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ON THE LANGMUIR OSCILLATIONS IN KERR PLASMA

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Abstract

Langmuir plasma oscillations around rotating gravitating objects such as black holes, neutron stars etc. in the Kerr metric, are investigated. A wave equation is derived which describes the various plasma modes in Kerr plasmas. The radial and theta dependence of the equilibrium quantities due to gravity and rotation are considered. GR effects on equilibrium quantities are significant near the event horizon. The Langmuir oscillations in the Kerr plasma with the system of perturbed equations are studied. The lapse function entering in the definition of plasma frequency describes the gravito-rotational effects on plasma oscillation. It is noted that the event horizon for Kerr metric lies inside the gravitational radius and Kerr parameter is always less than half of the gravitational radius. A strong GR effect is found at the event horizon. Langmuir oscillations go to zero near the event horizon, while it increases sharply near but away from the horizon and reaches a maximum value at a certain distance from it. A general equation describing the Langmuir oscillations in Kerr plasma is derived and a gravitoplasma dispersion relation is obtained. The results of this study are foundational and can be extended for further study of Kerr plasma phenomenon around gravito-rotating compact objects.

I. INTRODUCTION

Recently, the study of plasmas around black holes and other gravito-rotating compact objects has become of much interest [1-3]. It may yield the spectroscopic signatures of radiation emitted from plasma of a rotating black hole. MacDonald and Thorne [4,5] have introduced Maxwell's equations in 3+1 coordinates, which provides a foundation for the formulation of general relativistic (GR) set of plasma physics equations. It provides a means by which the electrodynamics equations and the plasma physics look more familiar to the usual formulations in flat space-time while taking account of general relativistic effects such as space time curvature. It is possible to obtain the various wave modes at a range of fixed values of the lapse function α. Secondly, gravitational and rotational effects on the equilibrium plasma are very much important to understand the Langmuir oscillations in such environment.

In this paper, we formulate the wave equation in the Kerr metric. As a first attempt, we concentrate only on the Langmuir oscillations in Kerr plasma. We study the radial and theta dependence of the equilibrium quantities. For the perturbed quantities, we study the Langmuir frequency of plasma oscillations. It is found that plasma frequency is zero at the event horizon, but at almost twice the distance of the gravitational radius, the plasma frequency is maximum, while at a larger distance it attains a constant value even if the background plasma density itself is assumed to be constant. When there is density variation, the plasma frequency varies additionally in the known fashion. We formulate a second order differential equation for Langmuir field under gravity and rotation. The gravito-plasma dispersion relation is obtained and it is analysed. It shows that away from the compact objects, we have usual plasma oscillations but near the event horizons, Langmuir oscillations go to zero near the event horizon, while it increases sharply near but away from the horizon and reaches a maximum value at a certain distance from it.

II. WAVE EQUATION IN KERR PLASMA

In this section, we review the 3+1 split of the Kerr metric. The continuity and motion equations are combined with the Maxwell system to derive the wave equation for plasmas around rotational gravitating compact objects such as black holes and neutron stars.

A. The 3+1 split of the Kerr metric

In the 3+1 split of the Kerr metric, Thorne et al. 1986 [5] space-time is split into a family of three dimensional differentially rotating hypersurfaces of constant time with internal curvature. These hypersurfaces of constant time are mentally collapsed into a single three dimensional "absolute space" in which time is *globally* measured by the Boyer Lindquist coordinate t. Physics is described in absolute space of locally non-rotating fiducial observers (FIDOs) with respect to their *local* proper time τ in their locally flat frames. The line element of the Kerr metric in 3+1 notation is given by

$$ds^2 = -\alpha^2 dt^2 + h_{ik}(dx^i + \beta^i dt)(dx^k + \beta^k dt) \tag{1}$$

where the lapse function is identified with the gravitational red shift.

$$\alpha = (\frac{d\tau}{dt})_{FIDO} = \frac{\varrho}{\Sigma}\sqrt{\Delta} \tag{2}$$

t is the time in absolute space and τ is the local proper time of locally nonrotating observer (FIDO).

The shift functions are the components of the gravitomagnetic potential $\vec{\beta}$ with

$$\beta^r = \beta^\theta = 0, \beta^\phi = -\omega = (\frac{d\phi}{dt})_{FIDO} = -\frac{ar_g r}{\Sigma^2} \tag{3}$$

which describes the differential rotation of the FIDOs relative to distant observer; ω is the angular velocity of the locally nonrotating observer (the so-called Lense-Thirring angular velocity). The components of the 3-metric h_{ik} are:

$$h^{ik} = g^{ik} + \beta^i \beta^k / \alpha^2. \tag{4}$$

$$h_{rr} = \frac{\varrho^2}{\Delta}, h_{\theta\theta} = \varrho^2, h_{\phi\phi} = \varpi, h_{ik} = 0, i \neq k.$$

Note that $h_{ik} = g_{ik}$. The metric functions appearing here are defined as

$$\Delta = r^2 - 2Mr + a^2 \equiv r^2 - r_g r + a^2, \rho^2 \equiv r^2 + a^2 cos^2\theta,$$
$$\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta sin^2\theta, \varpi = \frac{\Sigma}{\varrho} sin\theta. \tag{5}$$

The parameters of the rotational gravitating object are its mass M, its gravitational radius (Schwarzchild) radius r_g, and angular momentum J and the Kerr parameter $a = \frac{J}{M}$.

The FIDO- measured velocities are:

$$v^i = \frac{1}{\alpha}(\frac{dx^i}{dt} + \beta^j), e.g. v^\phi = \frac{\Omega - \omega}{\alpha}, (\Omega = \frac{d\phi}{dt}). \tag{6}$$

The physical velocity components v^i follow by multiplication with $(h_{ii})^{1/2}$. The metric has the sign $(- + + +)$ and $c = 1$.

B. Wave Equation for the Cold Plasma

We consider that radiation pressure in the plasma is much higher than the thermal plasma pressure. Thus we consider the cold plasma approximation in the Kerr plasma. Such a plasma is described by the two fluid MHD system of equations, Novikov et. al.[6], Mofiz et. al. [7], Khanna [3]:

$$\nabla \cdot \mathbf{E} = \Sigma_s n_s \gamma_s q_s, \tag{7}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{8}$$

$$\nabla \times (\alpha \mathbf{E}) = -(\frac{\partial}{\partial t} - \pounds_\beta)\mathbf{B}, \tag{9}$$

$$\nabla \times (\alpha \mathbf{B}) = -(\frac{\partial}{\partial t} - \pounds_\beta)\mathbf{E} + 4\pi\alpha \Sigma_s n_s \gamma_s \mathbf{v_s}, \tag{10}$$

$$(\frac{\partial}{\partial t} - \vec{\beta}\cdot\nabla)(\gamma_s n_s) + \nabla\cdot(\alpha n_s \gamma_s \mathbf{v_s}) = 0, \tag{11}$$

$$\frac{d_s \gamma_s \mathbf{v_s}}{d\tau} = \mathbf{g} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v_s}\times\mathbf{B}) + \mathbf{f}, \tag{12}$$

where,

$$\frac{d_s}{d\tau} = \frac{1}{\alpha}\{\frac{\partial}{\partial t} + (\alpha \mathbf{v_s} - \vec{\beta})\cdot\nabla\},$$
$$\mathbf{g} = -\nabla ln \nabla = -\frac{1}{\alpha}\frac{M}{r^2}\hat{\mathbf{e}}_\mathbf{r}, \gamma_s = (1 - v_s^2)^{-1/2},$$

$\mathbf{f}$ is an external force other than electromagnetic. For the Kerr plasma, it is the centrifugal force to keep the plasma in equilibrium with the gravitational force.

q_s is the particle charge, n_s is the particle number density and summation is over, for all the species. $\pounds_\beta$ denotes the Lie derivative along $\vec{\beta}$ i.e.

$$\pounds_\beta \mathbf{B} = -\nabla \times (\vec{\beta}\times\mathbf{B})$$
$$\pounds_\beta \mathbf{E} = -\{\nabla\times(\vec{\beta}\times\mathbf{E}) - \vec{\beta}(\nabla\cdot\mathbf{E})\} \tag{13}$$

is due to the rotation of locally non-rotating observer.

Gradient, curl, divergence are taken along the curvilinear coordinates:

$$e_{\hat{r}} = \frac{\sqrt{\nabla}}{\varrho}e_r = \frac{\sqrt{\nabla}}{\varrho}\frac{\partial}{\partial r}, e_\theta = \frac{1}{\varrho}e_\theta = \frac{1}{\varrho}\frac{\partial}{\partial\theta}, e_{\hat{\phi}} = \frac{1}{\varpi}e_\phi = \frac{1}{\varpi}\frac{\partial}{\partial\phi}. \tag{14}$$

From the above system of equations, we find the following Faraday's and Ampere's laws, respectively,

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times(\alpha\mathbf{E}) - \nabla\times(\vec{\beta}\times\mathbf{B}), \tag{15}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \nabla\times(\alpha\mathbf{B}) - \{\nabla\times(\vec{\beta}\times\mathbf{E}) - \vec{\beta}(\nabla\cdot\mathbf{E})\} - 4\pi\alpha\mathbf{J}, \tag{16}$$

which gives the wave equation

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\nabla\times\{\alpha\nabla\times(\alpha\nabla\times(\alpha\mathbf{E})\} - 4\pi\alpha\frac{\partial \mathbf{J}}{\partial t}$$
$$-\nabla\times\{\alpha\nabla\times(\vec{\beta}\times\mathbf{B}) + (\vec{\beta}\times\frac{\partial\mathbf{E}}{\partial t})\} + \vec{\beta}(\nabla\cdot\frac{\partial\mathbf{E}}{\partial t}). \tag{17}$$

From the above, we see that GR as Kerr effects on plasma dynamics is mostly played by the lapse function α. Therefore we do some analysis of the lapse function of the Kerr metric. It is given by the expression

$$\alpha^2 = \frac{(r^2 + a^2 cos^2\theta)(r^2 - r_g r + a^2)}{(r^2 + a^2)^2 - a^2(r^2 - r_g r + a^2 sin^2\theta)}. \tag{18}$$

For the Kerr parameter $a = 0$, we find the usual Schwarzchild expression

$$\alpha = \sqrt{1 - \frac{r_g}{r}}. \tag{19}$$

The distance at which $\alpha = 0$, describes the event horizons in Kerr space. In this case, we find two horizons, which are given by

$$r_1 = \frac{r_g}{2}(1 + \sqrt{1 - \frac{4a^2}{r_g^2}}), \tag{20}$$

$$r_2 = \frac{r_g}{2}(1 - \sqrt{1 - \frac{4a^2}{r_g^2}}). \tag{21}$$

It is to be noted that both r_1 and r_2 are less than r_g. Thus for Kerr metric, event horizons lie within the gravitational radius. $r_1 > r_2$ but $r_1 \sim r_2$ for $a \to \frac{r_g}{2}$. Secondly to satisfy $\alpha^2 > 0$, we find the Kerr parameter should satisfy the relation $a \leq \frac{r_g}{2}$.

Since $\alpha = \alpha(r, \theta)$, we consider all physical variables as the functions of r and θ only. Then the fluid equations for Langmuir oscillations may be written as

$$\frac{\alpha}{r^2}\frac{\partial}{\partial r}(r^2 E^r) + \frac{\alpha}{r sin\theta}\frac{\partial}{\partial\theta}(sin\theta E^\theta) = 4\pi\Sigma_s q_s \gamma_s n_s, \tag{22}$$

$$\frac{\partial}{\partial t}(\gamma_s n_s) + \frac{\alpha}{r^2}\frac{\partial}{\partial r}(\alpha r^2 n_s \gamma_s v_s^r) + \frac{\alpha}{r sin\theta}\frac{\partial}{\partial\theta}(\alpha sin\theta \gamma_s n_s v_s^\theta) = 0, \tag{23}$$

$$\frac{1}{\alpha}\frac{\partial}{\partial t}(\gamma_s v_s^r) + (v_s^r \frac{\partial}{\partial r} + v_s^\theta \frac{\partial}{r\partial\theta})(\gamma_s v_s^r) = -\frac{1}{\alpha}\frac{M}{r^2} + \frac{q_s}{m_s}E^r + (\hat{\mathbf{\Omega}} \times \mathbf{r}) \times \hat{\mathbf{\Omega}}, \tag{24}$$

$$\frac{1}{\alpha}\frac{\partial}{\partial t}(\gamma_s v_s^\theta) + (v_s^r \frac{\partial}{\partial r} + v_s^\theta \frac{\partial}{r\partial\theta})(\gamma_s v_s^\theta) = \frac{q_s}{m_s}E^\theta, \tag{25}$$

where, $\hat{\mathbf{\Omega}} = \frac{\mathbf{\Omega} - \omega}{\alpha}$.

For the simplicity, we consider $E^\theta = 0$, which implies that $v_s^\theta = 0$. Thus the above system of equations is more simplified and thus we may analyse the following system of equations for the study of Langmuir oscillations in Kerr plasma, i.e.

$$\frac{\alpha}{r^2}\frac{\partial}{\partial r}(r^2 E^r) = 4\pi\Sigma_s q_s \gamma_s n_s, \tag{26}$$

$$\frac{\partial}{\partial t}(\gamma_s n_s) + \frac{\alpha}{r^2}\frac{\partial}{\partial r}(\alpha r^2 n_s \gamma_s v_s^r) = 0, \tag{27}$$

$$\frac{1}{\alpha}\frac{\partial}{\partial t}(\gamma_s v_s^r) + v_s^r \frac{\partial}{\partial r}(\gamma_s v_s^r) = -\frac{1}{\alpha}\frac{M}{r^2} + \frac{q_s}{m_s}E^r + (\hat{\mathbf{\Omega}} \times \mathbf{r}) \times \hat{\mathbf{\Omega}}. \tag{28}$$

We linearize the above system of equations (26)-(28), using the following perturbations:

$$n_s(r,\theta,t) = n_{s0}(r,\theta) + n_{s1}(r,\theta,t), \tag{29}$$

$$v_s^r(r,\theta,t) = v_{s0}^r(r,\theta) + v_{s1}^r(r,\theta,t), \tag{30}$$

$$E^r(r,\theta,t) = E_1^r(r,\theta,t). \tag{31}$$

The equilibrium equations are:

$$4\pi\Sigma_s q_s \gamma_0 n_{s0} = 0, \tag{32}$$

$$\frac{\alpha}{r^2}\frac{\partial}{\partial r}(\alpha r^2 n_{s0}\gamma_0 v_{s0}^r) = 0, \tag{33}$$

$$v_{s0}^r \frac{\partial}{\partial r}(\gamma_0 v_{s0}^r) = -\frac{1}{\alpha}\frac{M}{r^2} + (\hat{\mathbf{\Omega}} \times \mathbf{r}) \times \hat{\mathbf{\Omega}}, \tag{34}$$

and the perturbed equations are:

$$\frac{\alpha}{r^2}\frac{\partial}{\partial r}(r^2 E_1^r) = 4\pi\Sigma_s q_s \gamma_0 n_{s1}, \tag{35}$$

$$\frac{\partial}{\partial t}(\gamma_0 n_{s1}) + \frac{\alpha}{r^2}\frac{\partial}{\partial r}(\alpha r^2 n_{s0}\gamma_0 v_{s1}^r) + \frac{\alpha}{r^2}\frac{\partial}{\partial r}(\alpha r^2 n_{s1}\gamma_0 v_{s0}^r) = 0, \tag{36}$$

$$\frac{1}{\alpha}\frac{\partial}{\partial t}(\gamma_0 v_{s1}^r) + v_{s0}^r\frac{\partial}{\partial r}(\gamma_0 v_{s1}^r) + v_{s1}^r\frac{\partial}{\partial r}(\gamma_0 v_{s0}^r) = \frac{q_s}{m_s}E^r. \tag{37}$$

III. RADIAL AND THETA DEPENDENCE OF EQUILIBRIUM QUANTITIES

The equilibrium quantities depend on r due to gravity and on θ due to rotation of the gravitating object. Equation (32) shows the quasineutrality of the equilibrium plasma, while from equation (33), we find $\alpha r^2 n_{s0} v_{s0}^r = f(\theta)$, which may be a constant or any arbitrary function θ.

We consider that in the equilibrium, the gravitational force is balanced by the centrifugal force. Thus from equation (34), it follows that $v_{s0}^r = v_{s0}^r(\theta)$, which also may be a constant including zero or any arbitrary function of θ. We calculate the Keppler frequency of rotation around a rotational gravitating object, which is

$$\Omega(r,\theta) = \omega(r,\theta) + (\frac{\alpha M}{r^3})^{1/2}, \tag{38}$$

and

$$\gamma_0 = (1 - \frac{v_\phi^2}{c^2})^{-1/2}, v_\phi = r\Omega(r,\theta). \tag{39}$$

We plot the Keppler frequency as a function of r for a fixed value of $\theta = \frac{\pi}{2}$ (equatorial region) and it is shown in Fig.1.

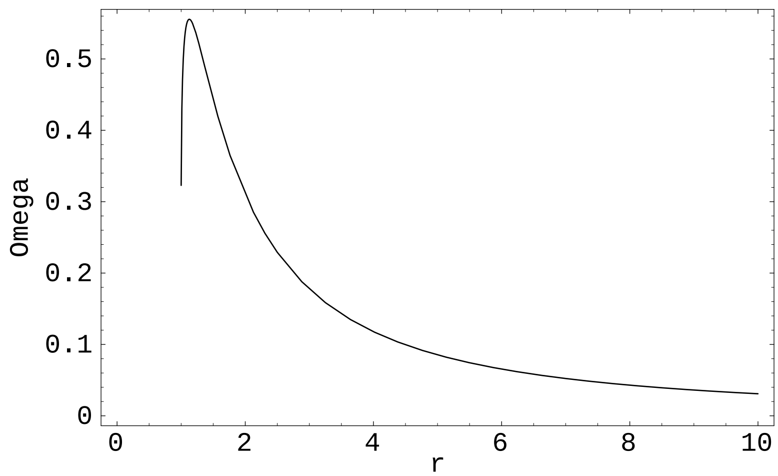

FIG. 1. Keppler frequency as a function of r for a fixed value $\theta = \frac{\pi}{2}$ (equatorial region), $a = .05, r_g = 1, M = \frac{r_g}{2}$.

From Fig.1, we see the that the Kepplerian rotation around a compact gravitating object is highly differential. Close to the event horizon the rotation is zero, but it sharply increases near the horizon, reaches a maximum value almost at twice the distance of the horizon and then falls as $\frac{1}{r^{3/2}}$.

IV. EQUATION FOR LANGMUIR OSCILLATION IN KERR PLASMA

From the perturbed system of equations (35)-(37), and considering the Lorentz invariant density, we find

$$\frac{\partial^2 n_{s1}}{\partial t^2} + \omega_p^2 n_{s1} + \frac{q_s}{m_s\gamma_0}(\alpha E_1^r)\frac{\partial}{\partial r}(\alpha^2 n_{s0})$$
$$-\frac{\alpha}{r^2}\{\alpha^2 r^2 n_{s0}\frac{\partial}{\partial r}(v_{s0}^r v_{s1}^r)\} + \frac{\alpha}{r^2}\frac{\partial}{\partial r}(\alpha r^2 v_{s0}^r \frac{\partial n_{s1}}{\partial t}) = 0, \tag{40}$$

where,

$$\omega_p^2 = \sum_s \frac{4\pi\alpha^2(r,\theta)n_{s0}(r,\theta)q_s^2}{m_s\gamma_0}, \tag{41}$$

is the plasma frequency around the rotational gravitating object.

Considering a uniform distribution of plasma, we put $n_{s0}(r,\theta) = n_0$. Then we may write

$$\omega_p^2 = \omega_{p\infty}^2\alpha^2(r,\theta), \tag{42}$$

where,

$$\omega_{p\infty}^2 = \sum_s \frac{4\pi n_0 q_s^2}{m_s\gamma_0}$$

is the plasma frequency at $r = r_\infty$.

We plot the radial dependence of plasma frequency for the fixed values $\theta = \pi/2, \theta = 0$ with the fixed values of other parameters. It shows for both the cases the curves are almost the same and it is shown in Fig. 2.

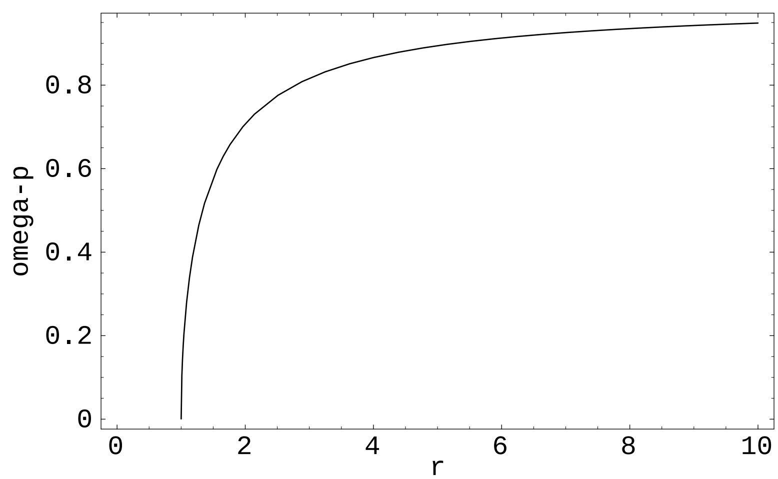

FIG. 2. Radial dependence of plasma frequency around a Kerr black hole.

From the above (Fig.2) we see that Langmuir plasma oscillation goes to zero near the Kerr event horizon i.e. at

$$r = r_0 = \frac{r_g}{2}(1 + \sqrt{1 - \frac{4a}{r_g^2}}).$$

Plasma oscillation sharply increases near the horizon and reaches the maximum at

$$r = r_m = \frac{5r_g}{6}(1 + \sqrt{1 - 36a^2/25r_g^2}),$$

and it attains a constant value away from the gravitating object.

For simplicity, we consider $v_{s0}^r(\theta) = 0$. Defining $\rho_{q1} = -en_{e1}$ and $n_i = n_0$, from equation (40), we get

$$\frac{\partial^2\rho_{q1}}{\partial t^2} + \omega_p^2\rho_{q1} = -\frac{2e^2n_0}{m}(\alpha E_1^r)\frac{\partial}{\partial r}(\alpha^2) \tag{43}$$

where, ρ_{q1} is defined by equation (35) i.e.

$$\rho_{q1} = \frac{1}{4\pi}\frac{\alpha}{r^2}\frac{\partial}{\partial r}(r^2E_1^r). \tag{44}$$

Now, eleminating ρ_{q1} from equations (43) by Eq.(44), we find

$$\frac{\partial}{\partial r}\{r^2(\frac{\partial^2 E_1^r}{\partial t^2}+\omega_{p\infty}^2\alpha^2 E_1^r)\}=0, \tag{45}$$

which describes the Langmuir field around the rotational gravitating objects.

Considering $E_1^r=\bar{E}_1^r(r,\theta)e^{-i\omega(r,\theta)t}$, equation (45) may be written as

$$\frac{\partial}{\partial r}\{r^2(\omega^2-\omega_{p\infty}^2\alpha^2)\bar{E}_1^r\}=0, \tag{46}$$

which yields a solution

$$r^2(\omega^2-{\omega_{p\infty}}^2\alpha^2)\bar{E}_1^r=F(\omega,\theta), \tag{47}$$

where, $F(\omega,\theta)$ is an arbitrary function including zero or constant. In the case $F(\omega,\theta)=0$, we get the usual Langmuir plasma oscillation with the eigen frequency

$$\omega^2=\omega_{p\infty}^2\alpha^2. \tag{48}$$

Thus the electric field associated with the Langmuir wave is given as

$$E(t,r,\theta)=\int\frac{F(\omega,\theta)}{r^2(\omega^2-\omega_{p\infty}^2\alpha^2)}e^{-i\omega t}\frac{d\omega}{2\pi}. \tag{49}$$

V. CONCLUSION

We have studied the Langmuir oscillations around rotato-gravitating compact objects such as black holes or neutron stars. The Kerr metric is utilised to study such a plasma. From the general system of equations, a wave equation is derived (equation (17)) which describes the various modes in the Kerr plasmas. For simplicity, we consider only the Langmuir oscillations. The system of equations for Langmuir oscillations is derived. The equations are linearized and the equilibrium as well as the perturbed quantities are studied in detail. We study the radial and theta dependence of physical quantities due to gravity and rotation of the plasma. It is found that GR effects on the equilibrium quantities are significant near the horizon. From the perturbed system of equation we derive the equation for Langmuir oscillation. The plasma frequency is obtained as a function of the lapse function. Lapse function is analysed in detail. It is noted that for a Kerr metric, the event horizons lie within the gravitational radius. Kerr parameter is always less than half of the gravitational radius. A strong GR effect is found on the plasma frequency near the event horizon. The lapse function for the Kerr metric enters in the expression of the plasma frequency similar to that in the Schwarzchild metric. Langmuir oscillation goes to zero at the event horizon, while it increases sharply near the horizon and reaches a maximum value near the horizon.The geometry of the Kerr event horizon is of course θ-dependent and inner than that of the Schwarzchild. A general equation describing the Langmuir oscillation in Kerr plasma is derived and the gravito-plasma dispersion relation is obtained. Results obtained in this study are foundational and can be extended for further study of Kerr plasma phenomenon around gravito-rotating compact objects.

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