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IC/2000/93

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

THE PLETHYSTIC HOPF ALGEBRA OF MACMAHON SYMMETRIC FUNCTIONS

Mercedes H. Rosas[1]
Departamento de Matemáticas, Universidad Simón Bolívar,
Apdo. Postal 89000, Caracas, Venezuela
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

A MacMahon symmetric function is a formal power series in a finite number of alphabets that is invariant under the diagonal action of the symmetric group.

We give a combinatorial overview of the plethystic Hopf algebra structure of the MacMahon symmetric functions relying on the construction of a plethystic Hopf algebra from any alphabet of neutral letters obtained by G.-C. Rota and J. Stein.

MIRAMARE – TRIESTE
August 2000

[1]E-mail: mrosas@usb.ve

1. Introduction.

In his seminal article [11] MacMahon initiated the systematic study of a new class of symmetric functions that he called symmetric functions of several systems of quantities. This class of functions had been previously considered by Cayley [2] and by Schläfli [20] in their investigations on the roots of polynomials. Following Ira Gessel [5, 6], and in honor of Major Percy MacMahon, we call this class of symmetric functions MacMahon symmetric functions.

The original motivation for the study of MacMahon symmetric functions comes from the following analogy with symmetric functions. On the other ahnd, symmetric functions appear when expressing a monic polynomial in terms of its roots. On the other hand, suppose that we can express the coefficients of a polynomial in two variables as a product of linear factors. That is, suppose that $e_{(0,0)} + \cdots + e_{(1,1)}xy + \cdots + e_{(n,n)}x^ny^n$ can be written as $(1+\alpha_1 x + \beta_1 y)\cdots(1+\alpha_n x+\beta_n y)$. Expanding the product of linear factors in the previous equation, we obtain symmetric functions like $e_{(0,0)} = 1$, $e_{(1,0)} = \alpha_1+\alpha_2+\cdots+\alpha_n$, and $e_{(0,1)} = \beta_1+\beta_2+\cdots+\beta_n$. But, we also get some things that are different, like

$$e_{(1,1)} = \alpha_1\beta_2 + \alpha_2\beta_1 + \cdots + \alpha_{n-1}\beta_n.$$
$$e_{(2,1)} = \alpha_1\alpha_2\beta_3 + \alpha_1\alpha_3\beta_2 + \cdots + \alpha_{n-2}\alpha_{n-1}\beta_n.$$

The relevant fact about this new class of symmetric functions is that they are invariant under the diagonal action of the symmetric group, but not under its full action. (The diagonal action of π in S_n on a monomial $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n}\beta_{j_1}\beta_{j_2}\cdots\beta_{j_n}$ is defined as $\alpha_{\pi i_1}\alpha_{\pi i_2}\cdots\alpha_{\pi i_n}\beta_{\pi j_1}\beta_{\pi j_2}\cdots\beta_{\pi j_n}$.) This class of functions that we just have found are the elementary MacMahon symmetric functions in two finite alphabets of size n. G.-C. Rota and J. Stein [19] have developed the analogy between symmetric functions and MacMahon symmetric functions even further in their studies on the theory of resultants.

As noted by MacMahon [10], in order to avoid syzygies between the elementary MacMahon symmetric functions, it is necessary to take the number of linear factors, or equivalently, the size of alphabets $X = \alpha_1 + \alpha_2 + \ldots$ and $Y = \beta_1 + \beta_2 + \cdots$, to be infinite. This observation leads to the following definition. A MacMahon symmetric function is a formal power series of bounded degree, in a finite number of infinite alphabets, that is invariant under the diagonal action of the symmetric group. As in the case of symmetric functions, each homogeneous piece of the algebra of MacMahon symmetric functions has the elementary MacMahon symmetric functions as an integral basis.

Probably, it is the fact that not all polynomials in several variables are expressable as products of linear forms, even over an algebraically closed field, which makes the MacMahon symmetric functions less ubiquitous than their symmetric relatives.

In this article we look at the plethystic Hopf algebra structure of the MacMahon symmetric functions from a combinatorial point of view. Barnabei, Brini, Joni, and Rota [3, 8, 16] have suggested a combinatorial interpretation of the product and the coproduct of a bialgebra. They have proposed that the product corresponds to the process of putting things together, and that the coproduct corresponds to the process of splitting them apart. We use their ideas as a starting point towards a combinatorial interpretation of some instances of the theory of plethystic Hopf algebras developed by Stein and Rota [7, 17, 18, 19]. In particular, we obtain a combinatorial interpretation for the plethystic Hopf algebra of MacMahon symmetric functions that extends the one developed by the author [13, 14, 15] for their vector space structure.

We assume that the reader is familiar with the basic notions of algebra, coalgebra, bialgebra, Hopf algebra, and with Sweedler's notation. A fine exposition of these topics is given in [1].

2. The plethystic Hopf algebra Gessel(A)

2.1. **The Hopf algebra** Super$[A]$**.** Grosshans, Stein, and Rota [7] have constructed a generalization of the ordinary algebra of polynomials in a set of variables A to the case where the variables can be of three different kinds: positively signed, neutral, and negatively signed. This new structure is called the supersymmetric algebra associated with the signed alphabet A, and denoted by Super$[A]$.

Let A^+ be a set of positive letters. The set of divided powers of A^+, denoted by $\mathrm{Div}(A^+)$, is constructed as follows. Let $\mathrm{Div}(A^+)$ be the quotient of $A^+ \times \mathbf{N}$ by the equivalence relation obtained by prescribing that $a^{(n)}$, the equivalente class of (a, n), behaves algebraically as $a^i/i!$. The map that sends the pair (a, n) to its equivalence class $a^{(n)}$ is called the divided powers operator. The details of this algebraic construction can be found in [7].

In this article, we introduce $\mathrm{Div}(A^+)$ in an equivalent way that has the advantage of showing its combinatorial nature. Let the pair (a, n) in $A^+ \times \mathbf{N}$ represent a set of n distinguishable balls of weight a. We define the divided powers operator acting on $A^+ \times \mathbf{N}$ as the operator that makes us to forget how to distinguish between objects of the same weight. In consequence, the image of (a, n) under the divided power operator, denoted by $a^{(n)}$, represents a set of n undistinguishable balls weight a.

To summarize: A positively signed letters represents the weight of an undistinguishable object. A monomial element in $\mathrm{Div}(A^+)$ has the form $a^{(i)}b^{(j)} \cdots c^{(k)}$ and represents a set consisting of i undistiguishable objects of weight a, j undistinguishable objects of weight b, and k undistinguishable objects of weight c. (We can always distinguish between object of different weights.) We interpret a sum of monomial elements of $\mathrm{Div}(A^+)$ as a disjoint union. In consequence, an element of $\mathrm{Div}(A^+)$ corresponds to a disjoint union of sets of undistinguishable objects.

Sometimes, it is convenient to think of an object of weight a as an object that has been colored a. In this framework, each positively signed letter correponds to a color. Moreover, (a, n) represents a set of n distinguishable balls that have been colored a, and $a^{(i)}$ corresponds to a set of i undistinguishable objects colored a. Again, the effect of the divided power operator is that we forget how to distinguish between objects of the same color.

Following Barnabei, Brini, Joni, and Rota [3, 8, 16], we interpret the product of element of $\mathrm{Div}(A^+)$ as the process of putting objects togheter. From this combinatorial interpretation, we get the following algebraic rule for the product of elements of $\mathrm{Div}(A^+)$:

$$a^{(i)}a^{(j)} = \binom{i+j}{i} a^{(i+j)},$$

because having a set consisting of i undistinguishable objects, together with another set that consists of j undistinguishable objects, is the same as having a set with $i+j$ undistinguishable objects, together with a distinguished subset of i elements.

Similarly, we the operation of exponentiation is defined by

$$(a^{(i)})^{(j)} = \frac{(ij)!}{j!(i!)^j} a^{(ij)},$$

because a set whose j elements are sets consisting of i objects is the same as a set with ij objects partitioned into j disjoint subsets, each of them consisting of i objects.

Finally, the following analog of Newton's Identity for $\mathrm{Div}(A^+)$ relates the sum with the product:

$$(a+b)^{(i)} = \sum_{j+k=i} a^{(j)} b^{(k)},$$

because a set with i objects, k of then of weight a and j of them of weight b, is equivalent to a set of k undistinguishable objects of weight a, together with a set of j undistinguishable objects of weight b.

The unit of this product is given by the weight of the empty set, and denoted by 1. Multiplying an element W of $\mathrm{Div}(A^+)$ by 1 corresponds to adding nothing to the object in W.

So far, we have described an algebra structure on Super$[A]$. To introduce its coalgebra structure, we follow Barnabei, Brini, Joni, and Rota [3, 8, 16] in interpreting the coproduct of $\mathrm{Div}(A^+)$ as the process of splitting objects apart. In consequence, the coproduct of a monomial element W, denoted by ΔW, describes all different ways of splitting the objects being weight by W into two different boxes. We interpret the boxes in terms of the tensor product. For instance, the term $a \otimes a^{(2)}b$ indicates that in the first box we have an object of weight a, and that in the second box we have two undistinguishable objects of weight a and an object of weight b.

The order in which we place balls of different weights do not affect the result. Hence, the coproduct should be multiplicative. We obtain the following algebraic rule for the coproduct of positive signed letters

$$\Delta a^{(i)} = \sum_{j+k=i} a^{(j)} \otimes a^{(k)}, \tag{1}$$

$$\Delta(WW') = \Delta W \Delta W' \tag{2}$$

For example,

$$\begin{aligned}\Delta a^{(2)}b &= (a^{(2)} \otimes 1 + a \otimes a + 1 \otimes a^{(2)})(b \otimes 1 + 1 \otimes b) \\ &= a^{(2)}b \otimes 1 + ab \otimes a + b \otimes a^{(2)} + a^{(2)} \otimes b + a \otimes ab + 1 \otimes a^{(2)}b.\end{aligned}$$

Let ϵ be the counit of our coalgebra. Using Sdweeler's notation, the counitary property says that $W = \sum W_{(1)}\epsilon(W_2) = \sum \epsilon(W_1)W_{(2)}$. Hencefort, from our combinatorial description we obtain that $\epsilon(W)$ equals 1 if $W = 0$, and $\epsilon(W)$ equals 0 if $W \neq 0$

The study of neutral variables is the algebraic analog of the study of weighted distinguishable objects. They behave as ordinary commuting variables. In consequence, the multiplication and the exponentiation are defined by $a^i a^j = a^{i+j}$ and $(a^i)^j = a^{ij}$, respectively. Moreover, Newton's identity relates the sum of neutral letters with their product.

The coproduct of neutral letters is defined in terms of placing distinguishable balls into distinguishable boxes. Therefore, we get the following rule: $\Delta a = a \otimes 1 + 1 \otimes a$. Moreover, the order in which we place distinguishable objects into the boxes does not affect the result, so the coproduct should be multiplicative. As in the case of positive letters, the coproduct is understood in terms of placing distinguishable objects into distinguishable boxes. For example, $\Delta a^2 b = a^2 b \otimes 1 + 2ab \otimes a + b \otimes a^2 + a^2 \otimes b + 2a \otimes ab + 1 \otimes a^2 b$, gives us all different ways of distributing two distinguishable objects of weight a, and a object of weight b between two different boxes. The occurrence of the factor 2 in terms $ab \otimes a$ and $a \otimes ab$ comes from the fact that objects of weight a are distinguishable. As in the case of positive letters, the counit is defined by saying that $\epsilon(W)$ equals 1 if $W = 0$, and $\epsilon(W)$ equals 0 if $W \neq 0$

Let A be an alphabet consisting of neutral and positively signed letters. (Negative letters do not appear in our study of the plethystic Hopf algebra structure of the MacMahon symmetric functions.) The superalgebra $\mathrm{Super}(A)$ is the algebra spanned by monomials in $\mathrm{Div}(A^+) \cup A^0$, where all letters commute and cocommute. So far, we have shown that $\mathrm{Super}(A)$ has structure of a bialgebra. But it has a richer structure, it is a graded, $\mathbf{Z}/2$ graded, commutative, cocommutative Hopf algebra, i.e., it is a supersymmetric algebra.

Theorem 1 (Rota-Stein)**.** *Let A be a signed alphabet, then* $\mathrm{Super}(A)$ *has the structure of a supersymmetric algebra.*

Given a monomial element W in $\mathrm{Super}(A)$, the degree W is defined as the number of objects that W is weighting, and denoted by $|W|$. There are no negatively signed letters, so it is automatically $\mathbf{Z}/2$ graded. The antipode sends W to $(-1)^{\deg(W)}W$.

2.2. **The plethystic Hopf algebra** $\mathrm{Gessel}(A)$**.** In this section we give a combinatorial overview of the construction of the plethystic Hopf algebra $\mathrm{Gessel}(A)$ from $\mathrm{Super}[A]$ introduced by G-C. Rota and J. Stein [17]. Since the objective of this paper is to study the plethystic Hopf algebra of MacMahon symmetric functions, from now on, we assume that alphabet A is composed of neutral letters. Positively signed letters will appear as the result of the construction of $\mathrm{Gessel}(A)$.

Let A be an alphabet of neutral letters. We consider the monomials in $\mathrm{Super}[A]$ to be the positive letters of a new alphabet, also denoted by $\mathrm{Super}[A]$. Using the procedure described in the previous section, we construct the supersymmetric algebra, $\mathrm{Super}(\mathrm{Div}(\mathrm{Super}[A]))$, associated to the set of positive letters $\mathrm{Div}(\mathrm{Super}[A])$. A monomial element in $\mathrm{Div}(\mathrm{Super}[A])$ looks like

$(\omega)^{(i)}(\omega')^{(j)}\cdots(\omega'')^{(k)}$ where ω, ω', and ω'' are different monomial elements in Super$[A]$. We set $(1) = 1$. Letters in Super$[A]$ correspond to weights of distinguishable balls. Similarly, letters in Div(Super$[A]$) correspond to weights of undistinguishable packages of balls. Monomial elements of Super(Div(Super$[A]$)) are made out of positive letters, this defines a supersymmetric algebra structure on them. In particular, Super(Div(Super$[A]$)) is graded by saying that the degree of W is the number of packages of balls that W is weighting. We denote the degree of W by $|W|$.

A fundamental part of the plethystic algebra structure of Gessel(A) is the existence of a Laplace pairing that we proceed to describe combinatorially. The Laplace pairing maps the element (W, W') of Super(Div(Super$[A]$))$\times$Super(Div(Super$[A]$)) to $(W|W')$ in Super(Div(Super$[A]$)) as follows. First, it finds all possible bijections between the set of packages of W and the set of packages of W'. Then, for each such bijection, if package (ω) corresponds to package (ω'), the Laplace pairing puts all balls in ω and ω' together in the same package of $(W|W')$.

Theorem 2. *Suppose that $(u_1)(u_2)\cdots(u_n)$ and $(v_1)(v_2)\cdots(v_n)$ are monomial elements of* Super(Div(Super$[A]$)).. *Let M be the square matrix obtained from $W = (u_1)(u_2)\cdots(u_n)$ and $W' = (v_1)(v_2)\cdots(v_n)$ by making (u_iv_j) be its ij entry. Then,*

$$((u_1)(u_2)\cdots(u_n)|(v_1)(v_2)\cdots(v_n)) = \mathrm{Per}(M).$$

Proof. The symmetric group S_n is the set of bijections of $[n]$ onto itself. Therefore,

$$((u_1)(u_2)\cdots(u_n)|(v_1)(v_2)\cdots(v_n)) = \sum_{\sigma\in S_n}(u_1v_{\sigma 1})(u_2v_{\sigma 2})\cdots(u_nv_{\sigma n}).$$

By definition, this is the permanent of matrix M, see [12]. □

From the combinatorial definition of the Laplace pairing, we can deduce a recursive definition [17].

1. Set $(1|1) = 1$.

   If we pair the empty package with itself, we obtain the empty package.
2. If $W = (\omega)^{(i)}$ and if $W' = (\omega')^{(j)}$, with $|\omega| > 0$ and $|\omega'| > 0$, then $(W|W') = (\omega\omega')^{(i)}$ if $i = j$ and $(W|W') = 0$ otherwise.

   Given packages $(\omega)^{(i)}$ and $(\omega')^{(j)}$. The Laplace pairing finds all bijections between packages of $(\omega)^{(i)}$ and packages of $(\omega')^{(j)}$. If i equals j, then there is exactly one such bijection. On the other hand, if i is different from j, there can be not such a bijection.
3. If W have packages of different weights, then $(W|W')$ can defined recursively by the Laplace identity:

   $$(UV|W) = \sum (U|W_{(1)})(V|W_{(2)})$$

   where $\Delta W = \sum W_{(1)} \otimes W_{(2)}$.

   Suppose that W have more than one class of packages. Let UV be an arbitrary partition of W into two non-empty parts. The Laplace pairing splits the packages weighted by W' in all possible ways by taking its coproduct. Then, it proceeds recursively. The only nonzero terms that it can possibly obtain are those where the degree of U equals the degree of $W_{(1)}$ and where the degree of V equals the degree of $W_{(2)}$. In particular, $(W|W')$ is equal to zero if the degree of W is different than the degree of W'.

Similarly, if W' have packages of different weights, then $(W|W')$ can defined recursively by the dual Laplace identities:

$$(U|VW) = \sum (U_{(1)}|V)(U_{(2)}|W),$$

where $\Delta U = \sum U_{(1)} \otimes U_{(2)}$.

The Laplace pairing allows us to define the circle product between elements of Super(Div(Super$[A]$)) by

$$U \circ V = \sum U_{(1)}(U_{(2)}|V_{(1)})V_{(2)} = \sum U_{(1)}V_{(1)}(U_{(2)}|V_{(2)}). \tag{3}$$

Note that the second equality follows from the fact that the coproduct is cocomutative. The pair (Super(Div(Super$[A]$)), $\circ$) is the Cliffordization of Super$[A]$, and is denoted by Pleth(Super(A)). The construction of Cliffordization of a supersymmetric algebra is studied in a more general setting in [17], where they obtained the following result.

Theorem 3 (Rota-Stein). *Let A be any signed alphabet. The Cliffordization of* Super$[A]$*, denoted by* Pleth(Super$[A]$) *is an associative Hopf algebra. The antipode is given by the Smith's formula*

$$s(W) = \epsilon(W) + \sum_{k>0} (-1)^k (W_{(1)} - \epsilon(W_{(1)})) \circ \cdots \circ (W_{(k)} - \epsilon(W_{(k)})). \tag{4}$$

If A is an alphabet of neutral letters, we follow Rota and Stein and denote Pleth(Super$[A]$) by Gessel(A). In the next section, we show that Gessel(A) is isomorphic, as a plethystic Hopf algebra, to the MacMahon symmetric functions.

Remark. There are two different products on Gessel(A). One that comes from the algebra structure of Super$[A]$, and is called the juxtaposition product, and another one that comes from the plethystic Hopf algebra structure, and is called the circle product. Later, we see that it is useful to introduce a third product on Gessel(A), the square product.

3. A combinatorial overview of the phletystic Hopf algebra of MacMahon symmetric functions.

Joel Stein and Gian-Carlo Rota have introduced an isomorphism between the plethystic Hopf algebra Gessel(A), where A is an alphabet consisiting of n neutral letters, and the MacMahon symmetric functions in n alphabets, denoted by $\mathfrak{M}^n$. They called this map the Gessel map [18]. In particular, if A consists of just one neutral letter, the Gessel map defines an isomorphism between Gessel(A) and the plethystic Hopf algebra of symmetric functions.

Rota and Stein have shown how to associate a plethystic Hopf algebra Pleth(H) to any supersymmetric algebra H [17] obtaining a generalization of the plethystic Hopf algebra of MacMahon symmetric functions. This is an intriguing object of study that we do not pursue here. A particularly striking case appears when A is an alphabet of negative letters. Then, Pleth(Super$[A]$) is isomorphic under the Gessel map to the skew-symmetric MacMahon symmetric functions. In this case, the role of the permanent will be played by the determinant function.

3.1. **The Gessel map.** As suggested by the work of the author [13, 14], we interpret the Gessel map as the generating function for a process of placing balls into boxes according to certain rules. Suppose that we have an infinite set of boxes labeled by the natural numbers. Let a be a letter in A. We write $(a|i)$ to indicate that we have placed a ball of weight a in box i. Sometimes, we denote $(a|i)$ by x_i, $(b|i)$ by y_i, $(c|i)$ by z_i, and so on.

Definition 4. We define the Gessel map

$$G : \mathrm{Gessel}(A) \to \mathfrak{M}^n$$

as the map that sends W in Gessel(A) to the generating function for the process of placings of the balls being weighted by W into distinguishable boxes labeled by the natural numbers according to the following rules:

- Balls that belong to different packages are placed into different boxes.
- Balls that belong to the same package are placed into the same box.

The Gessel map defines an isomorphism between Gessel(A) and the MacMahon symmetric functions on $|A|$ letters. To see that this important result holds, we introduce some definitions. There is a bijection between monomial elements in Gessel(A) and vector partitions. It associates to the monomial element $W = (\omega_1)^{(i_1)}(\omega_2)^{(i_2)} \cdots (\omega_l)^{(i_l)}$ the vector partition $\lambda = (a_1, b_1, \cdots, c_1)^{i_1}(a_2, b_2, \cdots, c_2)^{i_2} \cdots (a_l, b_l, \cdots, c_l)^{i_l}$, where a_j is the number of elements of weight a in ω_j, b_j is the number of elements of weight b in ω_j, and so on. We say that λ is the vector partition associated to the monomial element W. In particular, if there are no two balls in W of the same weight, then W is a set partition. In this case, we say that λ is a unitary vector partition.

Any vector partition $\lambda = (b_1, r_1, \ldots, w_1)(b_2, r_2, \ldots, w_2)\ldots$ determines a monomial $\mathbf{x}^\lambda = x_1^{b_1} y_1^{r_1} \cdots z_1^{w_1} x_2^{b_2} y_2^{r_2} \cdots z_2^{w_2} \cdots x_l^{b_l} y_l^{r_l} \cdots z_l^{w_l}$. The monomial MacMahon symmetric function indexed by λ is the sum of all distinct monomials that can be obtained from $\mathbf{x}^\lambda$ by a permutation π in S_∞, where the action of π in $\mathbf{x}^\lambda$ is the diagonal action. That is,

$$m_\lambda = \sum_{\text{different monomials}} x_{i_1}^{b_1} y_{i_1}^{r_1} \cdots z_{i_1}^{w_1} x_{i_2}^{b_2} y_{i_2}^{r_2} \cdots z_{i_2}^{w_2} \cdots.$$

Theorem 5. *Let W be a monomial element in* Gessel(A)*, and let λ be the associated vector partition. The image under the Gessel map of W is the monomial MacMahon symmetric function m_λ. Moreover, if λ is a partition of a number, then its image is the symmetric function m_λ.*

Proof. Let $W = (a_1b_1\cdots c_1)^{i_1}(a_2b_2\cdots c_2)^{i_2}\cdots(a_lb_l\cdots c_l)^{i_l}$ be a monomial element in Gessel(A). If f is one of the placing being weight by the Gessel map, then the weight of f is $x_1^{a_1}y_1^{b_1}\cdots z_1^{c_1}\cdots x_l^{a_l}y_l^{b_l}\cdots z_l^{c_l}$. Hencefort, the image of W under G is the monomial MacMahon symmetric function m_λ. □

Theorem 5 allows us to define the monomial MacMahon symmetric functions as the image under the Gessel map of monomial elements of Gessel(A). For instance, assume that $A = \{a, b\}$ is an alphabet with two neutral letters. Then, the monomial MacMahon symmetric function $m_{(2,1)(0,1)}$ is defined as $G(a^2b)(b) = \sum_{i\neq j}(a|i)^{(2)}(b|i)(b|j) = \sum_{i\neq j} x_i^2 y_i y_j$. Similarly, the monomial symmetric function $m_{(2)(1)}$ is defined as $G(a^2)(a) = \sum_{i\neq j}(a|i)^{(2)}(a|j) = \sum_{i\neq j} x_i^2 x_j$.

A monomial element in Gessel(A) is called elementary if it corresponds to the weight of a set of packages consisting of exactly one ball. Elementary monomials have the form $(a)^{(i)}(b)^{(j)}\cdots(c)^{(k)}$, with i, j, k greater than or equal to zero. A monomial element W in Gessel(A) is called primitive if it is the weight of exactly one package of balls. Primitive elements have the form $W = (\omega)$, for some monomial element ω in Super$[A]$.

We can use the previous theorem to define the Elementary MacMahon symmetric functions and the power sum Macmahon symmetric functions as the image under the Gessel map of elementary monomials, and of primitive elements of Gessel(A), respectively.

Let $W = (a)^{(i)}(b)^{(j)}\ldots(c)^{(k)}$ be an elementary monomial in Gessel(A). Then, the image of W under the Gessel map is the elementary MacMahon symmetric functions $e_{(i,j,\cdots,k)}$. It corresponds to the generating function for all different ways of placing the balls being weighted by W into different boxes.

Let (ω) be a primitive element in Gessel(A) corresponding to vector partition $(i, j, \cdots, k)$. Then, its image under the Gessel map is the power sum MacMahon symmetric functions $p_{(i,j,\cdots,k)}$. It corresponds to the generating function for all different ways of placing the balls being weighted by (ω) into one box.

For instance, the elementary symmetric function $e_{(n)}$ is defined as $G(a)^{(n)}$ and the elementary MacMahon symmetric functions $e_{(1,2)}$ is defined as $G(a)(b)^{(2)}$, it corresponds to $\sum_{\substack{i_i<i_2 \\ \text{all different}}} x_j y_{i_1} y_{i_2}$. Moreover, $G(a^n)$ equals the power sum symmetric function $p_{(n)}$ and $G(a^pb^q)$ equals the polarized power sum symmetric function $p_{(p,q)}$.

3.2. **The circle product of monomial elements in** Gessel(A)**.** The circle product of elements in Gessel(A) corresponds to the ordinary product of MacMahon symmetric functions, in this section we study some of the consequences of this fact. Any monomial element in Gessel(A) can be thought of as a partition of a multiset. For instance, $(a)^{(2)}(b)(ab)$ corresponds to the multiset partition $a|a|b|ab$. Given two monomial elements W and W' in Gessel(A), we associate them with a set of multiset partitions denoted by Par(W, W'), and defined as follows: If $|W| \geq |W'|$, then for each injection of W' into W, join the balls of each package of W with those of its image. On the other hand, if $|W| < |W'|$, set $\mathrm{Par}(W,W') = \mathrm{Par}(W',W)$. For instance,

$$\mathrm{Par}(a|a|b, ad) = \{a^3d|a|b, a|a|abd\}.$$

Let Π_M be the lattice of multiset partitions having underlying multiset M, ordered by the refinement relation. That is, for π and σ in Π_M, we say that π is a refinement of σ, written as $\pi \leq \sigma$, if every block of π is a sub multiset of some block of σ.

Theorem 6. *The circle product of two monomial elements W and W' is equal to the sum of all monomials V that are less than or equal to one of the members of* $\mathrm{Par}(W,W')$.

For instance, let $W = (a)^{(2)}(ab)$, and $W' = (ac)$. Then, $\mathrm{Par}(W,W') = \{a^2c|a|ab, a|a|a^2bc\}$. Hence,

$$W \circ W' = (a^2c)(a)(ab) + (a)^{(2)}(a^2bc) + (a)^{(2)}(ab)(ac).$$

Theorem 7 (Rota-Stein)**.** *The Gessel map is an algebra map:*

$$G(W \circ W') = G(W)G(W')$$

We define the elementary MacMahon symmetric functions as the image under the Gessel map of circle products of elementary monomials.

Definition 8. Let $W_1, W_2, \cdots, W_l$ be elementary monomial, and let $\lambda_1, \lambda_2, \cdots, \lambda_l$ be the associated vectors. We say that $\lambda = \lambda_1, \lambda_2, \cdots, \lambda_l$ is the vector partition associated to $W_1 \circ W_2 \circ \cdots \circ W_l$. If W is an elementary monomial, the the vector partition associated to W has exactly one part.

Corollary 9. *Let $W_1, W_2, \cdots, W_l$ be elementary monomials. Let λ be associated vector partition. The expression of $W_1 \circ W_2 \circ \cdots \circ W_l$ in the monomial basis is the following:*

$$W_1 \circ W_2 \circ \cdots \circ W_l = \sum_{\pi \wedge \lambda = \hat{0}} \pi.$$

It corresponds to all different ways of placing the balls being weighted by $W_1 \circ W_2 \circ \cdots \circ W_l$ into boxes, with the condition that balls in different packages of π go into different boxes.

For example,

$$\begin{aligned}(a)^{(2)}(b) \circ (c)^2 =& (ac)^{(2)}(b) + (a)(ac)(bc) \\ &+ (a)(c)(ac)(b) + (a)^{(2)}(c)(bc) + (a)^{(2)}(b)(c)^{(2)}.\end{aligned}$$

This is the characterization of the elementary MacMahon symmetric functions obtained in [13, 14].

We define the power sum MacMahon symmetric functions as the image under the Gessel map of the circle product of primitive elements.

Corollary 10. *The circle product of primitive elements $W, W', \cdots W''$ is the sum of all monomials that are bigger than or equal to the partition $WW' \cdots W''$. It corresponds to all different ways of placing the balls being weighted by π into boxes, with the condition that balls in the same package of $WW' \cdots W''$ go into the same box.*

For example,

$$(a^n) \circ (b^m) \circ (c^l) = (a^n b^m c^l) + (a^n b^m)(c^l) + (a^n c^l)(b^m) \\ + (a^n)(b^m c^l) + (a^n)(b^m)(c^l).$$

This is the characterization of the power sum MacMahon symmetric functions obtained in [13, 14].

3.3. **The plethysm of MacMahon symmetric functions.** The operation induced by the Laplace pairing on the algebra of MacMahon symmetric functions is very close to the permanent function. It has a beautiful description on the monomial basis.

Let $(\omega_1)(\omega_2)\cdots(\omega_k)$ and $(\omega_1')(\omega_2')\cdots(\omega_l')$ be monomial elements of Gessel(A). Let M be the matrix whose i,j entry is $(\omega_i\omega_j)$. We have seen that

$$((\omega_1)(\omega_2)\cdots(\omega_k)|(\omega_1')(\omega_2')\cdots(\omega_l'))$$

equals zero unless $k=l$. In this case, it equals the permanent of the matrix M whose i,j entry is $(\omega_i\omega_j)$.

4. Involution ω and the square product.

On the MacMahon symmetric functions there is a remarkable operation called involution ω [9, 21]. It is well-known that it corresponds to the antipode s_0 of Gessel(A). In this section, we use involution ω to define two remarkable basis for the MacMahon symmetric functions: the homogeneous and the forgotten MacMahon symmetric functions.

Joel Stein and Gian-Carlo Rota have looked at the pull-back of involution ω, and used it to define a new operation on Gessel(A), the bar product. We study their construction in this section.

Let $W_{<i>} = (1-\epsilon)W_{(i)}$. We rewrite the antipode s_0 as

$$s_0 W = \sum_{r\geq 1} (-1)^r W_{<1>} \circ W_{<2>} \circ \cdots \circ W_{<r>}. \tag{5}$$

See Eq.(4). The antipode acts as follows: For each k the antipode looks for all possible ways of splitting the packages of balls weighted by W into k different boxes, so that packages of balls weighted by $W_{<k>}$ are in box k, and such that no box remains empty. Then, it takes the signed circle product of the weight of the packages obtained in this way. For instance, $s_0(\omega) = -\omega$ and $s_0(\omega)^{(2)} = -(\omega)^{(2)} + (\omega)\circ(\omega) = (\omega)^{(2)} + (\omega^2)$.

We define the homogeneous MacMahon symmetric functions as the image under the antipode of the elementary MacMahon symmetric functions. To get a constructive definition, we start by describing those monomials of Gessel(A) that are the preimage of the homogenous MacMahon symmetric functions under the Gessel map.

We follow Rota and Stein [19], and define the box product of monomial elements in Gessel(A) by

$$W\Box W' = s_0[s_0[W]s_0[W']].$$

Note that $(a)\Box(a) = s_0[s_0[a]s_0[a]] = 2!s_0[(a)^{(2)}] = 2!((a^{(2)} + (a^2))$. The image under the Gessel map of $\frac{1}{2!}(a)\Box(a)$ is the homogeneous symmetric function h_2. Based on these observations, we define

$$(a)^{[n]} = (-1)^n \frac{(a)\Box(a)\Box\cdots\Box(a)}{n!}.$$

A Wronski element is an element of Gessel(A) of the form

$$(a)^{[i]}\Box(b)^{[j]}\cdots\Box(c)^{[k]}.$$

The image of a Wronski element under the Gessel map is a homogeneous MacMahon symmetric function, as we can easily check by taking the antipode at both sides of the defining equation. Then, we use induction. A combinatorial interpretation of this class of elements is obtained in the following theorem.

The image of $(a)^{[i]}\Box(b)^{[j]}\cdots(c)^{[k]}$ under the Gessel map corresponds to the MacMahon symmetric functions $h_{(i,j,\cdots,k)}$. It gives all different ways of placing the balls being weighted by the underlying monomial $(a)^{(i)}(b)^{(j)}\cdots(c)^{(k)}$ into boxes, with the condition that balls within the same box are linearly ordered. The combinatorial description follows from [13, 14].

The following Lemma [22] describes some properties of the bar product.

Lemma 11 (Rota-Stein). *Properties of the box product*

1. $s_0[W \circ W'] = s_0[W] \circ s_0[W']$
2. $s_o[WW'] = s_0[W]\Box s_0[W']$
3. $s_0[W\Box W'] = s_0[W]s_0[W']$

Moreover, the associativity of the juxtaposition product implies the associativity of the box product.

The first property of the box product described in the previous lemma implies that the image under the Gessel map of circle product of Wronski elements are the homogeneous MacMahon symmetric functions.

The forgotten MacMahon symmetric functions are defined as the image, under the antipode map, of the monomial MacMahon symmetric functions [4]. We have seen that the monomial MacMahon symmetric functions correspond to the image under the Gessel map of the monomial elements of Gessel(A). So, the previous lemma implies that the forgotten MacMahon symmetric functions are the image of monomial elements when using the bar product instead of the circle product.

Let $D = (\omega_1)^{[i]}\Box(\omega_2)^{[j]}\Box\cdots\Box(\omega_l)^{[k]}$ be a Doubilet element with underlying monomial $W = (\omega_1)^{(i)}(\omega_2)^{(j)}\cdots(\omega_l)^{(k)}$. Let λ be the vector partition associated with W. The image of D

under the Gessel map is the forgotten MacMahon symmetric function f_λ. It corresponds to the placement of the balls being weighted by W into boxes, where balls coming from the same package go to the same box. Moreover, we require that within each box, the packages appearing are linearly ordered.

The Gessel map is the generating function for the process of placing balls into boxes where balls in the same block go to the same box, and balls in different blocks go into different boxes. The antipode turns juxtapositions products into square product. The image under the Gessel map of square products elementary monomials correspond to placings with the only condition that balls that are in the same box, and that come from the same block are linearly ordered.

5. Operators on the plethystic Hopf algebra of MacMahon symmetric functions.

In the first part of the article, we gave a combinatorial picture of the plethystic Hopf algebra of MacMahon symmetric functions. The aim of this second part is to put some movement into this picture by defining some operators acting on it. We are particularly interested the polarization and the substitution operators, described by G.-C. Rota and J. Stein [18], and the projection and the lifting operator introduced by the author [13, 14].

Let A be an alphabet with k neutral letters. We define a finer grading on Gessel(A), called the homogeneous degree, by

$$\mathrm{Gessel}(A) = \bigcup_{i\geq 0} \bigcup_{\substack{u\in\mathbf{N}^k\\ \mathrm{weight}(u)=i}} \mathrm{Gessel}_u(A)$$

where $\mathrm{Gessel}_u(A)$ is the set of all monomial elements of Gessel(A) of type $u = (u_1, u_2, \cdots, u_n)$. The type of a monomial is the vector defined by the total number of balls of each color that the monomial is weight of, that is u_i is the number of balls of weight i on $\mathrm{Gessel}_u(A)$.

There are two homogeneous pieces on Gessel(A) that we want to emphasize. On the one hand, we have $\mathrm{Gessel}_{(1,1,\cdots,1)}(A)$ that consists of all monomial weights of packages of balls, with a total number of k balls, each of them colored using a different color. Their image under the Gessel map are called the unitary symmetric functions. They correspond to set partitions.

On the other hand, there is $\mathrm{Gessel}_{(n)}(A)$. It consists of all monomial weights of packages of balls, with a total number of k balls, and where all balls have been colored with the same color. Their image under the Gessel map are the symmetric functions. They correspond to partitions of a number.

5.1. **The polarization operator.** If A is an alphabet of neutral letters, we define the polarization operator $D(b, a)$ acting on Super[A], with a and b in A, as

$$D(b,a)a^k b^l \omega'' = \binom{k}{1} a^{k-1} b^{l+1} \omega''.$$

where ω'' is a monomial that does not contain letters a nor b. The polarization operator chooses one of the distinguishable balls of weight a and changes its weight to b in one out of k different ways. Then, we extend this definition to Gessel(A) by

$$D(b,a)(\omega)^{(p)} = (\omega)^{(p-1)}(D(a,b)\omega).$$

Here, the polarization operator acts on packages of balls. Packages of the same weight are undistinguishable, so there is only one way of choosing the package where the polarization operator acts on.

The polarization operator defines a derivation on Gessel(A),

$$D(b,a)\{WW'\} = \{D(b,a)W\}W' + W\{D(b,a)W'\}$$

because when we pick a ball from the set of packages weighted by WW', we either take it from W and leave W' fixed, or take it from W' and leave W fixed.

The polarization operator does not change the weight of a monomial element of Gessel(A). But, it does change its homogeneous degree. Suppose that the homogeneous degree of W is (u_1, u_2). Then, $D(b,a)W$ has homogeneous degree $(u_1 - 1, u_2 + 1)$.

The polarization operator acts on the plethystic Hopf algebra of MacMahon symmetric functions. The polarization operator on Gessel(A) induces an operator of the MacMahon symmetric functions. We abuse notation and also call the operator obtained in this way polarization operator. The induced operator can be defined explicitly by the following formula

$$\sum_i y_i \frac{\partial}{\partial x_i}.$$

as we can easily check on the power sum MacMahon symmetric functions basis. Note that the polarization operator on the MacMahon symmetric functions sends $p_{(n)}$ to $np_{(n-1,1)}$.

Theorem 12 (Rota-Stein). *The polarization operator commutes with the Gessel map.*

5.2. **The substitution operator.** The substitution operator $S(\omega, a)$ acts on Super$[A]$ as follows. Let $\omega' = a^k\omega''$, where ω'' contains no letter a. Then,

$$S(\omega,a)(\omega') = S(\omega,a)(a^k\omega'') = (\omega^k\omega'')$$

and extends multiplicatively to the whole algebra.

The substitution operator $S(\omega, a)$ acts in two stages. First, it selects all balls of weight a inside each of the packages, and then it substitutes each of the balls of weight a by a set of balls of weight ω. The substitution operator is an algebra map.

5.3. **The projection operator.** Let W be an element of Gessel(A) of degree n corresponding to the weight of packages of balls, where no weight is repeated. The image of W under the Gessel map is a unitary MacMahon symmetric functions. Moreover, W can be identified with a set partition. Given any composition $u = u_1 + u_2 + \cdots + u_k$ of n. The polarization operator ρ_u gives weight 1 to the first u_1 balls, weight 2 to the next u_2 balls, and so on. Moreover, balls in the same package of W are kept in the same block of the resulting multiset partition. For instance, let $u = (3, 1, 2)$. Then, ρ_u weights $1, 2$, and 3 by 1, 4 is weighted 2, and 5 and 6 are weighted 3.

The projection map defines an operator on Gessel(A). Note that if $u = (1, 1, \cdots, 1$, the projection map correponds to the identity. On the other hand, if u equals n, the images of the projection map are the symmetric functions of degree n. We can go from the unitary monomials to the one-color monomials by making all balls to be of the same color. More generally, we can move from one homogeneous piece of Gessel(A) to another one of the same weight by changing the colors of the balls. This idea was pursued by the author in [13, 14] where she defined the projection map

$$\rho_u : \text{Gessel}_{(1,1,\cdots,1)}(A) \to \text{Gessel}_u(A)$$

where u is a partition of weight n.

This map takes some of the balls that have different colors and makes them to have the same one. The projection map has been used to compute the connection coefficients between the different basis of the ring of polynomials [15].

The projection map can be defined through the iterated use of the substitution operator, but it is useful to do it this way.

5.4. **The lifting operator.** The main property of the polarization operator is that it changes the weights of the balls being weighted by a monomial element of Gessel(A). Applying it several times, we get all balls to have different colors. At this point, the monomial element obtained is unitary and we use the appendices appearing in [13, 4, 14] to work with them. They indicate how to make a change of basis, and how to take the internal and the Kronecker products.

The problem with this approach is that the unitary monomial obtained in this way depends on the choices of the polarization operator that we have made. The Lifting operator $\hat{\rho}_u$ does not have this problem. Let W be a monomial element on Gessel(A), and let u be its homogeneous degree. We define $\hat{\rho}_u(W)$ as the sum of all monomial elements in the preimage of W under the projection map. Note that $\hat{\rho}_u(W)$ belongs to $\text{Gessel}_{(1,1,\cdots,1)}(A)$.

5.5. **The permanent and Binet formula.** The permanent of an $n \times m$ matrix is the elementary MacMahon symmetric function in n alphabets of size m, evaluated at the entries of the matrix. For instance,

$$\text{Per}\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & y_n \\ z_1 & z_2 & z_3 & \cdots & z_n \end{pmatrix} = \sum_{i,j,k \text{ differents}}^{n} x_i y_j z_k = e^{(n)}_{(1,1,1)}$$

where we use the superindex n to indicate that the alphabets consist of n letters.

Hence, the permanent function gives us the number of ways of placing three balls, one of weight a, one of weight b, and one of weight c into n different distinguishable boxes.

In [13] the author showed how to use Doubilet's tables for the matrices of change of basis between the different basis for the MacMahon symmetric functions just mentioned. As an application, we obtain Binet's formula [12] for the evaluation of the permanent.

$$\mathrm{Per}(A) = \sum_{\sigma} \mu(\hat{0}, \sigma) p_{\sigma}.$$

6. Acknowlegments

I feel fortunate for having been introduced to the study of the MacMahon symmetric functions by Professors Ira Gessel, Joel Stein, and the late Gian-Carlo Rota.

It has been a pleasure to enjoy the hospitality of the people at the Abdus Salam International Centre for Theoretical Physics at Trieste, Italy during the elaboration of this article.

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