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PROJECTIVE NORMALITY OF THE WONDERFUL COMPACTIFICATION OF SEMISIMPLE ADJOINT GROUPS

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Abstract

We prove that if G is a semisimple adjoint group over an algebraically closed field of arbitrary characteristic, X is the wonderful compactification of G and $\widehat{G}$ is a simply connected covering of G, then for any ample line bundle L over X, the cone over X given by L is normal.

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1 Introduction

In [4], what are known as "wonderful compactifications" of symmetric varieties were constructed and studied by C. De Concini and C. Procesi. More precisely, if G is a semisimple adjoint group over the field of complex numbers, H is the subgroup of all fixed points of an involution σ of G that is induced by an involution $\widehat{\sigma}$ of the simply connected covering $\widehat{G}$ of G, then, they have constructed a complete embedding $\overline{G/H}$ of the homogeneous space G/H, with boundary being a union of normal crossing divisors. In particular, one gets such a compactification $\overline{G}$ for the group G (G being considered as $(G \times G)/\Delta(G)$). In [10], E.Strickland has proved that the wonderful compactifications for the group exists over algebraically closed fields of positive characteristics. In [5], C. De Concini and T.A. Springer have proved that these compactifications for arbitrary symmetric space G/H exists when the base field is of characteristic $p \neq 2$. In [6], G. Faltings raised the question: For what $\widehat{G}$- linearised line bundles L, the cone over $\overline{G/H}$ given by L is normal?

The aim of this paper is to provide an affirmative answer to his question for the case of the wonderful compactification of a semisimple adjoint group G over an algebraically closed field of arbitrary characteristic. To be more precise, we prove that if G is a semisimple adjoint group over an algebraically closed field of arbitrary characteritic, $\widehat{G}$ is a simply connected covering of G, then for any $\widehat{G} \times \widehat{G}$- linearised very ample line bundle L over the woderful compactification $\overline{G}$, the cone over $\overline{G}$ given by L is normal. In section 3, we prove this result when the base field is the field of complex numbers. For a more precise statement, see Theorem 3.4. In section 4, we prove the result for algebraically closed fields of positive characteristic by using the properties of good filtrations.

2 Notations and basic Theorems

Throughout sections 2 and 3, we fix the following notations: Let G be a semisimple adjoint group over the field of complex numbers. Let $\pi : \widehat{G} \longrightarrow G$ be a simply connected covering of G. Let $k[\widehat{G}]$ denote the co-ordinate ring of $\widehat{G}$. Let T be a maximal torus of G and B be a Borel subgroup of G containing T. Let B^- be the opposite Borel subgroup of G determined by T and B. Let $\widehat{T}$ denote the pull back of T in $\widehat{G}$. Let $\widehat{B}$ (resp. $\widehat{B^-}$) denote the pull back of B (resp. B^-) in $\widehat{G}$. Let W denote the Weyl group of G with respect to T.

Let Φ denote the set of roots with respect to T as above and $\Phi^+ \subset \Phi$ denote the set of positive roots with respect to B as above.

Let $\Delta \subset \Phi^+$ denote the set of simple roots. We label the elements of Δ by :

$$\Delta := \{\alpha_1, \alpha_2, ...\alpha_l\}, \; l = rank(G).$$

Let χ denote the weight lattice of G with respect to T. Let $E := \chi \bigotimes \mathbb{R}$ denote the $\mathbb{R}$ span of χ. Let $(.,.)$ denote the positive definite bilinear form on E induced by the Killing form of the Lie algebra of G and define $\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}$. Now, consider the fundamental weights. Since, they form a dual basis of the simple coroots, we label them:

$$\{\varpi_1, ...\varpi_l\},$$

where ϖ_i is dual to α_i. Let w_0 denote the longest element of W with respect to Δ.

In this paragraph, we recall some basic properties of the wonderful compactification of the symmetric spaces. Let G be a semisimple adjoint group, let σ be an involution of G that is induced by an involution $\widehat{\sigma}$ of $\widehat{G}$. Let H denote the subgroup of all fixed points of σ in G. Let H' denote the fixed points of $\widehat{\sigma}$ in $\widehat{G}$. Then, there exist a σ- stable maximal torus T of G and a Borel subgroup B of G containing T with the following property :

1. The dimension of $\{t \in T : \sigma(t) = t^{-1}\}$ is maximal.

2. For any positive root α (with respect to B), if $\sigma(\alpha)$ is a positive root, then $\sigma(\alpha) = \alpha$.

Let $\{\gamma_1, \gamma_2, \cdots \gamma_j\}$ be the set of all simple roots that are not fixed by σ. Then, σ induces a permutation $\widetilde{\sigma}$ on the set $\{1, 2, \cdots j\}$.

Also, there is only one closed G-orbit in the wonderful compactification $\overline{G/H}$, say G/P and the restriction map $Pic(\overline{G/H}) \longrightarrow Pic(G/P)$ is injective. For a dominant weight λ, let V_λ denote the irreducible representation of $\widehat{G}$ with highest weight λ. Let λ be a dominant weight such that $V_\lambda^{H'}$ is non zero. Then, [by Theorem 8.3, pp 30-31 [4]], the global sections of the line bundle L_λ on $\overline{G/H}$ is given by:

2.1 $H^0(\overline{G/H}, L_\lambda) = \bigoplus_{\nu \in \Gamma_\lambda} V_\nu^*$, where $,_\lambda$ denote the set of all dominant weights ν that are of the form $\nu = \lambda - (\sum_{i \leq \widetilde{\sigma}(i)} m_i(\gamma_i - \sigma(\gamma_i)))$, with $m_i \in \mathbb{Z}_{\geq 0}$.

Now, we apply these to our special involution $\sigma : G \times G \longrightarrow G \times G$ sending $(x, y) \mapsto (y, x)$.

Here, the symmetric space is the group $G = (G \times G)/\Delta(G)$. Let X denote the wonderful compactification of G.

Here, we should take $T \times T$ as a σ- stable maximal torus with the above maximal property. We should take $B \times B^-$ $(T \subset B)$ as a Borel subgroup of $G \times G$ containing $T \times T$ with the above property. Here, the unique closed $G \times G$- orbit is $G/B \times G/B^-$, and the image of $Pic(X) \longrightarrow Pic(G/B \times G/B^-)$ consists of line bundles associated to weights of the form $(\lambda, -\lambda)$. For any dominant weight λ, let L_λ denote the line bundle over X associated to λ and let

$H^0(X, L_\lambda)$ denote the global sections of L_λ over X. Let $\leq$ denote the dominant ordering on χ. That is $\nu \leq \lambda$ if and only if $\lambda - \nu = \sum_{i=1} m_i\alpha_i$ for some $m_i \in \mathbb{Z}_{\geq 0}$.

Now, applying the above statement 2.1 to our special involution of $G \times G$, we have

2.2 $H^0(X, L_\lambda) = \bigoplus_{\nu \leq \lambda} End(V_\nu^*)$, where the sum is taken over all dominant weights ν such that $\nu \leq \lambda$.

3 Projective normality

We now prove that for any line bundle L_λ on X, there is a unique (up to multiplication by a constant) injective homomorphism

$$\phi_\lambda : H^0(X, L_\lambda) \longrightarrow k[\widehat{G}] \tag{3.0.1}$$

of $\widehat{G} \times \widehat{G}$ modules.

For a proof of this fact: Since $G = G \times G/\Delta(G)$ is a $G \times G$ stable open subset of X, and the natural map $\pi : \widehat{G} = \widehat{G} \times \widehat{G}/\Delta(\widehat{G}) \longrightarrow G = G \times G/\Delta(G)$ is a surjection, the natural map $\pi : \widehat{G} = \widehat{G} \times \widehat{G}/\Delta(\widehat{G}) \longrightarrow X$ is a dominant map and hence for any line bundle L over X, $H^0(X, L)$ is a $\widehat{G} \times \widehat{G}$- sub module of the $\widehat{G} \times \widehat{G}$ module $H^0(\widehat{G}, L)$. Since $k[\widehat{G}]$ is a unique factorisation domain, the pull back $\pi^*(L)$ on $\widehat{G}$ is trivial and hence we have

$$H^0(\widehat{G}, L) = k[\widehat{G}].$$

Moreover, this identification is unique up to constants since the units of the k- algebra $k[\widehat{G}]$ are only nonzero constants. Thus, (3.0.1) holds.

We now prove that for any two line bundles L_λ and L_μ, the following diagram is commutative:

$$\begin{array}{ccc} H^0(X, L_\lambda) \bigotimes H^0(X, L_\mu) & \longrightarrow & H^0(X, L_\lambda \bigotimes L_\mu) \\ \downarrow & & \downarrow \\ k[\widehat{G}] \bigotimes k[\widehat{G}] & \longrightarrow & k[\widehat{G}] \end{array} \tag{3.0.2}$$

Here the horizontal arrows are the natural maps, the vertical map on the left is $\phi_\lambda \otimes \phi_\mu$ and the vertical map on the right is $\phi_{\lambda+\mu}$.

For a proof of this observation:

By the Peter-Weyl Theorem, the $\widehat{G} \times \widehat{G}$- module $k[\widehat{G}] = \bigoplus_\nu End(V_\nu^*)$, where the sum is taken over all dominant weights ν. Using this and the fact that $H^0(X, L_\lambda) = \bigoplus_{\nu \leq \lambda} End(V_\nu^*)$, it is easy to see that (3.0.2) holds.

Now, we prove the following Lemma.

Lemma 3.1. *Let λ and μ be two dominant weights. Let λ_1 be a dominant weight such that $\lambda_1 \leq \lambda$ and μ_1 be a dominant weight such that $\mu_1 \leq \mu$. Let ν be a dominant weight such that V_ν is a $\widehat{G}$- submodule of the tensor product $V_{\lambda_1} \bigotimes V_{\mu_1}$. Then, the image of $End(V^*_{\lambda_1}) \bigotimes End(V^*_{\mu_1})$ under the multiplication map $H^0(X, L_\lambda) \bigotimes H^0(X, L_\mu) \longrightarrow H^0(X, L_{\lambda+\mu})$ contains $End(V^*_\nu)$.*

Proof: Since $\widehat{G}$ is linearly reductive and V^*_ν is a quotient of $V^*_{\lambda_1} \bigotimes V^*_{\mu_1}$, we can assume that V^*_ν is a submodule of $V^*_{\lambda_1} \bigotimes V^*_{\mu_1}$.

Now, consider the $\widehat{B}$ action on $End(V^*_{\lambda_1})$ on the right and consider the $\widehat{B^-}$ action on $End(V^*_{\mu_1})$ on the right. By the observation 2.2 and the hypothesis $\lambda_1 \leq \lambda$ and $\mu_1 \leq \mu$, we have $End(V^*_{\lambda_1}) \subset H^0(X, L_\lambda)$ and $End(V^*_{\mu_1}) \subset H^0(X, L_\mu)$. So, by our identification (3.0.1), we have $End(V^*_{\lambda_1}) \subset H^0(X, L_\lambda) \subset k[\widehat{G}]$ and $End(V^*_{\mu_1}) \subset H^0(X, L_\mu) \subset k[\widehat{G}]$. Therefore, by the Borel-Weil Theorem [cf [1]], we have $V^*_{\lambda_1} = \{f \in End(V^*_{\lambda_1}) : f(xb) = \lambda_1(b).f(x) \ for \ x \in \widehat{G}, b \in \widehat{B}\} = \{f \in k[\widehat{G}] : f(xb) = \lambda_1(b).f(x) \ for \ x \in \widehat{G}, \ b \in \widehat{B}\}$ and $V^*_{\mu_1} = \{f \in End(V^*_{\mu_1}) : f(xb^-) = w_0(\mu_1)(b^-).f(x) \ for \ x \in \widehat{G}, \ b^- \in \widehat{B^-}\} = \{f \in k[\widehat{G}] : f(xb^-) = w_0(\mu_1)(b^-).f(x) \ for \ x \in \widehat{G}, \ b^- \in \widehat{B^-}\}$.

Fix our choice of $V^*_{\lambda_1}$ and $V^*_{\mu_1}$ as above and let $f = \sum_i f_i \otimes g_i \in V^*_{\lambda_1} \bigotimes V^*_{\mu_1}$, $f_i \in V^*_{\lambda_1}$, $g_i \in V^*_{\mu_1}$ be such that $(\sum_i f_i \otimes g_i)(x, x)$ is zero, for every $x \in \widehat{G}$. Then, we have $f((xb, xb^-)) = (\lambda_1(b))(w_0(\mu_1)(b^-))f(x, x) = 0$ for every $x \in \widehat{G}$, $b \in B$ and $b^- \in B^-$. Since the elements (xb, xb^-) form a dense open set in $\widehat{G} \times \widehat{G}$, f must vanish identically on $\widehat{G} \times \widehat{G}$. Thus, $V^*_{\lambda_1} \bigotimes V^*_{\mu_1}$ is mapped injectively via the multiplication map $End(V^*_{\lambda_1}) \bigotimes End(V^*_{\mu_1}) \subset k[\widehat{G}] \bigotimes k[\widehat{G}] \longrightarrow k[\widehat{G}]$. By the hypothesis $\lambda_1 \leq \lambda$, $\mu_1 \leq \mu$ and by Theorem(2.1), we have $End(V^*_{\lambda_1}) \subset H^0(X, L_\lambda)$ and $End(V^*_{\mu_1}) \subset H^0(X, L_\mu)$. Therefore, by the observation (3.0.2), the $\widehat{G}$ module $V^*_{\lambda_1} \bigotimes V^*_{\mu_1}$ is mapped injectively into $H^0(X, L_{\lambda+\mu})$ via the multiplication map $H^0(X, L_\lambda) \bigotimes H^0(X, L_\mu) \longrightarrow H^0(X, L_{\lambda+\mu})$. Since V^*_ν is a $\widehat{G}$- submodule of $V^*_{\lambda_1} \bigotimes V^*_{\mu_1}$, V^*_ν must be isomorphic to a $\widehat{G}$ submodule of the image of $End(V^*_{\lambda_1}) \bigotimes End(V^*_{\mu_1})$. Since the image of $End(V^*_{\lambda_1}) \bigotimes End(V^*_{\mu_1})$ is a $\widehat{G} \times \widehat{G}$ submodule of $k[\widehat{G}]$, it must contain $End(V^*_\nu)$.

Therefore, using the diagram (3.0.2), it is easy to see that $End(V^*_\nu)$ is a $\widehat{G} \times \widehat{G}$-submodule of the image of $H^0(X, L_\lambda) \bigotimes H^0(X, L_\mu) \longrightarrow H^0(X, L_{\lambda+\mu})$ under the multiplication map.

This completes the proof of the Lemma.

We now prove the following Lemma.

Lemma 3.2. *Let λ and μ be two dominant weights. Let ν be a dominant weight satisfying $\nu \leq \lambda + \mu$. Then, there is a dominant weight $\lambda_1 \leq \lambda$ and there is a dominant weight $\mu_1 \leq \mu$ such that V_ν is $\widehat{G}$- submodule of the tensor product $V_{\lambda_1} \bigotimes V_{\mu_1}$.*

Proof: Let λ and μ be two dominant weights. Let ν be a dominant weight such that $\nu \leq \lambda + \mu$. We wish to prove that there are dominant weights λ_1, μ_1 such that $\lambda_1 \leq \lambda$, $\mu_1 \leq \mu$

and an element w of the Weyl group W such that $\nu = \lambda_1 + w(\mu_1)$, and then using PRV conjecture [cf [9]], we prove that V_ν is a $\widehat{G}$- submodule of the tensor product $V_{\lambda_1} \bigotimes V_{\mu_1}$.

By [sections (13.4) and (21.3)] of [8]], ν is a weight of $V_{\lambda+\mu}$. Therefore, we have $\nu = \nu_1 + \nu_2$, where ν_1 is a weight of V_λ and ν_2 is a weight of V_μ. We have $\lambda - \nu_1 = (\sum_i^l m_i\alpha_i)$ for some $m_i \in \mathbb{Z}_{\geq 0}$.

We first prove that there is a dominant weight λ_1 such that $\lambda_1 \leq \lambda$ and there is a weight ν_2' of V_μ such that $\nu = \lambda_1 + \nu_2'$ by induction on the sum $\sum_i^l m_i$.

If $\sum_i^l m_i = 0$, there is nothing to prove.

If $\sum_i^l m_i = 1$, then, we have $\nu_1 = \lambda - \alpha_{i_0}$ for some $i_0 \in \{1, 2, \cdots l\}$. Now, if ν_1 is dominant, then there is nothing to prove. Otherwise, $\langle \nu_1, \alpha_{i_0} \rangle$ is negative. Since $\nu = \nu_1 + \nu_2$ is dominant, $\langle \nu_2, \alpha_{i_0} \rangle$ is positive. But, we know that $\nu_1 + k\alpha_{i_0}$ is a weight of V_λ for any $k \in \{0, 1, \cdots - \langle \nu_1, \alpha_{i_0} \rangle\}$ and $\nu_2 - k\alpha_{i_0}$ is a weight of V_μ for any $k \in \{0, 1, \cdots \langle \nu_2, \alpha_{i_0} \rangle\}$ [cf [sections (13.4) and (21.3) of [8]]]. In particular, $\nu_1 + \alpha_{i_0} = \lambda$ is a weight of V_λ (which is ofcourse trivial) and $\nu_2 - \alpha_{i_0}$ is a weight of V_μ and we also have $\nu = \lambda + \nu_2 - \alpha_{i_0}$. Therefore, in this case, we take $\lambda_1 = \lambda$ and $\nu_2' = \nu_2 - \alpha_{i_0}$.

Now, let N be a positive integer strictly bigger than 1. Let us assume that the statement is true for all ν such that $\nu = \nu_1 + \nu_2$, where $\nu_1 = \lambda - \sum_{i=1}^l m_i\alpha_i$ being a weight of V_λ with the sum $\sum_{i=1}^l m_i \leq N - 1$ and ν_2 being a weight of V_μ.

Let ν be a dominant weight such that $\nu \leq \lambda$ and $\nu = \nu_1 + \nu_2$ where ν_1 is a weight of V_λ, ν_2 is a weight of V_μ with $\nu_1 = \lambda - \sum_i m_i\alpha_i$ and the sum $\sum_{i=1}^l m_i = N$. If ν_1 is dominant, then, there is nothing to prove. Otherwise, there is a simple root α_{i_0} such that $\langle \nu_1, \alpha_{i_0} \rangle < 0$.

By a proof similar to the case when $\sum_i^l m_i = 1$, it is easy to see that $\nu_1 + \alpha_{i_0}$ is a weight of V_λ and $\nu_2 - \alpha_{i_0}$ is a weight of V_μ. Also, we have $\nu = (\nu_1 + \alpha_{i_0}) + (\nu_2 - \alpha_{i_0})$. Therefore, by induction, there exist a dominant weight $\lambda_1 \leq \lambda$ and a weight ν_2 of V_μ such that

$$\nu = \lambda_1 + \nu_2'.$$

We fix our choice of λ_1 and ν_2' as above. Since ν_2' is a weight of V_μ, $w(\nu_2')$ is also a weight of V_μ for every $w \in W$. Thus, the unique dominant weight (say $\mu_1 = w^{-1}(\nu_2')$) in the W orbit of ν_2' must also be a weight of V_μ.

Hence, we have $\nu = \lambda_1 + w(\mu_1)$ where λ_1 is a dominant weight satisfying $\lambda_1 \leq \lambda$, μ_1 is a dominant weight satisfying $\mu_1 \leq \mu$ and w is an element of the Weyl group W.

Now, the fact that V_ν is a $\widehat{G}$- submodule of the tensor product $V_{\lambda_1} \bigotimes V_{\mu_1}$ follows from PRV Conjecture [cf [9]].

This completes the proof of this Lemma.

We have

Corollary 3.3. *Let λ and μ be two dominant weights. Then, the multiplication map*

$$H^0(X, L_\lambda) \bigotimes H^0(X, L_\mu) \longrightarrow H^0(X, L_{\lambda+\mu}) \text{ is onto.}$$

Proof: Proof of this is an immediate consequence of Lemmas 3.1 and 3.2.

Theorem 3.4. *Let G be a semisimple adjoint group over the field of complex numbers. Let $\pi : \widehat{G} \longrightarrow G$ be a simply connected covering of G. Let X denote the wonderful compactification of G. Let L be an ample line bundle over X. Consider the embedding of X in $\mathbb{P}(H^0(X, L)^*)$. Let $\widehat{X}$ denote the cone over X with respect to this embedding of X. Then, $\widehat{X}$ is normal.*

Proof: By [4], we have $L = L_\lambda$ for some regular dominant weight λ. By [II (5.14)(d)] of [7], it suffices to prove that the multiplication map

$$H^0(X, L_\lambda)^{\otimes N} \longrightarrow H^0(X, L_{N\lambda}) \tag{3.4.1}$$

is surjective for every positive integer N. The proof of this is an immediate consequence of Corollary 3.3.

We have

Corollary 3.5. *Let w be an element of the Weyl group W. Let X_w denote the scheme theoretic closure of the cell BwB in X. Let L be a $\widehat{G} \times \widehat{G}$ linearised very ample line bundle over X and let $\widehat{X_w}$ denote the cone over the embedding given by L. Then, $\widehat{X_w}$ is normal.*

Proof: By [Corollary 3, [3]], the restriction map
$H^0(X, L^{\otimes n}) \longrightarrow H^0(X_w, L^{\otimes n})$ is surjective for every positive integer $n \cdots (3.5.1)$.

Now, consider the following commutative diagram of maps:

$$\begin{array}{ccc} H^0(X, L)^{\otimes n} & \longrightarrow & H^0(X, L^{\otimes n}) \\ \downarrow & & \downarrow \\ H^0(X_w, L)^{\otimes n} & \longrightarrow & H^0(X_w, L^{\otimes n}) \end{array} \tag{3.5.2}$$

Here the horizontal arrows are the natural multiplication maps and the vertical arrows are restriction maps.

The assertion of the corollary follows from the observations 3.5.1, 3.5.2 and Corollary 3.3.

4 Appendix:

In this section, we prove the analogue of Theorem 3.4 when the base field is of positive characteristic.

Let k be an algebraically closed field of positive characteristic. Let G be a semisimple adjoint group over k. Let $\widehat{G}$ be a simply connected covering of G. Let X denote the wonderful compactification of G. Let T, B, B^-, $\widehat{T}$, $\widehat{B}$, $\widehat{B^-}$ be as in section 2.

With the notations as in section 2, we introduce some more notations as follows: Let χ denote the weight lattice of G with respect to T. So, $\chi \times \chi$ is the weight lattice of $G \times G$ with respect to $T \times T$.

For any weight $\lambda \in \chi$, set $V_\lambda := H^0(G/B, L_{w_0(-\lambda)})$.

Definition 4.1. Let $\widehat{G}$ be as above. Let V be a $\widehat{G}$ module. A filtration of $\widehat{G}$- submodules $(0) = V_0 \subset V_1 \subset V_2 \subset \cdots V_n = V$ is a good filtration if each V_{i+1}/V_i is a direct sum of $V(\lambda_{i(j)})$'s for some dominant weights $\lambda_{i(j)}$'s.

As in section 2, let $\Delta = \{\alpha_1, \cdots \alpha_l\}$ denote the set of simple roots with respect to T and B.

Let λ and μ be two dominant weights in χ.

Let L_λ (resp. L_μ) denote the line bundle over X associated to $(\lambda, -\lambda)$ (resp. to $(\mu, -\mu)$. Let ν be a dominant weight such that $\nu \leq \lambda + \mu)$. Let $\phi : H^0(X, L_\lambda) \bigotimes H^0(X, L_\mu) \longrightarrow H^0(X, L_{\lambda+\mu})$ denote the multiplication map.

Then, we have

Lemma 4.2. *The $\widehat{G} \times \widehat{G}$- module $V_{w_0(-\nu)} \bigotimes V_\nu$ is a $\widehat{G} \times \widehat{G}$-subquotient of the image* $\phi(H^0(X, L_\lambda) \bigotimes H^0(X, L_\mu))$.

Proof: By Donkin's conjecture [cf Theorem 4.4.3, [11]], a tensor product of two modules with good filtrations has a good filtration. By Theorem 5.10 of [5], $H^0(X, L_{\lambda_1})$ (resp. $H^0(X, L_{\mu_1})$) has a good filtration of $\widehat{G} \times \widehat{G}$-submodules such that the successive quotients look like $V_{w_0(-\nu)} \bigotimes V_\nu$, $\nu \leq \lambda_1$ (resp. $\nu \leq \mu_1$). Therefore, the tensor product $H^0(X, L_{\lambda_1}) \bigotimes H^0(X, L_{\mu_1})$ has a good filtration of $\widehat{G} \times \widehat{G}$-submodules.

Now, let H be a semisimple Chevalley algebraic group over the field of complex numbers. Let H_p be the corresponding algebraic group in characteristic p. Now, if V_p is a H_p-module having a good filtration in characteristic p such that it comes from a H-module V modulo p, then the character of V is same as the chracter of V_p and so the H_p- subquotients of V_p are precisely the H- direct summands of V.

Now, let M_ν (resp. M'_ν) be the largest $\widehat{G} \times \widehat{G}$ submodule of the tensor product $H^0(X, L_{\lambda_1}) \bigotimes H^0(X, L_{\mu_1})$ all of whose $B \times B$- weights are $\leq (-w_0(\nu), \nu)$ (resp. $< (-w_0(\nu), \nu)$). Also, let N_ν and N'_ν be the corresponding submodules of $H^0(X, L_{\lambda_1+\mu_1})$. Therefore, from Lemmas 3.1 and 3.2, and by the observation above, ϕ induces a non zero $\widehat{G} \times \widehat{G}$- module homomorphism from the quotient $(V_{-w_0(\nu)} \bigotimes V_\nu)^m = M_\nu/M'_\nu \longrightarrow N_\nu/N'_\nu = V_{-w_0(\nu)} \bigotimes V_\nu$ for some positive integer m. Therefore, by the Frobenius reciprocity [cf Proposition 2.1.6 [11]], the assertion follows.

The Theorem (3.4) for positive characteristic follows from Lemma(4.2).

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