\cap_p Cl_p(K)) = K$$

by the assumption that the groups $G_i$ are big. Hence $(g^{-1} \circ f')(H) \subset K$. Similarly we have

$$(f'^{-1} \circ g)(K) \subset H,$$

i.e., $(f^{-1} \circ f')(H) = K$, and $H \simeq K$, hence $H = K$ since all groups $G_i$ are mutually non-isomorphic. Proposition 6 is proved. ■

Preceding observations show that the cardinality of $\mathcal{B}(G)$ is not greater than the cardinality of $M(\mathbf{Q})\backslash M(\mathbf{A})/\mathcal{C}(G)$. We want to show that the latter is finite. Define

$$\mathcal{D} = \mathcal{D}(G) := \{(a_p) \in \mathcal{C}(G) : a_p \in M(\mathbf{Z}_p)\ , \forall p\},$$

i.e., $\mathcal{D} = \mathcal{C}(G) \cap M(\mathbf{A}(\infty))$, where $\mathbf{A}(\infty)$ denotes the subring of finite adèles of $\mathbf{A}$. In particular we have

$$Card(M(\mathbf{Q})\backslash M(\mathbf{A})/\mathcal{C}(G)) \leq Card(M(\mathbf{Q})\backslash M(\mathbf{A})/\mathcal{D}.$$

The following proposition plays a crucial role in the proof of the finiteness of $Card(M(\mathbf{Q})\backslash M(\mathbf{A})/\mathcal{D})$.

**7. Proposition.** *There is only a finite number of subgroups of a given finite index $m$ in $\mathbf{G}(\mathbf{Z}_p)$.*

*Proof.* Let $R$ be a subgroup of index $m$ in $\mathbf{G}(\mathbf{Z}_p)$. First we assume that $R$ is a normal subgroup. Then by considering the factor group $\mathbf{G}(\mathbf{Z}_p)/R$ we conclude that $R$ contains the subgroup $\mathbf{G}(\mathbf{Z}_p)^m$ of $\mathbf{G}(\mathbf{Z}_p)$ generated by the $m$-powers. It suffices then only to prove that

$$[\mathbf{G}(\mathbf{Z}_p) : \mathbf{G}(\mathbf{Z}_p)^m] < \infty.$$

Passing to a uniform pro-$p$-subgroup $G'$ of $\mathbf{G}(\mathbf{Z}_p)$ we need only show that $G'^m$ is of finite index in $G'$. It is known ([DDMS], Theorem 3.16) that $G'$ is a pro-$p$-group of finite rank, say, $d$, and $G'$ is topologically generated by $g_1, ..., g_d$. Also, by (loc. cit., Theorem 4.9) there exists a homeomorphism

$$\psi : \mathbf{Z}_p^d \simeq G',$$

such that

$$\psi(x_1, ..., x_d) = g_1^{x_1} \cdots x_d^{g_d}.$$

Therefore $\psi((m\mathbf{Z}_p)^d)$ is an open subset of $G'$, since $m\mathbf{Z}_p$ is open in $\mathbf{Z}_p$. It is clear that $\psi((m\mathbf{Z}_p)^d) \subset G'^m$, hence $G'^m$ is open in $G'$ and also of finite index.

Now we assume that $R$ is not normal in $\mathbf{G}(\mathbf{Z}_p)$. Then it is well-known that $R$ contains a subgroup $R_0$ normal in $\mathbf{G}(\mathbf{Z}_p)$ and of index $[\mathbf{G}(\mathbf{Z}_p) : R_0]$ dividing $m!$, hence $R_0$ contains $\mathbf{G}(\mathbf{Z}_p)^{m!}$. Then the above proof shows that $[\mathbf{G}(\mathbf{Z}_p) : \mathbf{G}(\mathbf{Z}_p)^{m!}] < \infty$, therefore the proposition follows. ■

**8. Remarks.** The same proof of Proposition 7 gives the following (cf. also with [Seg]).

*a) For a given compact $p$-adic analytic group, the number of its subgroups of given index $m$ is finite.*

*b) There is only a finite number of subgroups of* $\mathbf{G}(\mathbf{Q}_p)$ *containing* $\mathbf{G}(\mathbf{Z}_p)$ *with given index* $m$. ∎

Now we denote by

$$M(\mathbf{Z}_p, Cl_p(G)) := \{f \in M(\mathbf{Z}_p) : f(Cl_p(G)) = Cl_p(G)\}.$$

From [MVW], Theorem 7.3, or [N], Theorem 5.4, we know that $Cl_p(G) = \mathbf{G}(\mathbf{Z}_p)$ for almost all $p$ (say, for all $p$ outside a finite set $W$ of primes). By the choice of the functions $f_j$ (in the proof of Proposition 4), they are $\mathbf{Z}$-polynomial functions. So if $f \in M(\mathbf{Z}_p)$ then we have $f(\mathbf{G}(\mathbf{Z}_p)) = \mathbf{G}(\mathbf{Z}_p)$. Hence for $p \notin W$ we have

$$M(\mathbf{Z}_p, Cl_p(G)) = M(\mathbf{Z}_p).$$

We need also the following

**9. Proposition.** *$M(\mathbf{Z}_p, Cl_p(G))$ is of finite index in $M(\mathbf{Z}_p)$.*

*Proof.* By assumption $G \subset \mathbf{G}(\mathbf{Z})$, so it follows that for all $p$ we have $Cl_p(G) \subset \mathbf{G}(\mathbf{Z}_p)$ and it is a subgroup of finite index in $\mathbf{G}(\mathbf{Z}_p)$ since $Cl_p(G)$ is an open subgroup of the compact group $\mathbf{G}(\mathbf{Z}_p)$. Let

$$t = [\mathbf{G}(\mathbf{Z}_p) : Cl_p(G)] < \infty,$$

and $Cl_p(G) = A_1, \dots, A_k$ be all subgroups of $\mathbf{G}(\mathbf{Z}_p)$ of index $t$ (see Prop. 7). Then we have for any $f \in M(\mathbf{Z}_p)$

$$[\mathbf{G}(\mathbf{Z}_p) : f(A_j)] = [f(\mathbf{G}(\mathbf{Z}_p)) : f(A_j)] = [\mathbf{G}(\mathbf{Z}_p) : A_j] = t,$$

so $f$ acts transitively on the set $\{A_1, \dots, A_k\}$. Thus we obtain a homomorphism

$$\psi : M(\mathbf{Z}_p) \to S_k,$$

where $S_k$ denotes the symmetric group on $k$ symbols. Consequently we have

$$[M(\mathbf{Z}_p) : \mathrm{Ker}\ \psi] < \infty.$$

It is obvious that $\mathrm{Ker}\ \psi \subset M(\mathbf{Z}_p, Cl_p(G))$ and the proposition follows. ∎

Now we are able to show

**10. Proposition.** *With the above notation we have*

$$Card(M(\mathbf{Q}) \backslash M(\mathbf{A}) / \mathcal{D}) < \infty.$$

*Proof.* We have

$$Card(M(\mathbf{Q})\backslash M(\mathbf{A})/\mathcal{D}) =$$

$$= Card(M(\mathbf{Q})\backslash M(\mathbf{A})/(\prod_{p\notin W} M(\mathbf{Z}_p) \times \prod_{p\in W} M(\mathbf{Z}_p, Cl_p(G))))$$

$$\leq Card(M(\mathbf{Q})\backslash M(\mathbf{A})/M(\mathbf{A}(\infty))) \times \prod_{p\in W}[M(\mathbf{Z}_p) : M(\mathbf{Z}_p, Cl_p(G))]$$

$$< \infty$$

by the main theorem of Borel [Bor] and by Proposition 9. ∎

Summing up all results above we have proved the following

**11. Theorem.** *Let $I$ be a set of indices and for $i \in I$, let $G_i$ be a Zariski-dense subgroup of a simply connected absolutely almost simple $\mathbf{Q}$-group $\mathbf{G}_i$, such that $G_i \subset \mathbf{G}_i(\mathbf{Z})$ are mutually non-isomorphic, but all their $p$-adic closures for all $p$ are topologically isomorphic, and each $G_i$ is big in $\mathbf{G}_i$. Then $I$ is finite.*

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