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BRAID RELATIONS IN THE YOKONUMA-HECKE ALGEBRA

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Abstract

In this note, we prove a theorem on another presentation for the algebra of the endomorphisms of the permutation representation (Yokonuma-Hecke algebra) of a simple Chevalley group with respect to a maximal unipotent subgroup. This presentation is done using certain non-standard generators.

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1. Introduction

Let G be a simple Chevalley group defined over $\mathbb{F}_q$. In this manuscript, we prove a theorem on a new presentation for the algebra of endomorphisms $\mathcal{Y}_n(q)$ associated to the induced representation of the trivial representaion of U up to G, where U is a maximal unipotent subgroup of G. In [6], this theorem was proved for the case when the Cartan matrix of G is symmetric, that is when G is of type A_l, D_l, E_6, E_7 or E_8. In this manuscript, we prove the theorem for the other simple Chevalley groups also. More precisely, we prove the nonstandard presentation theorem for the simple Chevalley groups of type B_l, C_l, F_4 and G_2.

In [7], T. Yokonuma has given a description (presentation) of this algebra $\mathcal{Y}_n(q)$ in terms of the standard generators, that is, in terms of generators given by the double cosets (see 11.30[3]). So, we call the algebra $\mathcal{Y}_n(q)$, the Yokonuma-Hecke algebra. The presentation of Yokonuma is analogous to the classical presentation of the Iwahori-Hecke algebra (see [5]).

In Theorem 2.18[6], the first author of this article has proved that this algebra $\mathcal{Y}_n(q)$ has a presentation with non standard generators for the simple Chevalley groups G whose Cartan matrix is symmetric. This presentation uses non-standard generators defined by a pre-fixed non-trivial additive character of $\mathbb{F}_q$, and a certain non-trivial linear combination involving the standard basis of $\mathcal{Y}_n(q)$ (see Definition 1). Originally, these generators were defined in a geometrical way for the group $GL_n(\mathbb{F}_q)$, that is, like Fourier Transforms on the space of functions of flags vectors on $\mathbb{F}_q^n$. As an application of our main theorem, we recall that abstracting the presentation in the case when G is of type A_l, it is posible to define a certain finite dimensional algebra, involving braids and ties, which give new matrix representation for the Artin group of type A, see [1] . It is a natural question to study the representation for the Artin groups of types B_l, C_l, F_4 and G_2 that arising from our theorem.

The aim of this note is to prove that the above mentioned non standard generators give a presentation for the algebra $\mathcal{Y}_n(q)$ for the simple Chevalley groups of type B_l, C_l, F_4 and G_2.

For more precise statement, see Theorem 2.

The layout of this manuscript is as follows:

Section 2 consists of preliminaries and statement of the main Theorem (for a more precise statement, see Theorem 2.) Section 3 consists of the proof for the case when G is of type B_l, C_l or F_4. Section 4 consists of the proof for the case when G is of type G_2.

2. Preliminares and statement of the main result

2.1. Let k denote a finite field with q elements. Let G be a simple simply connected Chevalley group defined over k. Let T be a "maximally split" torus of G. Let B be a Borel subgroup of G containing T. Let U be the unipotent radical of B. We will denote the rank of G by l.

We denote the set of all roots with respect to T by Φ.

Let Δ be the set of all simple roots with respect to T and B. Let N be the normaliser of T in G and let $W = N/T$ be the Weyl group of G with $S = \{s_\alpha : \alpha \in \Delta\}$ being the set of simple reflections. The pair (W, S) is a Coxeter system and we have the presentation:

$$W = \langle s_\alpha : (s_\alpha s_\beta)^{m_{\alpha\beta}} = 1, \alpha, \beta \in \Delta \rangle,$$

where $m_{\alpha\beta}$ denote the order of $s_\alpha s_\beta$.

Let π be the canonical homomorphism from N onto W. Using π, we have an action of the Weyl group W on T: $(w, t) \mapsto w(t) := \omega t \omega^{-1}$, where $\omega \in N$ is such that $\pi(\omega) = w$.

We recall that for any root $\alpha \in \Phi$, there is an $\omega_\alpha \in N$ such that $\pi(\omega_\alpha) = s_\alpha$ and there is a homomorphism $\phi_\alpha : SL_2 \longrightarrow G$ such that

$$\omega_\alpha = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad h_\alpha(r) = \phi_\alpha \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, \qquad (r \in k^\times).$$

2.2. Let $\mathcal{Y}_n(q)$ be the algebra of endomorphism of the induced (permutation) representation $\mathrm{Ind}_U^G 1$, over the field of complex numbers. We call the algebra $\mathcal{Y}_n(q)$ as the Yokonuma-Hecke algebra.

From the Bruhat decomposition, $G = \coprod_{n\in N} UnU$, we have that the standard basis of the Yokonuma-Hecke algebra is parametrised by N. Let $\{R_n \,|\, n \in N\}$ be the standard basis.

If $n = \omega_\alpha$, we denote R_n by R_α.

If $n = t \in T$, we call the elements R_t in $\mathcal{Y}_n(q)$ operators of homothety corresponding to t. In the case $t = h_\alpha(r)$, we denote R_t by $H_\alpha(r)$. With these notations, we define E_α by

$$E_\alpha := \sum_{r\in k^\times} H_\alpha(r) \quad (\alpha \in \Phi).$$

It is clear that the E_α's commute among themselves, and a direct computation shows that

$$E_\alpha^2 = (q-1)E_\alpha. \tag{1}$$

Now, we recall a Theorem due to T. Yokonuma.

Theorem 1. *(See [7]) The Yokonuma-Hecke algebra $\mathcal{Y}_n(q)$ is generated, as an algebra, by $R_\alpha\,(\alpha \in \Phi)$, and the homotheties $R_t\,(t \in T)$. Moreover, these generators with the relations below define a presentation for $\mathcal{Y}_n(q)$.*

(1.1) $$R_\alpha^2 = qH_\alpha(-1) + R_\alpha E_\alpha \quad \textit{(quadratic relation)}$$

(1.2) $$\underbrace{R_\alpha R_\beta R_\alpha R_\beta \cdots}_{m_{\alpha\beta}} = \underbrace{R_\beta R_\alpha R_\beta R_\alpha \cdots}_{m_{\alpha\beta}} \quad \textit{(braid relation)}$$

(1.3) $$R_t R_\alpha = R_\alpha R_{t'}, \quad \textit{where} \quad t' = \omega_\alpha t \omega_\alpha^{-1} \quad (t \in T)$$

(1.4) $$R_u R_v = R_{uv} \quad (u, v \in T).$$

2.3. In the following, we fix a non-trivial additive character ψ of $(k, +)$. For any α in Φ, we define Ψ_α as the following linear combination of elements in $\mathcal{Y}_n(q)$,

$$\Psi_\alpha := \sum_{r\in k^\times} \psi(r) H_\alpha(r).$$

From a direct computation, we have that Ψ_α commutes with E_α, and

$$\Psi_\alpha E_\alpha = -E_\alpha. \tag{2}$$

Definition 1. Let $\alpha \in \Psi$. We define the element L_α, as

$$L_\alpha := q^{-1}\left(E_\alpha + R_\alpha \Psi_\alpha\right).$$

Our main goal is to prove the following Theorem.

Theorem 2. *The Yokonuma-Hecke algebra $\mathcal{Y}_n(q)$ is generated (as an algebra), by $L_\alpha\,(\alpha \in \Phi)$, and the homotheties $R_t\,(t \in T)$. Moreover, these generators with the relations below define a presentation for $\mathcal{Y}_n(q)$.*

(2.1) $$L_\alpha^2 = 1 - q^{-1}\left(E_\alpha - L_\alpha E_\alpha\right) \quad \textit{(quadratic relation)}$$

(2.2) $$\underbrace{L_\alpha L_\beta L_\alpha L_\beta \cdots}_{m_{\alpha\beta}} = \underbrace{L_\beta L_\alpha L_\beta L_\alpha \cdots}_{m_{\alpha\beta}} \quad \textit{(braid relation)}$$

(2.3) $$R_t L_\alpha = L_\alpha R_{t'}, \quad \textit{where} \quad t' = \omega_\alpha t \omega_\alpha^{-1} \quad (t \in T)$$

(2.4) $$R_u R_v = R_{uv} \quad (u, v \in T).$$

To prove this Theorem, we introduce some notations and one useful Proposition. We denote by E_α^w the effect of w on E_α arising from the action of the Weyl group W on T. That is,

$$E_\alpha^w = \sum_{r\in k^\times} H_\gamma(r) \quad (\alpha \in \Phi, w \in W),$$

where γ is the root defined by $w(\alpha) = \gamma$.

In the similar way, we denote by Ψ_α^w the effect of w on Ψ_α.

Proposition 3. *Let s be the reflection corresponding to α, and let $\beta \in \Phi$. We have*

(3.1) $$E_\beta R_\alpha = R_\alpha E_\beta^s, \qquad \Psi_\beta R_\alpha = R_\alpha \Psi_\beta^s$$

(3.2) $$E_\alpha^s = E_\alpha$$

(3.3) $$E_\alpha \Psi_\alpha^s = -E_\alpha = E_\alpha \Psi_\alpha$$

(3.4) $$\Psi_\alpha \Psi_\alpha^s = \Psi_\alpha^s \Psi_\alpha = qH_\alpha(-1) - E_\alpha.$$

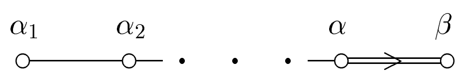
Proof. The proof of the assertions in 3.1 is an inmediate consequence of Yokonuma's Theorem, part 1.3 and the proofs of 3.2, 3.3 and 3.4 are straightforward computations. □

2.4. We are now going to sketch the proof of Theorem 2 for the simple Chevalley groups of type B_l, C_l, F_4 and G_2. The only statement of Theorem 2 that involves the Dynkin diagram of the group is the statement about the braid relation, that is 2.2. Since Theorem 2 was proved for the cases of type A_l, D_l, E_6, E_7 and E_8 in [6], to prove the Theorem, we need to prove only 2.2 for the cases when G is of type B_l, C_l, F_4 and G_2. In Section 2, we prove 2.2 for the case when G is of type B_l, C_l and F_4. In Section 3, we prove 3.2 for the case when G is of type G_2. The method of proof involves the one parameter subgroups $H_\alpha(t)$, $t \in k^\times$, $\alpha \in \Phi$, and some automorphisms of the two dimensional torus $k^\times \times k^\times$.

3. Cases B_l, C_l and F_4

3.1. Let $\Delta = \{\alpha_1, \ldots, \alpha_{l-1}, \alpha_l\}$ denote the set of all simple roots of type B_l. So, the Dynkin diagram is as follows:

B_l:



where $\alpha = \alpha_{l-1}$ and $\beta = \alpha_l$. Let s (respectively s') be the reflection corresponding to the root α (respectively β).

Notice that the simple roots $\alpha_1, \ldots, \alpha_{l-1}$ of B_l turn to the set of simple roots of A_{l-1} and so from Theorem 2.12[6], we deduce:

$$\begin{aligned} L_{\alpha_i} L_{\alpha_j} &= L_{\alpha_j} L_{\alpha_i} \quad \text{if} \quad |i-j| > 1 \\ L_{\alpha_i} L_{\alpha_j} L_{\alpha_i} &= L_{\alpha_j} L_{\alpha_i} L_{\alpha_j} \quad \text{if} \quad |i-j| = 1. \end{aligned}$$

Therefore, to prove Theorem 2, we need to prove only the relation $L_\alpha L_\beta L_\alpha L_\beta = L_\beta L_\alpha L_\beta L_\alpha$. In the proof of this braid relation, we will use the following lemma. The same proof holds for the cases: C_l and F_4. (The only difference is $\alpha = \alpha_l$, $\beta = \alpha_{l-1}$ in the case of C_l and $\alpha = \alpha_2$, $\beta = \alpha_3$ in the case of F_4).

Lemma 4. *We have*

(4.1) $$E_\alpha^{s'} E_\beta = E_\beta^s E_\alpha = E_\alpha E_\beta$$

(4.2) $$(E_\alpha^{s'})^s = E_\alpha^{s'}, \quad (E_\beta^s)^{s'} = E_\beta^s$$

(4.3) $$E_\beta^s E_\alpha^{s'} = E_\alpha E_\beta$$

(4.4) $$((E_\alpha^{s'})^s)^{s'} = E_\alpha, \qquad ((E_\beta^s)^{s'})^s = E_\beta$$

(4.5) $$(\Psi_\alpha^{s'})^s = \Psi_\alpha^{s'}, \qquad (\Psi_\beta^s)^{s'} = \Psi_\beta^s$$

(4.6) $$(H_\alpha(-1))^{s'} = H_\alpha(-1), \qquad (H_\beta(-1))^s = H_\alpha(-1) H_\beta(-1).$$

Proof. We now prove 4.1. We have $s'(\alpha)=\alpha+2\beta$ and so we have $E_\alpha^{s'}=E_{\alpha+2\beta}$. Hence, we have

$$\begin{aligned} E_\alpha^{s'}E_\beta &= \sum_{t\in k^\times} H_{\alpha+2\beta}(t)\sum_{r\in k^\times} H_\beta(r) \\ &= \sum_{(t,r)\in k^\times\times k^\times} H_\alpha(t)H_\beta(t^2\cdot r) \\ &= \sum_{(t,r)\in k^\times\times k^\times} H_\alpha(t)H_\beta(r) = E_\alpha E_\beta, \end{aligned}$$

since the map $(t,r)\mapsto(t,t^2\cdot r)$ is an automorphism of $k^\times\times k^\times$. This proves that $E_\alpha^{s'}E_\beta=E_\alpha E_\beta$.

The equality $E_\beta^s E_\alpha=E_\alpha E_\beta$ follows from the fact that $s(\beta)=\alpha+\beta$ and the map $(t,r)\mapsto(tr,r)$ is an automorphism of $k^\times\times k^\times$.

We now prove 4.2. We have $s(s'(\alpha))=s(\alpha+2\beta)=-\alpha+2(\beta+\alpha)=\alpha+2\beta=s'(\alpha)$. This proves that $(E_\alpha^{s'})^s=E_\alpha^{s'}$.

Proof of $(E_\beta^s)^{s'}=E_\beta^s$ follows from the fact:

$$s'(s(\beta))=s'(\alpha+\beta)=(\alpha+2\beta)-\beta=\alpha+\beta=s(\beta).$$

Proof of 4.3 follows from the facts that $s(\beta)=\alpha+\beta$, $s'(\alpha)=\alpha+2\beta$ and the map $(t,r)\mapsto(tr,tr^2)$ is an automorphism of $k^\times\times k^\times$.

Proof of 4.4 follows from the facts that

$$s'(s(s'(\alpha)))=s'(s(\alpha+2\beta))=s'(\alpha+2\beta)=(\alpha+2\beta)-2\beta=\alpha$$

and

$$s(s'(s(\beta)))=s(s'(\alpha+\beta))=s(\alpha+\beta)=-\alpha+\alpha+\beta=\beta.$$

Proof of 4.5 is similar to the proof of 4.2. We note here that Ψ does not play an important role in this situation.

We now prove 4.6. We have $s'(\alpha)=\alpha+2\beta$ and hence, we have

$$(H_\alpha(-1))^{s'}=H_\alpha(-1)(H_\beta(-1))^2=H_\alpha(-1)H_\beta((-1)^2)=H_\alpha(-1).$$

Since $s(\beta)=\alpha+\beta$, we have $(H_\beta)(-1)^s=H_\alpha(-1)H_\beta(-1)$. □

We now prove the following Lemma which will complete the proof of Theorem 2 for the cases when G is of type B_l, C_l and F_4.

Lemma 5. $L_\alpha L_\beta L_\alpha L_\beta=L_\beta L_\alpha L_\beta L_\alpha$.

Proof. First, we compute the products:
$p_{\alpha\beta}:=q^2L_\alpha L_\beta$, and $p_{\beta\alpha}:=q^2L_\beta L_\alpha$.
From the definition of L_α and L_β, we have

$$\begin{aligned} p_{\alpha\beta} &= (E_\alpha+R_\alpha\Psi_\alpha)(E_\beta+R_\beta\Psi_\beta) \\ &= E_\alpha E_\beta+E_\alpha R_\beta\Psi_\beta+R_\alpha\Psi_\alpha E_\beta+R_\alpha\Psi_\alpha R_\beta\Psi_\beta \\ &= \underbrace{E_\alpha E_\beta}_{a}+\underbrace{R_\beta E_\alpha^{s'}\Psi_\beta}_{b}+\underbrace{R_\alpha\Psi_\alpha E_\beta}_{c}+\underbrace{R_\alpha R_\beta\Psi_\alpha^{s'}\Psi_\beta}_{d}. \end{aligned}$$

Notice that b (respectively d) is obtained from $E_\alpha R_\beta\Psi_\beta$ (respectively $R_\alpha\Psi_\alpha R_\beta\Psi_\beta$) using proposition 3.

Now, we compute $p_{\alpha\beta}^2$:

$$p_{\alpha\beta}^2=a^2+b^2+c^2+d^2+ab+ac+ad+ba+bc+bd+ca+cb+cd+da+db+dc.$$

In the same way, we obtain an analogous expression for $p_{\beta\alpha}^2$, but in the symbols a', b', c' and d'.

The proof of this Lemma is as follows. In the expression of $p_{\alpha\beta}^2$ and $p_{\beta\alpha}^2$, we first bring the monomials 1, R_α, R_β, $R_\alpha R_\beta$, $R_\beta R_\alpha$, $R_\alpha R_\beta R_\alpha$, $R_\beta R_\alpha R_\beta$, and $R_\alpha R_\beta R_\alpha R_\beta=R_\beta R_\alpha R_\beta R_\alpha$ to the left. After this procedure, we will check that the coefficients of these monomials in $p^2_{\alpha\beta}$ with the corresponding coefficients in $p^2_{\beta\alpha}$ are the same. Then, the lemma follows.

Let X_0, X_α, X_β, $X_{\alpha\beta}$, $X_{\beta\alpha}$, $X_{\alpha\beta\alpha}$, $X_{\beta\alpha\beta}$ and $X_{\alpha\beta\alpha\beta}$ be the coefficient of 1, R_α, R_β, $R_\alpha R_\beta$, $R_\beta R_\alpha$, $R_\alpha R_\beta R_\alpha$, $R_\beta R_\alpha R_\alpha$ and $R_\alpha R_\beta R_\alpha R_\beta$, respectively in $p^2_{\alpha\beta}$. Let Y_0, Y_α, Y_β, $Y_{\alpha\beta}$, $Y_{\beta\alpha}$, $Y_{\alpha\beta\alpha}$, $Y_{\beta\alpha\beta}$ and $Y_{\alpha\beta\alpha\beta}$ be the coefficient of 1, R_α, R_β, $R_\alpha R_\beta$, $R_\beta R_\alpha$, $R_\alpha R_\beta R_\alpha$, $R_\beta R_\alpha R_\alpha$ and $R_\alpha R_\beta R_\alpha R_\beta$ respectively in $p^2_{\beta\alpha}$.

We need to prove that $X_\gamma = Y_\gamma$ for all γ (words in α and β) as above. To do this, we will compute X_γ and Y_γ using essentially the Lemma 4. Now, as the computations are all very similar, we are going to compute only X_0, $X_{\alpha\beta\alpha}$, $Y_{\alpha\beta\alpha}$, X_α, Y_α, $X_{\alpha\beta}$, $Y_{\alpha\beta}$, $X_{\alpha\beta\alpha\beta}$ and $Y_{\alpha\beta\alpha\beta}$.

Computation of X_0 and Y_0. It is easy to see that the terms contributing to the constant coefficient in the expression of $p^2_{\alpha\beta}$ (resp. in $p^2_{\beta\alpha}$) are only a^2, b^2 and c^2 (resp. $(a')^2$, $(b')^2$ and $(c')^2$).

We have $a^2 = (E_\alpha E_\beta)^2 = (E_\beta E_\alpha)^2 = (a')^2$.

We now compute b^2.

We have

$$b^2 = R_\beta E_\alpha^{s'} \Psi_\beta R_\beta E_\alpha^{s'} \Psi_\beta \quad = \quad R^2_\beta ((E_\alpha^{s'} \Psi_\beta)^{s'} E_\alpha^{s'} \Psi_\beta$$

From the observation 1.1 of Theorem 1, we have $R^2_\beta = qH_\beta(-1) + R_\beta E_\beta$.

Thus, the constant coefficient yielded by b^2 is $qH_\beta(-1)(E_\alpha^{s'} \Psi_\beta)^{s'} E_\alpha^{s'} \Psi_\beta$.

By a similar computation, it is easy to see that $(c')^2$ yields the same constant coefficient as that yielded by b^2.

A similar proof shows that the constant coeffiicient yielded by c^2 and that yielded by $(b')^2$ are the same and both are equal to $qH_\alpha(-1)(E_\beta^s \Psi_\alpha)^s E_\beta^s \Psi_\alpha$.

Thus, we have $X_0 = Y_0$.

Computation of $X_{\alpha\beta\alpha}$ and $Y_{\alpha\beta\alpha}$. It is clear that the terms having $R_\alpha R_\beta R_\alpha$ in $p_{\alpha\beta}$ (respectively $p_{\beta\alpha}$) is only dc (respectively $b'd'$). We have $dc = b'd'$. Namely,

$$\begin{aligned}
b'd' &= (R_\alpha E_\beta^s \Psi_\alpha)(R_\beta R_\alpha \Psi_\beta^s \Psi_\alpha) \\
&= R_\alpha R_\beta R_\alpha ((E_\beta^s)^{s'})^s (\Psi_\alpha^{s'})^s \Psi_\beta^s \Psi_\alpha \\
&= R_\alpha R_\beta R_\alpha E_\beta (\Psi_\alpha^{s'})^s \Psi_\beta^s \Psi_\alpha \qquad \text{(from 4.4)} \\
&= R_\alpha R_\beta R_\alpha (\Psi_\alpha^{s'})^s \Psi_\beta^s \Psi_\alpha E_\beta \\
&= (R_\alpha R_\beta \Psi_\alpha^{s'} \Psi_\beta)(R_\alpha \Psi_\alpha E_\beta) \\
&= dc.
\end{aligned}$$

Computation of X_α and Y_α. The terms having R_α in $p^2_{\alpha\beta}$ are: c^2, ac, ca, and db. We now compute c^2. We have

$$\begin{aligned}
c^2 &= (R_\alpha \Psi_\alpha E_\beta)(R_\alpha \Psi_\alpha E_\beta) \\
&= R_\alpha^2 \Psi_\alpha^s E_\beta^s \Psi_\alpha E_\beta \quad \text{(from proposition 3)} \\
&= (qH_\alpha(-1) + R_\alpha E_\alpha)\Psi_\alpha^s E_\beta^s \Psi_\alpha E_\beta \quad \text{(from 1.1)}.
\end{aligned}$$

Hence, c^2 yields the coefficient $E_\alpha \Psi_\alpha^s E_\beta^s \Psi_\alpha E_\beta$. Now, using proposition 3 and lemma 4, we get

$$\begin{aligned}
E_\alpha \Psi_\alpha^s E_\beta^s \Psi_\alpha E_\beta &= -E_\alpha E_\beta^s \Psi_\alpha E_\beta \qquad \text{(from 3.3)} \\
&= -E_\alpha E_\beta \Psi_\alpha E_\beta \qquad \text{(from ¿ 4.1)} \\
&= -E_\beta (E_\alpha \Psi_\alpha) E_\beta \\
&= E_\beta E_\alpha E_\beta \qquad \text{(from 3.3)} \\
&= (q-1) E_\alpha E_\beta \qquad \text{(from 1)}.
\end{aligned}$$

On the other hand, using Proposition 3, we get

$$ac = (E_\alpha E_\beta)(R_\alpha \Psi_\alpha E_\beta) = R_\alpha E_\alpha^s E_\beta^s \Psi_\alpha E_\beta,$$

and

$$ca = R_\alpha \Psi_\alpha E_\beta E_\alpha E_\beta.$$

Using Lemma 4, we deduce:

$$E_\alpha^s E_\beta^s \Psi_\alpha E_\beta = \Psi_\alpha E_\beta E_\alpha E_\beta = -(q-1)E_\alpha E_\beta.$$

Let us compute db,

$$\begin{aligned} db &= (R_\alpha R_\beta \Psi_\alpha^{s'} \Psi_\beta)(R_\beta E_\alpha^{s'} \Psi_\beta) \\ &= R_\alpha R_\beta^2 (\Psi_\alpha^{s'})^{s'} \Psi_\beta^{s'} E_\alpha^{s'} \Psi_\beta \quad \text{(from proposition 3)}. \end{aligned}$$

Therefore, from 1.1, we have

$$db = R_\alpha (qH_\beta(-1) + R_\beta E_\beta)(\Psi_\alpha^{s'})^{s'} \Psi_\beta^{s'} E_\alpha^{s'} \Psi_\beta.$$

Thus, db yields the coefficient $qH_\beta(-1)(\Psi_\alpha^{s'})^{s'} \Psi_\beta^{s'} E_\alpha^{s'} \Psi_\beta = qH_\beta(-1)\Psi_\alpha \Psi_\beta^{s'} E_\alpha^{s'} \Psi_\beta$. Now,

$$\begin{aligned} qH_\beta(-1)\Psi_\alpha \Psi_\beta^{s'} E_\alpha^{s'} \Psi_\beta &= qH_\beta(-1)\Psi_\alpha E_\alpha^{s'} \Psi_\beta^{s'} \Psi_\beta \\ &= qH_\beta(-1)\Psi_\alpha E_\alpha^{s'} (qH_\beta(-1) - E_\beta) \quad \text{(from 3.4)} \\ &= q^2 \Psi_\alpha E_\alpha^{s'} - qH_\beta(-1)\Psi_\alpha E_\alpha^{s'} E_\beta \\ &= q^2 \Psi_\alpha E_\alpha^{s'} - qH_\beta(-1)\Psi_\alpha E_\alpha E_\beta \quad \text{(from 4.1)} \\ &= q^2 \Psi_\alpha E_\alpha^{s'} + qH_\beta(-1) E_\alpha E_\beta \quad \text{(from 3.3)}. \end{aligned}$$

Thus, db yields

$$q^2 \Psi_\alpha E_\alpha^{s'} + qE_\alpha E_\beta.$$

Therefore, we have

$$X_\alpha = q^2 \Psi_\alpha E_\alpha^{s'} + E_\alpha E_\beta.$$

It is easy to see that the terms having R_α in $p_{\beta\alpha}$ are precisely $a'b'$, $b'a'$, $c'd'$, and $(b')^2$.

Let us compute $(b')^2$,

$$\begin{aligned} (b')^2 &= (R_\alpha E_\beta^s \Psi_\alpha)(R_\alpha E_\beta^s \Psi_\alpha) \\ &= R_\alpha^2 (E_\beta^s)^s \Psi_\alpha^s E_\beta^s \Psi_\alpha \\ &= R_\alpha^2 E_\beta E_\beta^s \Psi_\alpha^s \Psi_\alpha \\ &= (qH_\alpha(-1) + R_\alpha E_\alpha) E_\beta E_\beta^s (qH_\alpha(-1) - E_\alpha). \end{aligned}$$

Hence $(b')^2$ yield the coefficient $E_\alpha E_\beta E_\beta^s (qH_\alpha(-1) - E_\alpha) = qE_\alpha E_\beta E_\beta^s - E_\alpha^2 E_\beta E_\beta^s$. Then $(b')^2$ yield precisely

$$(q-1)E_\alpha E_\beta.$$

Now, we have $a'b' = (E_\beta E_\alpha)(R_\alpha E_\beta^s \Psi_\alpha) = R_\alpha E_\beta^s E_\alpha^s E_\beta^s \Psi_\alpha$. Therefore, using Lemma 4, we deduce that $a'b'$ yield the coefficient

$$E_\beta^s E_\alpha^s E_\beta^s \Psi_\alpha = -(q-1)E_\alpha E_\beta.$$

It is easy to see that $b'a'$ also yield the same coefficient of $a'b'$.

Let us now compute $c'd'$,

$$\begin{aligned} c'd' &= (R_\beta \Psi_\beta E_\alpha)(R_\beta R_\alpha \Psi_\beta^s \Psi_\alpha) \\ &= R_\beta^2 R_\alpha (\Psi_\beta^{s'})^s (E_\alpha^{s'})^s \Psi_\beta^s \Psi_\alpha \\ &= (qH_\beta(-1) + R_\beta E_\beta) R_\alpha (\Psi_\beta^{s'})^s (E_\alpha^{s'})^s \Psi_\beta^s \Psi_\alpha \\ &= qH_\beta(-1) R_\alpha (\Psi_\beta^{s'})^s (E_\alpha^{s'})^s \Psi_\beta^s \Psi_\alpha + R_\beta E_\beta R_\alpha (\Psi_\beta^{s'})^s (E_\alpha^{s'})^s \Psi_\beta^s \Psi_\alpha. \end{aligned}$$

That is,

$$c'd' = qR_\alpha(H_\beta(-1))^s(\Psi_\beta^{s'})^s(E_\alpha^{s'})^s\Psi_\beta^s\Psi_\alpha + R_\beta R_\alpha E_\beta^s(\Psi_\beta^{s'})^s(E_\alpha^{s'})^s\Psi_\beta^s\Psi_\alpha.$$

Thus, $c'd'$ yields the coefficient

$$q(H_\beta(-1))^s(\Psi_\beta^{s'})^s(E_\alpha^{s'})^s\Psi_\beta^s\Psi_\alpha = q(H_\beta(-1))^s(\Psi_\beta^{s'}\Psi_\beta)^s(E_\alpha^{s'})^s\Psi_\alpha.$$

Now, from the observation 3.4 of Proposition 3, and the observation 4.2 of Lemma 4, we have

$$\begin{aligned} q(H_\beta(-1))^s(\Psi_\beta^{s'}\Psi_\beta)^s(E_\alpha^{s'})^s\Psi_\alpha &= q(H_\beta(-1))^s(qH_\beta(-1) - E_\beta)^s E_\alpha^{s'}\Psi_\alpha \\ &= q(H_\beta(-1))^s(q(H_\beta(-1))^s - E_\beta^s)E_\alpha^{s'}\Psi_\alpha \\ &= q^2E_\alpha^{s'}\Psi_\alpha - qE_\beta^sE_\alpha^{s'}\Psi_\alpha \\ &= q^2E_\alpha^{s'}\Psi_\alpha + qE_\alpha E_\beta. \end{aligned}$$

Thus, we have proved $Y_\alpha = q^2E_\alpha^{s'}\Psi_\alpha + E_\alpha E_\beta = X_\alpha$.

Computation of $X_{\alpha\beta}$ and $Y_{\alpha\beta}$. First, notice that there is only one term having $R_\alpha R_\beta$ in $p_{\beta\alpha}^2$, which is $b'c'$. Now,

$$\begin{aligned} b'c' &= (R_\alpha E_\beta^s\Psi_\alpha)(R_\beta\Psi_\beta E_\alpha) \\ &= R_\alpha R_\beta(E_\beta^s)^{s'}\Psi_\alpha^{s'}\Psi_\beta E_\alpha \\ &= R_\alpha R_\beta E_\beta^s\Psi_\alpha^{s'}\Psi_\beta E_\alpha \quad \text{(from 4.2)} \\ &= R_\alpha R_\beta\Psi_\alpha^{s'}\Psi_\beta E_\alpha E_\beta \quad \text{(from 4.1)} \\ &= -R_\alpha R_\beta\Psi_\alpha^{s'}E_\alpha E_\beta \quad \text{(from 3.3)}. \end{aligned}$$

Therefore, we have $Y_{\alpha\beta} = -\Psi_\alpha^{s'}E_\alpha E_\beta$.

On the other side, the only terms having the monomial $R_\alpha R_\beta$ in $p_{\alpha\beta}^2$ are: ad, cb, cd, da and db. We have:

$$\begin{aligned} ad &= (E_\alpha E_\beta)(R_\alpha R_\beta\Psi_\alpha^{s'}\Psi_\beta) \\ &= R_\alpha R_\beta(E_\alpha^s)^{s'}(E_\beta^s)^{s'}\Psi_\alpha^{s'}\Psi_\beta \\ &= R_\alpha R_\beta E_\alpha^{s'}E_\beta^s\Psi_\alpha^{s'}\Psi_\beta \\ &= R_\alpha R_\beta(E_\alpha\Psi_\alpha)^{s'}E_\beta^s\Psi_\beta \\ &= R_\alpha R_\beta(-E_\alpha^{s'})E_\beta^s\Psi_\beta \\ &= -R_\alpha R_\beta E_\alpha E_\beta\Psi_\beta \quad \text{(from 4.3)} \\ &= R_\alpha R_\beta E_\alpha E_\beta. \end{aligned}$$

$$\begin{aligned} da &= (R_\alpha R_\beta\Psi_\alpha^{s'}\Psi_\beta)(E_\alpha E_\beta) \\ &= R_\alpha R_\beta\Psi_\alpha^{s'}\Psi_\beta E_\alpha E_\beta \\ &= -R_\alpha R_\beta\Psi_\alpha^{s'}E_\alpha E_\beta. \end{aligned}$$

$$\begin{aligned} cb &= (R_\alpha\Psi_\alpha E_\beta R_\beta)(E_\alpha^{s'}\Psi_\beta) \\ &= R_\alpha R_\beta\Psi_\alpha^{s'}E_\beta^{s'}E_\alpha^{s'}\Psi_\beta \\ &= R_\alpha R_\beta(\Psi_\alpha E_\alpha)^{s'}E_\beta\Psi_\beta \\ &= R_\alpha R_\beta E_\alpha^{s'}E_\beta \\ &= R_\alpha R_\beta E_\alpha E_\beta \quad \text{(from 4.1)}. \end{aligned}$$

From the observation 1.1 of Theorem 1, it is easy to see that db yields the coefficient

$$\begin{aligned} E_\beta(\Psi_\alpha^{s'})^{s'}\Psi_\beta^{s'}E_\alpha^{s'}\Psi_\beta &= (\Psi_\alpha^{s'})^{s'}(E_\beta\Psi_\beta)^{s'}E_\alpha^{s'}\Psi_\beta \\ &= -\Psi_\alpha E_\beta E_\alpha^{s'}\Psi_\beta \\ &= -\Psi_\alpha E_\beta E_\alpha \Psi_\beta \\ &= -E_\alpha E_\beta. \end{aligned}$$

Again from the observation 1.1 of Theorem 1, we deduce that cd yield the coefficient

$$\begin{aligned} E_\alpha^{s'}(\Psi_\alpha^{s})^{s'}(E_\beta^{s})^{s'}\Psi_\alpha^{s'}\Psi_\beta &= (E_\alpha\Psi_\alpha^{s})^{s'}E_\beta^{s}\Psi_\alpha^{s'}\Psi_\beta \\ &= -E_\alpha^{s'}\Psi_\alpha^{s'}E_\beta^{s}\Psi_\beta \\ &= -(E_\alpha^{s'}\Psi_\alpha^{s'})(E_\beta^{s}\Psi_\beta) \\ &= -E_\alpha E_\beta. \end{aligned}$$

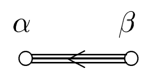
Thus, we have $X_{\alpha\beta} = Y_{\alpha\beta} = -\Psi_\alpha^{s'}E_\alpha E_\beta = E_\alpha E_\beta$.

Computation of $X_{\alpha\beta\alpha\beta}$ and $Y_{\alpha\beta\alpha\beta}$. It is easy to see that d^2 is the only term that yields $X_{\alpha\beta\alpha\beta}$ and this coefficient is $\Psi_\alpha^{s'}\Psi_\beta((\Psi_\alpha^{s'}\Psi_\beta)^{s})^{s'}$.

Also, it is easy to see that $Y_{\alpha\beta\alpha\beta}$ is equal to $\Psi_\alpha^{s'}\Psi_\beta\Psi_\alpha\Psi_\beta^{s}$. By Lemma 4, it is clear that $((\Psi_\alpha^{s'})^{s})^{s'} = \Psi_\alpha$ and $(\Psi_\beta^{s})^{s'} = \Psi_\beta^{s}$. Hence, we have $X_{\alpha\beta\alpha\beta} = Y_{\alpha\beta\alpha\beta}$. □

4. Case G_2

Let $\Pi = \{\alpha, \beta\}$ be a system of positive simple root of Φ. Let us put

G_2:


the Dynkin diagram. Let W denote the Weyl group of G_2. Let s (respectively s') denote the reflection corresponding to the root α (respectively β). We have, $\langle\alpha,\beta\rangle = -1$ and $\langle\beta,\alpha\rangle = -3$. Hence, we have

(3) $$s(\beta) = 3\alpha + \beta, \quad s'(\alpha) = \alpha + \beta.$$

In the proof of the braid relation of type G_2 (Lemma 7), we will use the following Lemma.

Lemma 6. *We have,*

(6.1) $$((E_\alpha^{s'})^{s})^{s'} = (E_\alpha^{s'})^{s}, \quad ((E_\beta^{s})^{s'})^{s} = (E_\beta^{s})^{s'}$$

(6.2) $$((\Psi_\alpha^{s'})^{s})^{s'} = (\Psi_\alpha^{s'})^{s}, \quad ((\Psi_\beta^{s})^{s'})^{s} = (\Psi_\beta^{s})^{s'}$$

(6.3) $$E_\alpha E_\alpha^{s'} = E_\alpha E_\beta, \quad E_\alpha E_\beta E_\beta^{s} = (q-1)E_\alpha E_\beta$$

(6.4) $$H_\alpha(-1)E_{3\alpha+2\beta} = E_{3\alpha+2\beta}, \quad H_{\alpha+\beta}(-1)E_\beta E_{3\alpha+2\beta} = E_\beta E_{3\alpha+2\beta}$$

(6.5) $$E_{\alpha+\beta}E_{3\alpha+2\beta} = E_\alpha E_\beta, \quad E_{3\alpha+\beta}E_{3\alpha+2\beta} = E_\beta E_{3\alpha+2\beta}$$

(6.6) $$((\Psi_\beta^{s'}\Psi_\beta)((\Psi_\alpha\Psi_\alpha^{s})^{s'})(E_\beta^{s})^{s'})^{s} = (\Psi_\beta^{s'}\Psi_\beta)(\Psi_\alpha\Psi_\alpha^{s})^{s'}(E_\beta^{s})^{s'}$$

(6.7) $$(E_\alpha E_\beta)^{w} = E_\alpha E_\beta, \quad \Psi_\alpha^{w}E_\alpha E_\beta = \Psi_\beta^{w}E_\alpha E_\beta = -E_\alpha E_\beta, \quad w \in W$$

(6.8) $$H_\alpha(-1)E_\alpha^{s'} = H_\beta(-1)E_\alpha^{s'}$$

(6.9) $$\Psi_\alpha^{s'}E_\beta + \Psi_\alpha\Psi_\beta^{s'}E_\beta = 0.$$

Proof. We have $s(s'(\alpha)) = s(\alpha+\beta) = -\alpha+(3\alpha+\beta) = 2\alpha+\beta$. Therefore, we have $s'(s(s'(\alpha))) = s'(2\alpha+\beta) = 2(\alpha+\beta)-\beta = 2\alpha+\beta = s(s'(\alpha))$. Using a similar argument, it is easy to see that $s(s'(s(\beta))) = s'(s(\beta))$. These observations prove 6.1 and 6.2.

Now, we will prove 6.3. We have $s'(\alpha) = \alpha+\beta$, and so we have $E_\alpha^{s'} = \sum_{r\in k^\times} H_\alpha(r)H_\beta(r)$. Then

$$\begin{aligned} E_\alpha E_\alpha^{s'} &= \sum_{t\in k^\times} H_\alpha(t) \sum_{r\in k^\times} H_\alpha(r)H_\beta(r) \\ &= \sum_{r,t\in k^\times} H_\alpha(rt)H_\beta(r) \\ &= \sum t, r \in k^\times H_\alpha(t)H_\beta(r) \\ &= E_\alpha E_\beta \end{aligned}$$

In the above assertion, we use the fact that $(t,r)\mapsto(rt,r)$ is an automorphism of $k^\times \times k^\times$.

Proof of the other assertion of 6.3 is similar to this proof.

We now prove 6.4.

We have

$$\begin{aligned} H_\alpha(-1)E_{3\alpha+2\beta} &= \sum_{t\in k^\times} H_\alpha(-t^3)H_\beta(t^2) \\ &= \sum_{t\in k^\times} H_\alpha((-t)^3)H_\beta((-t)^2) \\ &= E_{3\alpha+2\beta} \end{aligned}$$

We note that here, we use the fact that $t\mapsto -t$ is a bijection of $k^\times$ onto itself.

Once again using this fact, we have $H_\beta E_\beta = E_\beta$. Therefore, we have

$$\begin{aligned} H_{\alpha+\beta}(-1)E_\beta E_{3\alpha+2\beta} &= (H_\beta(-1)E_\beta)(H_\alpha(-1)E_{3\alpha+2\beta}) \\ &= E_\beta E_{3\alpha+2\beta} \end{aligned}$$

This proves 6.4.

We now prove 6.5. We have

$$\begin{aligned} E_{\alpha+\beta}E_{3\alpha+2\beta} &= \sum_{(t,s)\in k^\times\times k^\times} H_\alpha(ts^3)H_\beta(ts^2) \\ &= E_\alpha E_\beta \end{aligned}$$

Here, we use the fact that $(t,s)\mapsto(ts^3,ts^2)$ is an automorphism of the group $k^\times\times k^\times$.

Similarly, the other assertion $E_{3\alpha+\beta}E_{3\alpha+2\beta} = E_\beta E_{3\alpha+2\beta}$ of 6.5 follows from the fact that $(t,s)\mapsto(ts, s^-1)$ is an automorphism of $k^\times\times k^\times$.

We now prove the assertion 6.6.

We first compute $(\Psi_\alpha\Psi_\alpha^s)^{s'}(\Psi_\beta\Psi_\beta^{s'})E_{3\alpha+2\beta}$.

We have $\Psi_\alpha\Psi_\alpha^s = qH_\alpha(-1) - E_\alpha$ and $s'(\alpha) = \alpha+\beta$. Therefore, we have $(\Psi_\alpha\Psi_\alpha^s)^{s'} = qH_{\alpha+\beta}(-1) - E_{\alpha+\beta}$.

We also have $\Psi_\beta\Psi_\beta^{s'} = qH_\beta(-1) - E_\beta$.

Thus, we get

$$((\Psi_\alpha\Psi_\alpha^s))^{s'}(\Psi_\beta\Psi_\beta^{s'})(((E_\beta)^s)^{s'})^s = (qH_{\alpha+\beta}(-1) - E_{\alpha+\beta})(qH_\beta(-1) - E_\beta)E_{3\alpha+2\beta}.$$

Then using 6.4, we have

$$\begin{aligned} ((\Psi_\alpha\Psi_\alpha^s))^{s'}(\Psi_\beta\Psi_\beta^{s'})(((E_\beta)^s)^{s'})^s &= q^2E_{3\alpha+2\beta} - qE_\beta E_{3\alpha+2\beta} - qE_\alpha E_\beta + (q-1)E_\alpha E_\beta \\ &= q^2E_{3\alpha+2\beta} - qE_\beta E_{3\alpha+2\beta} - E_\alpha E_\beta. \end{aligned}$$

We now prove that $q^2E_{3\alpha+2\beta} - qE_\beta E_{3\alpha+2\beta} - E_\alpha E_\beta$ is s- invariant.

To prove this, we prove each of the summand is s- invariant.

First, we have $s(3\alpha+2\beta) = -3\alpha + 2(3\alpha+\beta) = 3\alpha+2\beta$ and so we have $E^s_{3\alpha+2\beta} = E_{3\alpha+2\beta}$.
Secondly, we have

$$\begin{aligned}(E_\beta E_{3\alpha+2\beta})^s &= E_{3\alpha+\beta} E_{3\alpha+2\beta} \\ &= E_\beta E_{3\alpha+2\beta}. \quad \text{(from 6.5)}\end{aligned}$$

Thirdly, we have

$$\begin{aligned}(E_\alpha E_\beta)^s &= E_{-\alpha} E_{3\alpha+\beta} \\ &= \sum_{(t,s)\in k^\times \times k^\times} H_\alpha(t^{-1}s^3) H_\beta(s) \\ &= E_\alpha E_\beta.\end{aligned}$$

Here, we use the fact that $(t,s) \mapsto (t^{-1}s^3, s)$ is an automorphism of the group $k^\times \times k^\times$.
Thus, we have proved 6.6.
We now prove 6.7.
First, we prove $(E_\alpha E_\beta)^w = E_\alpha E_\beta$ for any $w \in W$.
Since the Weyl group of G_2 is generated by s and s', it is sufficient to prove that

$$(E_\alpha E_\beta)^s = E_\alpha E_\beta = (E_\alpha E_\beta)^{s'}.$$

For a a proof of the first equality, we have

$$\begin{aligned}(E_\alpha E_\beta)^s &= E_{-\alpha} E_{3\alpha+\beta} \\ &= \sum_{(t,s)\in k^\times \times k^\times} H_\alpha(t^{-1}s^3) H_\beta(s) \\ &= \sum_{(t,s)\in k^\times \times k^\times} H_\alpha(t) H_\alpha(s) \\ &= E_\alpha E_\beta.\end{aligned}$$

We note that in this proof, we use the fact that the map $(t,s) \mapsto (t^{-1}s^3, s)$ is an automorphism of $k^\times \times k^\times$.

The proof of the second equality follows from the fact that the map $(t,s) \mapsto (t, ts^{-1})$ is an automorphism of $k^\times \times k^\times$.

We now prove that $\Psi^w_\alpha E_\alpha E_\beta = -E_\alpha E_\beta$ for any $w \in W$..
Since $(E_\alpha E_\beta)^w = E_\alpha E_\beta$, we have

$$\begin{aligned}\Psi^w_\alpha E_\alpha E_\beta &= (\Psi_\alpha E_\alpha E_\beta)^w \\ &= (-E_\alpha E_\beta)^w \quad \text{(from (6.5))} \\ &= -E_\alpha E_\beta.\end{aligned}$$

We now prove 6.8.
We have

$$\begin{aligned}H_\alpha(-1) E^{s'}_\alpha &= H_\alpha(-1) E_{\alpha+\beta} \\ &= H_\alpha(-1)(\sum_{t\in k^\times} H_\alpha(t) H_\beta(t)) \\ &= \sum_{t\in k^\times} H_\alpha(-t) H_\beta(t) \\ &= \sum_{t\in k^\times} H_\alpha(t) H_\beta(-t) \\ &= H_\beta(-1)(\sum_{t\in k^\times} H_\alpha(t) H_\beta(t)) \\ &= H_\beta(-1) E^{s'}_\alpha.\end{aligned}$$

We now prove 6.9.

We have $s'(\alpha) = \alpha + \beta$, and so we have

$$\begin{aligned} \Psi_\alpha^{s'} E_\beta &= \sum_{(t,s)\in k^\times \times k^\times} \Psi(t) H_\alpha(t) H_\beta(ts) \\ &= \sum_{(t,s)\in k^\times \times k^\times} \Psi(t) H_\alpha(t) H_\beta(s) = \Psi_\alpha E_\beta. \end{aligned}$$

Here, we use the fact that the map $(t,s) \mapsto (t,ts)$ is an automorphism of $k^\times \times k^\times$.

On the other hand, we have

$$\Psi_\alpha(\Psi_\beta^{s'} E_\beta) = \Psi_\alpha(-E_\beta) = -\Psi_\alpha E_\beta.$$

Hence, we have

$$\Psi_\alpha^{s'} E_\beta + \Psi_\alpha \Psi_\beta^{s'} E_\beta = (\Psi_\alpha - \Psi_\alpha) E_\beta = 0.$$

Thus, we have proved 6.9. □

Lemma 7. *We have* $L_\alpha L_\beta L_\alpha L_\beta L_\alpha L_\beta = L_\beta L_\alpha L_\beta L_\alpha L_\beta L_\alpha$.

Proof. Set $p_{\alpha\beta} = q^3 L_\alpha L_\beta L_\alpha$, and $p_{\beta\alpha} = q^3 L_\beta L_\alpha L_\beta$. Then, one can re-write the braid relation of the lemma as

$$p_{\alpha\beta} p_{\beta\alpha} = p_{\beta\alpha} p_{\alpha\beta}. \tag{7.1}$$

According to Proposition 3, Lemma 6 and 1, we obtain

$$\begin{aligned} p_{\alpha\beta} = {} & \underbrace{(q-1)E_\alpha E_\beta}_{a} + \underbrace{qH_\alpha(-1)\Psi_\alpha \Psi_\alpha^s E_\beta^s}_{b} - \underbrace{R_\alpha E_\alpha E_\beta}_{c} + \underbrace{R_\beta E_\alpha E_\alpha^{s'} \Psi_\beta}_{d} \\ & + \underbrace{R_\alpha R_\beta \Psi_\alpha^{s'} \Psi_\beta E_\alpha}_{e} + \underbrace{R_\beta R_\alpha (E_\alpha^{s'} \Psi_\beta)^s \Psi_\alpha}_{f} + \underbrace{R_\alpha R_\beta R_\alpha (\Psi_\alpha^{s'} \Psi_\beta)^s \Psi_\alpha}_{g}, \end{aligned}$$

and

$$\begin{aligned} p_{\beta\alpha} = {} & \underbrace{(q-1)E_\alpha E_\beta}_{a'} + \underbrace{qH_\beta(-1)\Psi_\beta \Psi_\beta^{s'} E_\alpha^{s'}}_{b'} - \underbrace{R_\beta E_\beta E_\alpha}_{c'} + \underbrace{R_\alpha E_\beta E_\beta^{s} \Psi_\alpha}_{d'} \\ & + \underbrace{R_\beta R_\alpha \Psi_\beta^{s} \Psi_\alpha E_\beta}_{e'} + \underbrace{R_\alpha R_\beta (E_\beta^{s} \Psi_\alpha)^{s'} \Psi_\beta}_{f'} + \underbrace{R_\beta R_\alpha R_\beta (\Psi_\beta^{s} \Psi_\alpha)^{s'} \Psi_\beta}_{g'}. \end{aligned}$$

We are now going to compare the coefficients of the monomials on R_α and R_β obtained in both sides of equation 7.1. For any word γ in α and β, let X_γ (resp. Y_γ) be the coefficient of R_γ in the expression of L.H.S (resp. R.H.S) of 7.1.

To prove the Lemma, it is sufficient to prove that $X_\gamma = Y_\gamma$ for all words γ in α and β.

Computation of $X_{\alpha\beta\alpha\beta\alpha\beta}$ and $Y_{\alpha\beta\alpha\beta\alpha\beta}$. On the left in the product of 7.1 the monomial $R_\alpha R_\beta R_\alpha R_\beta R_\alpha R_\beta$ appears only in the multiplication gg', and then the coefficient $X_{\alpha\beta\alpha\beta\alpha\beta}$ of this monomnial is $((((\Psi_\alpha^{s'}\Psi_\beta)^s \Psi_\alpha)^{s'})^s)^{s\prime}((\Psi_\beta^s \Psi_\alpha)^{s'} \Psi_\beta)$.

We have

$$\begin{aligned} X_{\alpha\beta\alpha\beta\alpha\beta} &= ((((\Psi_\alpha^{s'})^s)^{s'})^s)^{s\prime}(((\Psi_\beta^{s})^{s'})^s)^{s\prime}((\Psi_\alpha^{s'})^s)^{s\prime}(\Psi_\beta^{s})^{s'}\Psi_\alpha^{s'}\Psi_\beta \\ &= \Psi_\alpha \Psi_\beta^s (\Psi_\alpha^{s'})^s (\Psi_\beta^s)^{s'} \Psi_\alpha^{s'} \Psi_\beta \quad \text{(from 6.2)} \\ &= ((((\Psi_\beta^s \Psi_\alpha)^{s'} \Psi_\beta))^s)^{s'})^s ((\Psi_\alpha^{s'} \Psi_\beta)^s \Psi_\alpha). \quad \text{(from 6.2)} \end{aligned}$$

This is the coefficient $Y_{\alpha\beta\alpha\beta\alpha\beta}$ of $R_\alpha R_\beta R_\alpha R_\beta R_\alpha R_\beta$ on the right of 7.1. Notice that we have $gg' = g'g$

Computation of $X_{\alpha\beta\alpha\beta\alpha}$ and $Y_{\alpha\beta\alpha\beta\alpha}$. It is to check that the monomial $R_\alpha R_\beta R_\alpha R_\beta R_\alpha$ occurs only in the product ge' on the left of 7.1, and only in the product $f'g$ on the right of 7.1. Now, the coefficient $X_{\alpha\beta\alpha\beta\alpha}$ is

$$\begin{aligned} X_{\alpha\beta\alpha\beta\alpha} &= ((\Psi_\alpha^{s'}\Psi_\beta)^s\Psi_\alpha)^{s'})^s(\Psi_\beta^s\Psi_\alpha E_\beta) \\ &= ((((E_\beta^s\Psi_\alpha)^{s'}\Psi_\beta)^s)^{s'})^s(\Psi_\alpha^{s'}\Psi_\beta)^s\Psi_\alpha \\ &= Y_{\alpha\beta\alpha\beta\alpha}. \end{aligned}$$

(notice that from 6.1, $(((E_\beta^s)^{s'})^s)^{s'})^s = E_\beta)$.

Computation of $X_{\alpha\beta\alpha\beta}$ and $Y_{\alpha\beta\alpha\beta}$. On the right of the equation 7.1 the monomial $R_\alpha R_\beta R_\alpha R_\beta$ appears only in the product $f'e$. We now compute this coefficient.

We have $s'(s(s'(s(\beta)))) = s(\beta)$ and so $(((E_\beta^s)^{s'})^s)' = E_\beta^s$. Using this observation, we have:

$$\begin{aligned} Y_{\alpha\beta\alpha\beta} &= (((E_\beta^s\Psi_\alpha)^{s'}\Psi_\beta)^s)^{s'}(\Psi_\alpha^{s'}\Psi_\beta E_\alpha) \\ &= (((E_\beta^s)^{s'})^s)^{s'}((\Psi_\alpha^{s'}\Psi_\beta)^s)^{s'}(\Psi_\alpha^{s'}\Psi_\beta E_\alpha) \\ &= (E_\beta^s E_\alpha)(\Psi_\alpha^{s'}\Psi_\beta)((\Psi_\alpha^{s'}\Psi_\beta)^s)^{s'} \quad \text{(from above)} \\ &= (E_\beta E_\alpha)^s)(\Psi_\alpha^{s'}\Psi_\beta)((\Psi_\alpha^{s'}\Psi_\beta)^s)^{s'} (\text{since } E_\alpha^s = E_\alpha) \\ &= (-1)^4 E_\alpha E_\beta \quad \text{(from 6.9)} \\ &= E_\alpha E_\beta. \end{aligned}$$

Hence, we have

$$Y_{\alpha\beta\alpha\beta} = E_\alpha E_\beta. \tag{7.2}$$

On the other side, the terms on the left of the equation 7.1. that contain the monomial $R_\alpha R_\beta R_\alpha R_\beta$, are the products: cg', ef', gc', eg', and gf'.

We now prove that cg' yields the coefficient $E_\alpha E_\beta$.

We have

$$\begin{aligned} cg' &= -R_\alpha R_\beta R_\alpha R_\beta(((E_\alpha E_\beta)^{s'})^s)^{s'}(\Psi_\beta^s\Psi_\alpha)^{s'}\Psi_\beta \\ &= -R_\alpha R_\beta R_\alpha R_\beta((-1)^3 E_\alpha E_\beta). \quad \text{(from 6.7)} \end{aligned}$$

Therefore, the coefficient of $R_\alpha R_\beta R_\alpha R_\beta$ in the expression of cg' is

$$-E_\alpha E_\beta. \tag{7.3}$$

We now prove that ef' yields the coefficient $E_\alpha E_\beta$. We have

$$\begin{aligned} ef' &= R_\alpha R_\beta R_\alpha R_\beta((\Psi_\alpha^{s'}\Psi_\beta E_\alpha)^s)^{s'}(E_\beta^s\Psi_\alpha)^{s'}\Psi_\beta \\ &= R_\alpha R_\beta R_\alpha R_\beta((E_\alpha E_\beta)^s)^{s'}((\Psi_\alpha^{s'}\Psi_\beta)^s)^{s'}\Psi_\alpha^{s'}\Psi_\beta \\ &= R_\alpha R_\beta R_\alpha R_\beta(-1)^4 E_\alpha E_\beta. \quad \text{(from 6.7)} \end{aligned}$$

Therefore, the coefficient yielded by ef' is

$$E_\alpha E_\beta. \tag{7.4}$$

By using 6.7, it is easy to see that gc' yields the coefficient

$$-(-1)^3 E_\alpha E_\beta = E_\alpha E_\beta. \tag{7.5}$$

Now, we compute the coefficient yielded by eg'.

We have

$$\begin{aligned} eg' &= (R_\alpha R_\beta\Psi_\alpha^{s'}\Psi_\beta E_\alpha)(R_\beta R_\alpha R_\beta(\Psi_\beta^s\Psi_\alpha)^{s'}\Psi_\beta) \\ &= R_\alpha R_\beta^2 R_\alpha R_\beta(((\Psi_\alpha^{s'}\Psi_\beta E_\alpha)^{s'})^s)^{s'}(\Psi_\beta^s\Psi_\alpha)^{s'}\Psi_\beta. \end{aligned}$$

Now, notice that $(((\Psi_\alpha^{s'}\Psi_\beta E_\alpha)^{s'})^s)^{s'}(\Psi_\beta^s\Psi_\alpha)^{s'}\Psi_\beta = (\Psi_\alpha\Psi_\alpha^s\Psi_\beta\Psi_\beta^s)^{s'}E_\alpha\Psi_\beta$. Then using 1.1, we get

$$\begin{aligned} R_\alpha R_\beta^2 R_\alpha R_\beta &= R_\alpha(qH_\beta(-1) + R_\beta E_\beta)R_\alpha R_\beta \\ &= R_\alpha^2 R_\beta((qH_\beta(-1))^s)^{s'} + R_\alpha R_\beta R_\alpha R_\beta((E_\beta)^s)^{s'}. \end{aligned}$$

But, we are interested in computing only the coefficient of $R_\alpha R_\beta R_\alpha R_\beta$. From the above computations, we first compute the E's (without (Ψ's) in coefficient of $R_\alpha R_\beta R_\alpha R_\beta$ in the product eg'

$$((E_\beta)^s)^{s'}(((E_\alpha)^{s'})^s)^{s'} = (((E_\beta E_\alpha)^{s'})^s)^{s'},$$

since $E_\beta^{s'} = E_\beta$. We now compute the coefficient together with Ψ's.

$$\begin{aligned} (((E_\beta E_\alpha)^{s'})^s)s'(\Psi_\alpha\Psi_\alpha^s\Psi_\beta^s)^{s'}((\Psi_\beta^{s'})^s)^{s'}\Psi_\beta &= (-1)^5 E_\alpha E_\beta \quad \text{(from 6.7)} \\ &= -E_\alpha E_\beta. \end{aligned}$$

Therefore, eg' yields the coefficient

(7.6) $$-E_\alpha E_\beta.$$

By a similar computation, we can see that the coefficient of $R_\alpha R_\beta R_\alpha R_\beta$ in the expression of gf' is equal to

(7.7) $$(-1)^5 E_\alpha E_\beta = -E_\alpha E_\beta.$$

Summing up these five coefficients (from 7.3 to 7.7), we have

(7.8) $$X_{\alpha\beta\alpha\beta} = 3E_\alpha E_\beta - 2E_\alpha E_\beta = E_\alpha E_\beta.$$

From the observations 7.2 and 7.8, we have $X_{\alpha\beta\alpha\beta} = Y_{\alpha\beta\alpha\beta}$.

Computation of $X_{\alpha\beta\alpha}$ and $Y_{\alpha\beta\alpha}$. In the left hand side of 7.1, the products that contain the monomials $R_\alpha R_\beta R_\alpha$ are: ce', ed', ee', ga', gb' and gd'.

Since $E_\alpha E_\beta$ is occurring in c, by using 6.7, it is easy to see that the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of the product ce' is

(7.9) $$-(-1)^2 E_\alpha E_\beta^2 = -(q-1)E_\alpha E_\beta.$$

Since $E_\alpha E_\beta$ is also occurring in a', one can check that the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of ga' to be equal to

(7.10) $$(-1)^3(q-1)E_\alpha E_\beta = -(q-1)E_\alpha E_\beta.$$

We now compute ed'. We have

$$\begin{aligned} ed' &= R_\alpha R_\beta R_\alpha(\Psi_\alpha^{s'}\Psi_\beta)^s(E_\alpha E_\beta)^s E_\beta\Psi_\alpha \\ &= R_\alpha R_\beta R_\alpha(-1)^3 E_\alpha E_\beta^2 \quad \text{(from 6.7)} \\ &= -R_\alpha R_\beta R_\alpha(q-1)E_\alpha E_\beta. \end{aligned}$$

Therefore, ed' yields the coefficient:

(7.11) $$-(q-1)E_\alpha E_\beta.$$

We now compute the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of ee'.

We have

$$\begin{aligned} ee' &= (R_\alpha R_\beta\Psi_\alpha^{s'}\Psi_\beta E_\alpha)(R_\beta R_\alpha\Psi_\beta^s\Psi_\alpha E_\beta) \\ &= R_\alpha R_\beta^2 R_\alpha((\Psi_\alpha^{s'}\Psi_\beta E_\alpha)^{s'})^s\Psi_\beta^s\Psi_\alpha E_\beta. \end{aligned}$$

We also have

$$R_\alpha R_\beta^2 R_\alpha = qR_\alpha H_\beta(-1)R_\alpha E_\alpha E_\beta + R_\alpha R_\beta R_\alpha(E_\beta)^s E_\alpha E_\beta. \quad \text{(from 1.1)}$$

We are interested in computing only the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of ee'. We first compute only the product of E's (without Ψ's).

This coefficient is equal to

$$((E_\alpha^{s'})^s)(E_\beta E_\beta^s) = ((E_\alpha E_\beta)^s)^{s'} E_\beta,$$

since $E_\beta^{s'} = E_\beta$.

We now compute the coefficient of $R_\alpha R_\beta R_\alpha$ (together with the product of Ψ's) in the expression of ee'.

The coeficient of $R_\alpha R_\beta R_\alpha$ in the expression of ee' is

$$\begin{aligned} ((\Psi_\alpha^{s'}\Psi_\beta)^{s'})^s \Psi_\beta^s \Psi_\alpha ((E_\alpha E_\beta)^{s'})^s E_\beta &= (-1)^4 E_\alpha E_\beta^2 \quad \text{(from 6.7)} \\ &= (q-1)E_\alpha E_\beta. > \quad \text{(from 1)} \end{aligned}$$

Hence ee' yields the coefficient

(7.12) $$(q-1)E_\alpha E_\beta.$$

By a similar computation, one can check that the coeficient of $R_\alpha R_\beta R_\alpha$ in the expression of gd' is

(7.13) $$(q-1)E_\alpha E_\beta.$$

We now compute gb'.

It is easy to see that gb' yields the coefficient

(7.14) $$q(\Psi_\alpha^{s'}\Psi_\beta)^s \Psi_\alpha \Psi_\beta \Psi_\beta^{s'} (H_\beta(-1)E_\alpha^{s'}).$$

Summing up all these coefficients (using the observations from 7.9 to 7.14): it is easy to see that the sum of the coefficients coming from ce', ed' and ga' is $-3(q-1)E_\alpha E_\beta$ and that the sum of the coefficients coming from ee' and gd' is $2(q-1)E_\alpha E_\beta$.

Therefore, we have

$$X_{\alpha\beta\alpha} = -(3(q-1)E_\alpha E_\beta) + 2(q-1)E_\alpha E_\beta + q(H_\beta(-1)E_\alpha^{s'})(\Psi_\alpha^{s'}\Psi_\beta)^s \Psi_\alpha \Psi_\beta \Psi_\beta^{s'}.$$

Thus, we have

(7.15) $$X_{\alpha\beta\alpha} = -(q-1)E_\alpha E_\beta + q(H_\beta(-1)E_\alpha^{s'}(\Psi_\alpha^{s'}\Psi_\beta)^s \Psi_\alpha \Psi_\beta \Psi_\beta^{s'}.$$

Now, we will compute $Y_{\alpha\beta\alpha}$. The products on the right that contain the monomials $R_\alpha R_\beta R_\alpha$ are: $a'g$, $b'g$, $d'f$, $d'g$, $f'c$, and $f'f$.

Computations are similar to the computations of $X_{\alpha\beta\alpha}$.

Since a' contains $E_\alpha E_\beta$ as a factor, by using 6.7, it is easy to see that the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of $a'g$ is

(7.16) $$(-1)^3(q-1)E_\alpha E_\beta = -(q-1)E_\alpha E_\beta.$$

By the same argument, it is easy to see that the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of $f'c$ is

(7.17) $$-(-1)^2(q-1)E_\alpha E_\beta = -(q-1)E_\alpha E_\beta.$$

(Here, we use the fact that $E_\alpha E_\beta$ is a factor of c).

We now compute $d'f$.

We have

$$\begin{aligned} d'f &= R_\alpha R_\beta R_\alpha ((E_\beta E_\beta^s \Psi_\alpha)^{s'})^s (E_\alpha^{s'}\Psi_\beta)^s \Psi_\alpha \\ &= R_\alpha R_\beta R_\alpha ((E_\alpha E_\beta)^{s'})^s ((E_\beta^s)^{s'})^s (\Psi_\alpha^{s'})^s \Psi_\beta^s \Psi_\alpha. \end{aligned}$$

Therefore, by using 6.7, it is easy to see that the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of $d'f$ is

(7.18) $$(-1)^3(q-1)E_\alpha E_\beta = -(q-1)E_\alpha E_\beta.$$

By a similar computation in ee', using 1.1, and 6.7, one can check that the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of $f'f$ is equal to

(7.19) $$(-1)^4(q-1)E_\alpha E_\beta = (q-1)E_\alpha E_\beta.$$

By a similar computation in gd', using 1.1 and 6.7, one can check that the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of $d'g$ is equal to

$$(-1)^4(q-1)E_\alpha E_\beta = (q-1)E_\alpha E_\beta. \tag{7.20}$$

We now compute the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of $b'g$.

Using Theorem 1, it is easy to see that the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of the product $b'g$ is:

$$\begin{aligned} & (((qH_\beta(-1)\Psi_\beta\Psi_\beta^{s'}E_\alpha^{s'})^s)^{s'})^s(\Psi_\alpha^{s'}\Psi_\beta)^s\Psi_\alpha \\ = \; & qH_\alpha(-1)^3H_\beta(-1)^2E_\alpha^{s'}(\Psi_\beta^s)^{s'}(((\Psi_\beta^{s'})^s)^{s'})^s(\Psi_\alpha^{s'}\Psi_\beta)^s\Psi_\alpha \\ = \; & qH_\alpha(-1)E_\alpha^{s'}((\Psi_\beta)^s)^{s'}(((\Psi_\beta^{s'})^s)^{s'})^s(\Psi_\beta^{s'}\Psi_\beta)^s\Psi_\alpha. \end{aligned}$$

Then, from 6.8 we obtain that the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of the product $b'g$ is

$$qH_\beta(-1)E_\alpha^{s'})((\Psi_\beta^s)^{s'}(((\Psi_\beta^{s'})^s)^{s'})^s(\Psi_\alpha^{s'}\Psi_\beta)^s\Psi_\alpha \tag{7.21}$$

Summing up these coefficients (using observations 7.16 to 7.21), we have

$$Y_{\alpha\beta\alpha} = -(q-1)E_\alpha E_\beta + qH_\beta(-1)(((((\Psi_\beta)^{s'})^s)^{s'})^s((\Psi_\beta)^s)^{s'}E_\alpha^{s'})(\Psi_\alpha^{s'}\Psi_\beta)^s\Psi_\alpha. \tag{7.22}$$

To prove $X_{\alpha\beta\alpha} = Y_{\alpha\beta\alpha}$, from the observations 7.15 and 7.22, it is sufficient to prove that

$$(\Psi_\beta^s)^{s'}((((\Psi_\beta)^{s'})^s)^{s'})^sE_\alpha^{s'} = \Psi_\beta\Psi_\beta^{s'}E_\alpha^{s'}.$$

We now prove this assertion.

By a similar proof of 6.1, it is easy to see that

$$s(s'(s(s'(\beta)))) = -(3\alpha+2\beta) = s'(s(s'(\beta))).$$

Hence, we have

$$\begin{aligned} ((((\Psi_\beta)^{s'})^s)^{s'})^s((\Psi_\beta)^s)^{s'} &= (((\Psi_\beta)^{s'})^s)^{s'} \\ &= ((\Psi_\beta\Psi_\beta^{s'})^s)^{s'} \\ &= ((qH_\beta - E_\beta)^s)^{s'} \quad \text{(from Theorem 1)} \\ &= q(H_\alpha(-1))^3(H_\beta(-1))^2 - ((E_\beta)^s)^{s'} \quad (\text{since } s's(\beta) = 3\alpha+2\beta) \end{aligned}$$

Using this and fact that $(H_\alpha(-1))^2 = (H_\beta(-1))^2 = 1$, we have

$$\begin{aligned} ((((\Psi_\beta)^{s'})^s)^{s'})^s((\Psi_\beta)^s)^{s'}E_\alpha^{s'} &= (qH_\alpha(-1)E_\alpha^{s'} - (E_\beta^sE_\alpha)^{s'} \\ &= qH_\beta(-1)E_\alpha^{s'} - (E_\beta^sE_\alpha)^{s'} \quad \text{(from 6.8))} \\ &= qH_\beta(-1)E_\alpha^{s'} - ((E_\beta E_\alpha)^s)^{s'} \quad (\text{since } E_\alpha^s = E_\alpha) \\ &= qH_\beta(-1)E_\alpha^{s'} - (E_\alpha E_\beta)^{s'} \quad \text{(from 6.7)} \\ &= qH_\beta E_\alpha^{s'} - E_\alpha^{s'}E_\beta \quad (\text{since } E_\beta^{s'} = E_\beta) \\ &= \Psi_\beta\Psi_\beta^{s'}E_\alpha^{s'} \quad \text{(from Theorem 1).} \end{aligned}$$

Thus, we have proved $X_{\alpha\beta\alpha} = Y_{\alpha\beta\alpha}$.

Computation of X_α and Y_α. We first compute X_α. On the left of 7.1, the terms containing the monomials R_α are: $(a+b)d'$, $c(a'+b')$, cd', de', ec', ee', and gf'.

Now, we deduce that de' yields $q(q-1)E_\alpha E_\beta$. In fact,

$$\begin{aligned} de' &= (R_\beta E_\alpha E_\alpha^{s'}\Psi_\beta)(R_\beta R_\alpha \Psi_\beta^s\Psi_\alpha E_\beta) \\ &= R_\beta^2R_\alpha(E_\alpha^{s'})^s((E_\alpha)^({s'^2}))^s(\Psi_\beta^s)^{s'}\Psi_\beta^s\Psi_\alpha E_\beta) \\ &= R_\beta^2R_\alpha(E_\alpha^{s'})^sE_\alpha(\Psi_\beta^s)^{s'}\Psi_\beta^s\Psi_\alpha E_\beta \quad (\text{since } (s')^2 = 1 \text{ and } E_\alpha^s = E_\alpha). \end{aligned}$$

Now, using 1.1, it is easy to see that de' yields the coefficient $qH_\beta(-1)(E_\alpha^{s'})^s E_\alpha (\Psi_\beta^s)^{s'} \Psi_\beta^s \Psi_\alpha E_\beta$. We also have $(E_\alpha^{s'})^s E_\alpha E_\beta = (q-1)E_\alpha E_\beta$. Therefore, using Lemma 6 we deduce that de' yields the coefficent

(7.23) $$(-1)^3 q(q-1)E_\alpha E_\beta = -q(q-1)E_\alpha E_\beta.$$

In the same way, one can check that ee' yields the same coefficient

(7.24) $$(-1)^4 q(q-1)E_\alpha E_\beta = q(q-1)E_\alpha E_\beta.$$

Since $E_\alpha E_\beta$ is a factor of c', using 6.7 and the fact that $R_\beta^2 = qH_\beta(-1) + R_\beta E_\beta$, it is easy to see that the coefficient of R_α in the expression of ec' is

(7.25) $$(-1)^3(q(q-1))E_\alpha E_\beta = -q(q-1)E_\alpha E_\beta.$$

We now compute the coefficient of R_α in the expression of cd'. We have

$$\begin{aligned} cd' &= (R_\alpha E_\alpha E_\beta)(R_\alpha E_\beta E_\beta^s \Psi_\alpha) \\ &= R_\alpha^2 E_\alpha E_\beta^s E_\beta E_\beta^s \Psi_\alpha \\ &= (qH_\alpha(-1) + R_\alpha E_\alpha)E_\alpha E_\beta^s E_\beta E_\beta^s \Psi_\alpha. \end{aligned}$$

Therefore, the coefficient of R_α in the expression of cd' is $E_\alpha E_\alpha E_\beta^s E_\beta E_\beta^s \Psi_\alpha$, which turns out to be equal to

(7.26). $$(-1)^2(q-1)^3 E_\alpha E_\beta = (q-1)^3 E_\alpha E_\beta.$$

(by using 6.7).

We now compute the coefficient of R_α in the expression of $c(a'+b')$.

Since $E_\alpha E_\beta$ is a factor of c, using 6.7, it is easy to see that the coefficient of R_α in the expression of ca' yields $-(q-1)^3 E_\alpha$. Now, again using 6.7, it is easy to see that the coefficient of R_α in the expression of cb' is $(-1)^3 q(q-1)E_\alpha E_\beta = -q(q-1)E_\alpha E_\beta$.

Therefore $c(a'+b')$ yields the coefficient

(7.27) $$-((q-1)^3 + q(q-1))E_\alpha E_\beta = -(q-1)(q^2-q+1)E_\alpha E_\beta.$$

We now compute the coefficient of R_α in the expression of $(a+b)d'$.

Since $E_\alpha E_\beta$ is a factor of a, using 6.7, it is easy to see that $-(q-1)^3 E_\alpha E_\beta$ is the coefficient of R_α in the expression of ad'.

On the other hand, we have

$$\begin{aligned} bd' &= (qH_\alpha(-1)\Psi_\alpha \Psi_\alpha^s E_\beta^s)(R_\alpha E_\beta E_\beta^s \Psi_\alpha) \\ &= qR_\alpha H_\alpha(-1)\Psi_\alpha \Psi_\alpha^s E_\beta E_\beta E_\beta^s \Psi_\alpha \quad (\text{since } s^2 = 1 \text{ and } H_\alpha(-1)^s = H_\alpha(-1)) \\ &= q(q-1)R_\alpha H_\alpha(-1)\Psi_\alpha \Psi_\alpha^s E_\beta E_\beta^s \Psi_\alpha. \end{aligned}$$

Thus, $(a+b)d'$ yields

(7.28) $$-(q-1)^3 E_\alpha E_\beta + q(q-1)H_\alpha(-1)\Psi_\alpha \Psi_\alpha^s E_\beta E_\beta^s \Psi_\alpha.$$

We now compute the coefficient of R_α in the expression of gf'. We have

$$\begin{aligned} gf' &= (R_\alpha R_\beta R_\alpha (\Psi_\alpha^{s'} \Psi_\beta)^s \Psi_\alpha)(R_\alpha R_\beta (E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta) \\ &= R_\alpha R_\beta R_\alpha^2 R_\beta (\Psi_\alpha^{s'} \Psi_\beta)^{s'} (\Psi_\alpha^s)^{s'} (E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta. \end{aligned}$$

Using $R_\alpha^2 = qH_\alpha(-1) + R_\alpha E_\alpha$, we get

$$gf' = qR_\alpha R_\beta^2 H_\alpha(-1)^{s'} (\Psi_\alpha^{s'} \Psi_\beta)^{s'} (\Psi_\alpha^s)^{s'} (E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta + R_\alpha R_\beta R_\alpha R_\beta E_\alpha^{s'} (\Psi_\alpha^{s'} \Psi_\beta)^{s'} \Psi_\alpha^{s'} (E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta.$$

Now, using $R_\beta^2 = qH_\beta(-1) + R_\beta E_\beta$, and using the fact that

$$H_\beta(-1)(H_\alpha(-1))^{s'} = H_\alpha(-1)H_\beta(-1)^2 = H_\alpha(-1),$$

one can see that the coefficient of R_α in the expression of gf' is

(7.29) $$q^2 H_\alpha(-1)\Psi_\alpha \Psi_\beta^{s'} (\Psi_\alpha^s)^{s'} (E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta$$

Therefore, using the observations from 7.23 to 7.29, we have
(7.30)

$$X_\alpha = -(q-1)(q^2+1)E_\alpha E_\beta + q(q-1)H_\alpha(-1)\Psi_\alpha\Psi_\alpha^s E_\beta E_\beta^s\Psi_\alpha$$
$$+q^2 H_\beta(-1)H_\alpha(-1)^{s'}\Psi_\alpha\Psi_\beta^{s'}(\Psi_\alpha^s)^{s'}(E_\beta^s\Psi_\alpha)^{s'}\Psi_\beta.$$

We now compute Y_α. On the right of 7.1, the terms having the monomials R_α are: $(a'+b')c$, $d'(a+b)$, $d'c$, $c'f$, $f'd$, $f'f$, and $e'g$.

Since $E_\alpha E_\beta$ is a factor of c, using 6.7 and the facts that $E_\alpha^2=(q-1)E_\alpha$, $E_\beta^2=(q-1)E_\beta$, it is easy to see that the coefficient of R_α in the expression of $a'c$ is $-(q-1)^3E_\alpha E_\beta$. Again, since $E_\alpha E_\beta$ is a factor of c, using 6.7 and the facts that $E_\alpha^2=(q-1)E_\alpha$, $H_\beta(-1)E_\beta=E_\beta$, it is easy to see that the coefficient of R_α in the expression of $b'c$ is

$$(-1)^3q(q-1)E_\alpha E_\beta = -(q(q-1))E_\alpha E_\beta.$$

Thus, the coefficient of R_α in the expression of $(a'+b')c$ is

(7.31) $$-((q-1)^3+q(q-1))E_\alpha E_\beta = -(q-1)(q^2-q+1)E_\alpha E_\beta.$$

We now compute the coefficient of R_α in the expression of $d'(a+b)$. Since $E_\alpha E_\beta$ is a factor of a using 6.7 and the fact that $E_\beta^2=(q-1)E_\beta$, it is easy to check that the coefficient of R_α in the expression of $d'a$ is $-(q-1)^3E_\alpha E_\beta$. Also, it is clear that the coefficient of R_α in the expression of $d'b$ is

$$(E_\beta E_\beta^s\Psi_\alpha)(qH_\alpha(-1)\Psi_\alpha\Psi_\alpha^sE_\beta^s) = q(q-1)H_\alpha(-1)\Psi_\alpha\Psi_\alpha\Psi_\alpha^sE_\beta E_\beta^s.$$

Thus, the coefficient of R_α in the expression of $d'(a+b)$ is

(7.32) $$-(q-1)^3E_\alpha E_\beta + q(q-1)H_\alpha(-1)(\Psi_\alpha)^2\Psi_\alpha^sE_\beta(E_\beta^s)^2.$$

We now compute the coefficient of R_α in the expression of $d'c$. We have

$$d'c = (R_\alpha E_\beta E_\beta^s\Psi_\alpha)(R_\alpha E_\alpha E_\beta)$$
$$= R_\alpha^2E_\beta^sE_\beta\Psi_\alpha^sE_\alpha E_\beta.$$

Using $R_\alpha^2 = qH_\alpha(-1)+R_\alpha E_\alpha$ and 6.7, it is easy to see that the coefficient of R_α in the expression of $d'c$ is

(7.33) $$E_\alpha E_\beta^sE_\beta\Psi_\alpha^sE_\alpha E_\beta = (q-1)^3E_\alpha E_\beta.$$

Let us compute the coefficient the yields $f'd$. We have

$$f'd = (R_\alpha R_\beta(E_\beta^s\Psi_\alpha)^{s'}\Psi_\beta)(R_\beta E_\alpha E_\alpha^{s'}\Psi_\beta)$$
$$= R_\alpha R_\beta^2E_\beta^s\Psi_\alpha\Psi_\beta^{s'}E_\alpha E_\alpha^{s'}\Psi_\beta.$$

Therefore from 1.1, we get

$$f'd = qR_\alpha H_\beta(-1)E_\beta^s\Psi_\alpha\Psi_\beta^{s'}E_\alpha E_\alpha^{s'}\Psi_\beta + R_\alpha R_\beta E_\beta E_\beta^s\Psi_\alpha\Psi_\beta^{s'}E_\alpha E_\alpha^{s'}\Psi_\beta.$$

We have $E_\alpha^s=E_\alpha$ and so we have

$$H_\beta(-1)E_\alpha E_\beta^s = H_\beta(-1)(E_\alpha E_\beta)^s$$
$$= E_\alpha(H_\beta(-1)E_\beta) \quad \text{(from 6.7)}$$
$$= E_\alpha E_\beta \quad (\text{since } H_\beta(-1)E_\beta=E_\beta).$$

Therefore, $E_\alpha E_\beta$ is a factor of the coefficient of R_α in the expression of $f'd$ and hence using 6.7 and the fact that $E_\beta^2=(q-1)E_\beta$, it is easy to see that the coefficient of R_α in the expression of $f'd$ is

(7.34) $$(-1)^3(q(q-1))E_\alpha E_\beta = -q(q-1)E_\alpha E_\beta.$$

Since $E_\alpha E_\beta$ is a factor of c', using 6.7, 1.1 and the fact that $E_\alpha^2=(q-1)E_\alpha$, it is easy to see that the coefficient of R_α in the expression of $c'f$ is

(7.35) $$(-1)^3q(q-1)E_\alpha E_\beta = -(q(q-1))E_\alpha E_\beta.$$

We now compute the coefficient of R_α in the expression of the product $f'f$. We have

$$\begin{aligned} f'f &= (R_\alpha R_\beta (E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta)(R_\beta R_\alpha (E_\alpha^{s'} \Psi_\beta)^s \Psi_\alpha) \\ &= R_\alpha R_\beta^2 R_\alpha (E_\beta^s \Psi_\alpha \Psi_\beta^{s'} E_\alpha^{s'} \Psi_\beta)^s \Psi_\alpha \\ &= q R_\alpha^2 H_\beta(-1)^s E_\beta (\Psi_\alpha \Psi_\beta^{s'} E_\alpha^{s'} \Psi_\beta)^s \Psi_\alpha + R_\alpha R_\beta R_\alpha E_\beta^s E_\beta (\Psi_\alpha \Psi_\beta^{s'} E_\alpha^{s'} \Psi_\beta)^s \Psi_\alpha. \end{aligned}$$

Thus, from 1.1 we deduce that $f'f$ yields the coefficient

$$qE_\alpha (H_\beta(-1) E_\beta^s \Psi_\alpha \Psi_\beta^{s'} E_\alpha^{s'} \Psi_\beta)^s \Psi_\alpha = -qE_\alpha E_\beta (H_\beta(-1) \Psi_\alpha \Psi_\beta^{s'} E_\alpha^{s'} \Psi_\beta)^s.$$

Using 6.7, we deduce that $f'f$ yields

$$(-1)^4 q(q-1) E_\alpha E_\beta = q(q-1) E_\alpha E_\beta. \tag{7.36}$$

We now compute the coefficient of R_α in the expression of $e'g$. We have

$$\begin{aligned} e'g &= (R_\beta R_\alpha \Psi_\beta^s \Psi_\alpha E_\beta)(R_\alpha R_\beta R_\alpha (\Psi_\alpha^{s'} \Psi_\beta)^s \Psi_\alpha) \\ &= R_\beta R_\alpha^2 R_\beta R_\alpha (((\Psi_\beta^s \Psi_\alpha E_\beta)^s)^{s'})^s (\Psi_\alpha^{s'} \Psi_\beta)^s \Psi_\alpha. \end{aligned}$$

Using twice the relation 1.1 we deduce that $e'g$ yields the coefficient

$$q^2 (H_\beta(-1)(H_\alpha)^{s'})^s (((\Psi_\beta^s \Psi_\alpha E_\beta)^s)^{s'})^s (\Psi_\alpha^{s'} \Psi_\beta)^s \Psi_\alpha,$$

which can be written by

$$q^2 H_\alpha(-1) (\Psi_\beta^{s'} \Psi_\beta)^s ((\Psi_\alpha \Psi_\alpha^s)^{s'})^s ((E_\beta^s)^{s'})^s \Psi_\alpha. \tag{7.37}$$

(Here, we use $(H_\beta(-1)(H_\alpha(-1))^{s'})^s = (H_\beta(-1))^2 (H_\alpha(-1))^s = H_\alpha(-1)^s = H_\alpha(-1)$).

Therefore, using the observations from 7.31 to 7.37, we conclude that

$$\begin{aligned} Y_\alpha &= -q(q-1)^2 + q(q-1) H_\alpha(-1) \Psi_\alpha \Psi_\alpha \Psi_\alpha^s E_\beta E_\beta^s \\ &+ q^2 H_\beta(-1)(H_\alpha(-1))^{s'} (\Psi_\beta^{s'} \Psi_\beta)^s (\Psi_\alpha \Psi_\alpha^s)^{s'} ((E_\beta^s)^{s'})^s \Psi_\alpha \\ &= X_\alpha \quad \text{(from 7.30)}. \end{aligned}$$

Computation of $X_{\alpha\beta}$ and $Y_{\alpha\beta}$. The products on the left of equation 7.1 involving the monomial $R_\alpha R_\beta$ are: $(a+b)f'$, cc', cf', $e(a'+b')$, ec', eg', dg', gd', and gf'.

We now compute the coefficient of $R_\alpha R_\beta$ in the expression of af'.

Since $E_\alpha E_\beta$ is a factor of a, using 6.7, and the fact that $E_\beta^2 = (q-1)E_\beta$, it is easy to see that the coefficient of $R_\alpha R_\beta$ in the expression of af' is

$$(-1)^2 (q-1)^2 E_\alpha E_\beta = (q-1)^2 E_\alpha E_\beta. \tag{7.38}$$

We now compute bf'. We have

$$\begin{aligned} bf' &= qH_\alpha(-1) \Psi_\alpha \Psi_\alpha^s E_\beta^s R_\alpha R_\beta (E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta \\ &= R_\alpha R_\beta ((qH_\alpha(-1)^s)^{s'} (\Psi_\alpha \Psi_\alpha^s)^{s'} (E_\beta E_\beta^s)^{s'} \Psi_\alpha^{s'} \Psi_\beta \quad \text{(from 1.3)}. \end{aligned}$$

Therefore, using the fact that $(H_\alpha(-1)^s)^{s'} E_\beta^{s'} = -H_\alpha(-1) E_\beta$ and the fact that $\Psi_\beta E_\beta^{s'} = -E_\beta$, it is easy to see that the coefficient of $R_\alpha R_\beta$ in the expression of bf' is

$$-q(H_\alpha(-1))(\Psi_\alpha \Psi_\alpha^s)^{s'} E_\beta (E_\beta^s)^{s'} \Psi_\alpha^{s'}. \tag{7.39}$$

We now compute cc'.

Since $E_\alpha E_\beta$ is a factor of c, using 6.7 and the facts that $E_\alpha^2 = (q-1)E_\alpha$ and $E_\beta^2 = (q-1)E_\beta$, it is easy to see that the coefficient of $R_\alpha R_\beta$ in the expression of cc' is

$$(-1)^2 (q-1)^2 E_\alpha E_\beta = (q-1)^2 E_\alpha E_\beta. \tag{7.40}$$

We now compute cf'.

We have

$$\begin{aligned} cf' &= -R_\alpha E_\alpha E_\beta R_\alpha R_\beta (E_\alpha^s \Psi_\alpha)^{s'} \Psi_\beta \\ &= -R_\alpha^2 R_\beta E_\alpha E_\beta (E_\beta^s \Psi_\alpha)^{s'} \quad \text{(from 1.3)} \\ &= (qH_\alpha(-1) + R_\alpha E_\alpha) R_\beta E_\alpha E_\beta (E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta. \quad \text{(from 1.1)} \end{aligned}$$

But, we are interested only in the coefficient of $R_\alpha R_\beta$, which is equal to

$$\begin{aligned} -E_\alpha^{s'} E_\alpha E_\beta (E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta &= (-1)^3 E_\alpha E_\beta (E_\alpha E_\beta^s)^{s'} \quad \text{(from 6.7)} \\ &= ((-1)^3 (E_\alpha^2 E_\beta E_\beta^s)^{s'} \quad \text{(from 6.7)} \\ &= -((q-1)(E_\alpha E_\beta E_\beta^s)^{s'} \quad (\text{since } E_\alpha^2 = (q-1)E_\alpha) \\ &= -(q-1)((E_\alpha E_\beta^2)^s)^{s'} \quad \text{(from 6.7)} \\ &= -(q-1)^2((E_\alpha E_\beta)^s)^{s'} \quad (\text{since } E_\beta^2 = (q-1)E_\beta) \\ &= -(q-1)^2 E_\alpha E_\beta \quad \text{(from 6.7)}. \end{aligned}$$

Therefore, the coefficient of $R_\alpha R_\beta$ in the expression of cf' is

(7.41) $$-(q-1)^2 E_\alpha E_\beta.$$

We now compute ec'.
We have

$$\begin{aligned} ec' &= -R_\alpha R_\beta^2 (\Psi_\alpha^{s'} \Psi_\beta E_\alpha)^{s'} E_\alpha E_\beta \\ &= R_\alpha (qH_\beta(-1) + R_\beta E_\beta)(\Psi_\alpha^{s'} Psi_\beta E_\alpha)^{s'} E_\alpha E_\beta \quad \text{(from 1 .1)} \end{aligned}$$

But, we are interested only in the coefficient of $R_\alpha R_\beta$, which is equal to

$$\begin{aligned} -(\Psi_\alpha \Psi_\beta) E_\alpha^{s'} E_\alpha E_\beta^2 &= (-1)^3 E_\alpha E_\beta^2 E_\alpha^{s'} \quad \text{(from 6.7)} \\ &= -(q-1) E_\alpha E_\beta E_\alpha^{s'} \quad (\text{since } E_\beta^2 = (q-1)E_\beta) \\ &= -(q-1)(E_\alpha^2 E_\beta)^{s'} \quad \text{(from 6.7)} \\ &= -(q-1)^2 (E_\alpha E_\beta)^{s'} \quad (\text{since } E_\alpha^2 = (q-1)E_\alpha) \\ &= -(q-1)^2 E_\alpha E_\beta > \quad \text{(from 6.7)}. \end{aligned}$$

Therefore, the coefficient of $R_\alpha R_\beta$ in the expression of ec' is

(7.42) $$-(q-1)^2 E_\alpha E_\beta.$$

We now compute eg'.
We have

$$\begin{aligned} eg' &= R_\alpha R_\beta (\Psi_\alpha^{s'} \Psi_\beta E_\alpha R_\beta R_\alpha R_\beta (\Psi_\beta^s \Psi_\alpha)^{s'} \Psi_\beta \\ &= R_\alpha R_\beta^2 R_\alpha R_\beta (((\Psi_\alpha^{s'} \Psi_\beta E_\alpha)^{s'})^s)^{s'} \Psi_\beta^s \Psi_\alpha^{s'} \Psi_\beta \\ &= R_\alpha^2 R_\beta ((qH_\beta(-1) + R_\beta E_\beta)^s)^{s'} (((\Psi_\alpha^{s'} \Psi_\beta E_\alpha)^{s'})^s)^{s'} \Psi_\beta^s \Psi_\alpha^{s'} \Psi_\beta \quad \text{(from 1.1 and 1.3)}. \end{aligned}$$

Using $R_\alpha^2 = qH_\alpha(-1) + R_\alpha E_\alpha$, it is easy to see that the coefficient of $R_\alpha R_\beta$ in the expression of eg' is

$$E_\alpha^{s'} ((qH_\beta(-1))^s)^{s'} (((\Psi_\alpha^{s'} \Psi_\beta E_\alpha)^{s'})^s)^{s'} \Psi_\beta^s \Psi_\alpha^{s'} \Psi_\beta.$$

Here, we first consider the term $E_\alpha^{s'}((E_\alpha^{s'})^s)^{s'}$. This can be written as

$$\begin{aligned} (E_\alpha((E_\alpha)^{s'})^s)^{s'} &= ((E_\alpha E_\alpha^{s'})^s)^{s'} \quad (\text{since } E_\alpha^s = E_\alpha) \\ &= ((E_\alpha E_\beta)^s)^{s'} \\ &= E_\alpha E_\beta \end{aligned}$$

Now, from the above two observations, using 6.7, it is easy to see that the coefficient of $R_\alpha R_\beta$ in the expression of eg' is

$$(-1)^5 q E_\alpha E_\beta = -q E_\alpha E_\beta. \tag{7.43}$$

We now compute ea'.
We have

$$\begin{aligned} ea' &= R_\alpha R_\beta((q-1)\Psi_\alpha^{s'}\Psi_\beta E_\alpha^2 E_\beta \\ &= R_\alpha R_\beta (q-1)(-1)^2 E_\alpha E_\beta \quad \text{(from 6.7)} \\ &= (q-1)^2 E_\alpha E_\beta \quad (\text{since}\, E_\alpha^2 = (q-1)E_\alpha) \end{aligned}$$

Therefore, the coefficient of $R_\alpha R_\beta$ in the expression of ea' is

$$(q-1)^2 E_\alpha E_\beta. \tag{7.44}$$

We now compute the coefficient of $R_\alpha R_\beta$ in the expression of eb'. Using the fact that $E_\alpha E_\alpha^{s'} = E_\alpha E_\beta$, we have

$$\begin{aligned} eb' &= R_\alpha R_\beta q H_\beta(-1)\Psi_\beta^2(\Psi_\alpha\Psi_\beta)^{s'} E_\alpha E_\beta \\ &= R_\alpha R_\beta q H_\beta(-1)(-1)^4 E_\alpha E_\beta \quad \text{(from 6.7)} \\ &= R_\alpha R_\beta q E_\alpha E_\beta \quad (\text{since}\, H_\beta(-1)E_\beta = E_\beta) \end{aligned}$$

Therefore, the coefficient of $R_\alpha R_\beta$ in the expression of eb' is

$$q E_\alpha E_\beta. \tag{7.45}$$

We now compute dg'.
We have

$$\begin{aligned} dg' &= R_\beta E_\alpha E_\alpha^{s'}\Psi_\beta R_\beta R_\alpha R_\beta (\Psi_\beta^s \Psi_\alpha)^{s'}\Psi_\beta \\ &= R_\beta^2 R_\alpha R_\beta(((E_\alpha E_\alpha^{s'}\Psi_\beta)^{s'})^s)^{s'}(\Psi_\beta^s\Psi_\alpha)^{s'}\Psi_\beta \\ &= R_\alpha R_\beta((qH_\beta(-1) + R_\beta E_\beta)^s)^{s'}(((E_\alpha E_\beta \Psi_\beta)^{s'})^s)^{s'}(\Psi_\beta^s\Psi_\alpha)^{s'}\Psi_\beta \quad \text{(from 1.1, 1.3)} \\ &= R_\alpha R_\beta((qH_\beta(-1) + R_\beta E_\beta)^s)^{s'}(-1)^4 E_\alpha E_\beta \quad \text{(from 6.7)} \end{aligned}$$

Therefore, the coefficient of $R_\alpha R_\beta$ in the expression of dg' is

$$q E_\alpha E_\beta. \tag{7.46}$$

Here, we use the fact that $H_\beta(-1)E_\beta = E_\beta$.
We now compute $R_\alpha R_\beta$ in the expression of gd'.
We have

$$\begin{aligned} gd' &= R_\alpha R_\beta R_\alpha(\Psi_\alpha^{s'}\Psi_\beta)^s \Psi_\alpha R_\alpha E_\beta E_\beta^s \Psi_\alpha \\ &= R_\alpha R_\beta R_\alpha^2 (\Psi_\alpha^{s'}\Psi_\beta)^{s^2}\Psi_\alpha^s\Psi_\alpha E_\beta E_\beta^s. \end{aligned}$$

Using the quadratic relation $R_\alpha^2 = qH_\alpha(-1) + R_\alpha E_\alpha$ and the fact that $s^2 = 1$, it is easy to see that the coefficient of $R_\alpha R_\beta$ in the expression of gd' is

$$qH_\alpha(-1)(\Psi_\alpha\Psi_\alpha^s\Psi_\alpha^{s'}\Psi_\beta)E_\beta E_\beta^s. \tag{7.47}$$

We now compute the coefficient of $R_\alpha R_\beta$ in the expression of gf'.
We have

$$gf' R_\alpha R_\beta R_\alpha^2 R_\beta \Psi_\alpha \Psi_\beta^{s'}(\Psi_\alpha\Psi_\alpha^s)^{s'}\Psi_\beta E_\beta^s.$$

Using $R_\alpha^2 = qH_\alpha(-1) + R_\alpha E_\alpha$ and $R_\beta^2 = qH_\beta(-1) + R_\beta E_\beta$ and the fact that $H_\alpha(-1)^{s'}\Psi_\beta E_\beta = -H_\alpha(-1)E_\beta$, it is easy to see that the coefficient of $R_\alpha R_\beta$ in the expression of gf' is

$$-qH_\alpha(-1)\Psi_\alpha(\Psi_\alpha\Psi_\alpha^s)^{s'}\Psi_\beta^{s'}E_\beta E_\beta^s. \tag{7.48}$$

Using the observations 7.39, 7.48, and the observation 6.9 (of Lemma 6), it is easy to see that the sum of coefficients yielded by bf' and gf' is equal to

$$-qH_\alpha(-1)(\Psi_\alpha\Psi_\alpha^s)^{s'}E_\beta^s(\Psi_\alpha^{s'}E_\beta+\Psi_\alpha\Psi_\beta^{s'}E_\beta)=0.$$

Therefore, summing up all the other coefficients (using the observations from 7.38 to 7.48), we have

$$X_{\alpha\beta}=(q^2-q+1)E_\alpha E_\beta+qH_\alpha(-1)\Psi_\alpha\Psi_\alpha^s\Psi_\alpha^{s'}\Psi_\beta E_\beta E_\beta^s. \tag{7.49}$$

We now compute the coefficient of $R_\alpha R_\beta$ in the expression of $p_{\beta\alpha}p_{\alpha\beta}$.

In the product $p_{\beta\alpha}p_{\alpha\beta}$, the terms involving $R_\alpha R_\beta$ are $a'e$, $b'e$, $d'd$, $d'e$, $f'a$, $f'b$ and $f'd$.

We now compute $a'e$.

Since $E_\alpha E_\beta$ is a factor of a', using 6.7, and the fact that $E_\alpha^2=(q-1)E_\alpha$, it is easy to see that the coefficient of $R_\alpha R_\beta$ in the expression of $a'e$ is

$$(-1)^2(q-1)^2E_\alpha E_\beta. \tag{7.50}$$

We now compute $b'e$.

We have

$$\begin{aligned}
b'e &= qH_\beta(-1)\Psi_\beta\Psi_\beta^{s'}E_\alpha^{s'}R_\alpha R_\beta\Psi_\alpha^{s'}\Psi_\beta E_\alpha\\
&= R_\alpha R_\beta q((H_\beta(-1)\Psi_\beta\Psi_\beta^{s'})^s)^{s'}((E_\alpha^{s'})^s)^{s'}\Psi_\alpha^{s'}\Psi_\beta E_\alpha \quad \text{(from 1.3)}\\
&= R_\alpha R_\beta q((H_\beta(-1)\Psi_\beta\Psi_\beta^{s'})^s)^{s'}(E_\alpha E_\beta)\Psi_\alpha^{s'}\Psi_\beta \quad \text{(from 6.1 and 6.3)}\\
&= R_\alpha R_\beta(-1)^4q((E_\alpha E_\beta H_\beta(-1))^s)^{s'} \quad \text{(from 6.7)}\\
&= R_\alpha R_\beta q((E_\alpha E_\beta)^s)^{s'} \quad (\text{since } H_\beta(-1)E_\beta=E_\beta)\\
&= R_\alpha R_\beta qE_\alpha E_\beta \quad \text{(from 6.7))}
\end{aligned}$$

Here, we use the fact that

$$\begin{aligned}
E_\alpha((E_\alpha^{s'})^s)^{s'} &= E_\alpha((E_\alpha)^{s'})^s \quad \text{(from 6.1)}\\
&= (E_\alpha E_\alpha^{s'})^s \quad (\text{since } E_\alpha^s=E_\alpha)\\
&= E_\alpha E_\beta \quad \text{(from 6.3 and 6.7).}
\end{aligned}$$

Therefore, the coefficient of $R_\alpha R_\beta$ in the expression of $b'e$ is

$$qE_\alpha E_\beta. \tag{7.51}$$

We now compute $d'd$.

We have

$$\begin{aligned}
d'd &= R_\alpha E_\beta E_\beta^s\Psi_\alpha R_\beta E_\alpha E_\alpha^{s'}\Psi_\beta\\
&= R_\alpha R_\beta(E_\beta E_\beta^s\Psi_\alpha)^{s'}E_\alpha E_\alpha^{s'}\Psi_\beta\\
&= R_\alpha R_\beta(E_\beta E_\beta^s\Psi_\alpha)^{s'}E_\alpha E_\beta\Psi_\beta \quad \text{(from 6.3)}\\
&= R_\alpha R_\beta(-1)^2(q-1)^2E_\alpha E_\beta \quad (\text{from 6.7 and } E_\beta^3=(q-1)^2E_\alpha E_\beta)
\end{aligned}$$

Therefore, the coefficient of $R_\alpha R_\beta$ in the expression of $d'd$ is

$$(q-1)^2E_\alpha E_\beta. \tag{7.52}$$

We now compute the coefficient of $R_\alpha R_\beta$ in the expression of $d'e$.

We have

$$\begin{aligned}
d'e &= R_\alpha E_\beta E_\beta^s\Psi_\alpha R_\alpha R_\beta\Psi_\alpha^{s'}\Psi_\beta E_\alpha\\
&= R_\alpha^2R_\beta((E_\beta E_\beta^s\Psi_\alpha)^s)^{s'}\Psi_\alpha^{s'}\Psi_\beta E_\alpha \quad \text{(from 1.3)}\\
&= (qH_\alpha(-1)+R_\alpha E_\alpha)R_\beta((E_\beta E_\beta^s\Psi_\alpha)^s)^{s'}\Psi_\alpha^{s'}\Psi_\beta E_\alpha \quad \text{(from 1.1)}
\end{aligned}$$

But, we are interested in computing only the coefficient of $R_\alpha R_\beta$, which is equal to

$$(E_\alpha^{s'} E_\alpha)((E_\beta E_\beta^s \Psi_\alpha)^s)^{s'} \Psi_\alpha^{s'} \Psi_\beta = (-1)^3 (q-1)^2 E_\alpha E_\beta \quad \text{(from 6.3 and 6.7)}.$$

Therefore, the coefficient of $R_\alpha R_\beta$ in the expression of $d'e$ is

$$-(q-1)^2 E_\alpha E_\beta. \tag{7.53}$$

We now compute $f'a$.
We have

$$\begin{aligned} f'a &= R_\alpha R_\beta (E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta (q-1)(E_\alpha E_\beta) \\ &= R_\alpha R_\beta (-1)^2 (q-1)^2 ((E_\alpha E_\beta^2)^s)^{s'} \quad \text{(from 6.7)} \\ &= R_\alpha R_\beta (q-1)^2 E_\alpha E_\beta \quad (\text{since } E_\beta^2 = (q-1)E_\beta) \end{aligned}$$

Therefore, the coefficient of $R_\alpha R_\beta$ in the expression of $f'a$ is

$$(q-1)^2 E_\alpha E_\beta. \tag{7.54}$$

We now compute $f'b$.
A straightforward computation shows that the coefficient of $R_\alpha R_\beta$ in the expression of $f'b$ is

$$q(E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta H_\alpha(-1) \Psi_\alpha \Psi_\alpha^s E_\beta^s. \tag{7.55}$$

We now compute the coefficient of $R_\alpha R_\beta$ in the expression of $f'd$.
We have

$$\begin{aligned} f'd &= R_\alpha R_\beta (E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta R_\beta (E_\alpha E_\alpha^{s'}) \Psi_\beta \\ &= R_\alpha R_\beta^2 ((E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta)^{s'} (E_\alpha E_\alpha^{s'}) \Psi_\beta \quad \text{(from 1.3)} \\ &= R_\alpha (q H_\alpha(-1) + R_\beta E_\beta) ((E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta)^{s'} (E_\alpha E_\alpha^{s'}) \Psi_\beta. \end{aligned}$$

But, we are interested in computing only the coefficient of $R_\alpha R_\beta$, which is equal to

$$\begin{aligned} E_\beta ((E_\beta^s \Psi_\alpha)^{s'} \Psi_\beta)^{s'} (E_\alpha E_\alpha^{s'}) \Psi_\beta &= (-1)^3 ((E_\alpha E_\beta^2)^s) E_\alpha^{s'} \quad (\text{from 6.7 and } (s')^2 = 1) \\ &= -(q-1)(E_\alpha E_\beta)^s E_\alpha^{s'} \quad (\text{since } E_\beta^2 = (q-1)E_\beta) \\ &= -(q-1)(E_\alpha E_\beta E_\alpha)^{s'} \quad \text{(from 6.7)} \\ &= -(q-1)^2 (E_\alpha E_\beta)^{s'} \quad (\text{since } E_\alpha^2 = (q-1)E_\alpha) \\ &= -(q-1)^2 E_\alpha E_\beta \quad \text{(from 6.7)}. \end{aligned}$$

Therefore, the coefficient of $R_\alpha R_\beta$ in the expression of $f'd$ is

$$-(q-1)^2 E_\alpha E_\beta. \tag{7.56}$$

Summing up these coefficients (using the observations from 7.50 to 7.56), we have

$$Y_{\alpha\beta} = (q^2 - q + 1) E_\alpha E_\beta + q H_\alpha(-1) \Psi_\alpha \Psi_\alpha^s \Psi_\alpha^{s'} \Psi_\beta E_\beta^s (E_\beta^s)^{s'}. \tag{7.57}$$

We have $E_\alpha^s = E_{3\alpha+\beta}$ and $(E_\beta^s)^{s'} = E_{3\alpha+2\beta}$.
Using a similar proof 6.5, it is easy to see that

$$E_\beta (E_\beta)^s E_\beta E_{3\alpha+\beta} = E_{3\alpha+\beta} E_{3\alpha+2\beta} = E_\beta^s ((E_\beta)^s)^{s'}. \tag{7.58}$$

Using the observations, 7.49, 7.57 and 7.58, it is easy to see that $X_{\alpha\beta} = Y_{\alpha\beta}$.
We now prove that

$$X_0 = Y_0.$$

The terms yielding the constant coefficients in the expression of $p_{\alpha\beta} p_{\beta\alpha}$ are aa', ab', ba', bb', cd', dc', ee' and ff'.

The terms yielding the constant coefficients in the expression of $p_{\beta\alpha} p_{\alpha\beta}$ are $a'a$, $a'b$, $b'a$, $b'b$, $c'd$, $d'c$, $e'e$ and $f'f$.

It is easy to see that these terms yield the following coefficients.

$$\begin{aligned}
aa' &= a'a = (q-1)^4 E_\alpha E_\beta \\
ab' &= b'a = q(q-1)E_\alpha E_\beta \\
ba' &= a'b = q(q-1)E_\alpha E_\beta \\
bb' &= b'b = q^2 H_{\alpha+\beta}(-1)\Psi_\alpha \Psi_\beta (\Psi_\alpha E_\beta)^s (\Psi_\beta E_\alpha)^{s'} \\
cd' &= d'c = q(q-1)^2 E_\alpha E_\beta \\
dc' &= c'd = q(q-1)^2 E_\alpha E_\beta \\
ee' &= f'f = q^2 H_\beta(-1)^s H_\alpha(-1)\Psi_\alpha \Psi_\alpha^s (\Psi_\beta \Psi_\beta^{s'})^s (E_\alpha^{s'})^s E_\beta \\
ff' &= e'e = q^2 H_\alpha(-1)^{s'} H_\beta(-1)\Psi_\beta \Psi_\beta^{s'} (\Psi_\alpha \Psi_\alpha^s)^{s'} (E_\beta^s)^{s'} E_\alpha.
\end{aligned}$$

Hence, we have

$$X_0 = Y_0.$$

Since the computations of X_β, $X_{\beta\alpha}$, $X_{\beta\alpha\beta}$, $X_{\beta\alpha\beta\alpha}$, $X_{\beta\alpha\beta\alpha\beta}$ are similar to the computations of X_α, $X_{\alpha\beta}$, $X_{\alpha\beta\alpha}$, $X_{\alpha\beta\alpha\beta}$, $X_{\alpha\beta\alpha\beta\alpha}$ respectively and the same is true for the Y's, we will only quote the coefficients.

We first quote the terms involving R_β in the expression of $p_{\alpha\beta}p_{\beta\alpha}$ and the terms involving R_β in the expression of $p_{\beta\alpha}p_{\alpha\beta}$.

The terms involving R_β in the expression of $p_{\alpha\beta}p_{\beta\alpha}$ are ac', bc', cf', da', db', dc', eg' fd' and ff'.

On the other hand, the terms involving R_β in the expression of $p_{\beta\alpha}p_{\alpha\beta}$ are $a'd$, $b'd$, $c'a$, $c'b$, $c'd$, $d'e$, $e'c$, $e'e$ and $g'f$.

The coefficients yielded by these are as follows:

$$\begin{aligned}
ac' &= c'a = -(q-1)^2 E_\alpha E_\beta \\
bc' &= c'b = -(q(q-1))E_\alpha E_\beta \\
cf' &= ec' = -(q-1)E_\alpha E_\beta \\
da' &= a'd = -(q-1)^2 E_\alpha E_\beta \\
db' &= b'd = -(q(q-1))E_\alpha E_\beta \\
dc' &= c'd = (q-1)^2 E_\alpha E_\beta \\
eg' &= g'f = q^2 H_\beta(-1)(\Psi_\alpha \Psi_\alpha^s)^{s'} (\Psi_\beta \Psi_\beta^{s'})^s)^{s'} \Psi_\beta (E_\alpha^{s'})^s \\
fd' &= d'e = -(q-1)E_\alpha E_\beta \\
ff' &= e'e = q(q-1)E_\alpha E_\beta.
\end{aligned}$$

We note that here, we write the only coefficients (not with the monomials).

Therefore, we have

$$X_\beta = Y_\beta.$$

We now do the same for $R_\beta R_\alpha$.

The terms involving $R_\beta R_\alpha$ in the expression of $p_{\alpha\beta}p_{\beta\alpha}$ are $(a+b)e'$, dd', de', $f(a'+b')$ and fd'.

On the other hand, the terms involving $R_\beta R_\alpha$ in the expression of $p_{\beta\alpha}p_{\alpha\beta}$ are $a'f$, $b'f$, $c'c$, $c'f$, $d'g$, $e'a$, $e'b$, $e'c$, $e'g$, $g'd$ and $g'f$.

The coefficient yielded by these are as follows:

$$\begin{aligned}
ae' &= e'a = (q-1)^2 E_\alpha E_\beta \\
be' &= e'b = q(E_\beta E_\beta^s - (q-1)E_\alpha E_\beta) \\
dd' &= c'c = (q-1)^2 E_\alpha E_\beta \\
de' &= c'f = -(q(q-1))E_\alpha E_\beta \\
fa' &= a'f = (q-1)^2 E_\alpha E_\beta \\
fb' &= b'f = qE_\alpha E_\beta \\
fd' &= e'c = -(q-1)^2 E_\alpha E_\beta \\
d'g &= -(e'g) = q(H_\alpha(-1)^{s'})^s((\Psi_\alpha^s)^{s'})^s((\Psi_\alpha^{s'}\Psi_\beta)^s)\Psi_\alpha E_\beta^s (E_\beta^s)^{s'} \\
g'd &= -(g'f) = E_\alpha E_\beta.
\end{aligned}$$

(We note that we write only the coefficients)

Hence, we have

$$X_{\beta\alpha} = Y_{\beta\alpha}.$$

We now do the same for $R_{\beta\alpha\beta}$.

The terms involving $R_\beta R_\alpha R_\beta$ in the expression of ag', bg', df', dg', fc' and ff'.

On the other hand, terms involving $R_\beta R_\alpha R_\beta$ in the expression of $p_{\beta\alpha}p_{\alpha\beta}$ are $c'e$, $e'd$, $e'e$, $g'a$, $g'b$, $g'd$.

The coefficients yielded by these are as follows:

$$\begin{aligned}
ag' &= g'a = -(q-1)E_\alpha E_\beta \\
bg' &= g'b = qH_\alpha(-1)(\Psi_\beta^s \Psi_\alpha)^{s'}\Psi_\beta \Psi_\alpha \Psi_\alpha^s E_\beta^s \\
dg' &= g'd = (q-1)E_\alpha E_\beta \\
df' &= e'd = -(q-1)E_\alpha E_\beta \\
fc' &= c'e = -(q-1)E_\alpha E_\beta \\
ff' &= e'e = (q-1)E_\alpha E_\beta.
\end{aligned}$$

Hence, we have

$$X_{\beta\alpha\beta} = Y_{\beta\alpha\beta}.$$

We now do the same for $R_\beta R_\alpha R_\beta R_\alpha$.

The terms involving $R_\beta R_\alpha R_\beta R_\alpha$ in the expression of $p_{\alpha\beta}p_{\beta\alpha}$ is fe' only.

On the otherhand, the terms involving $R_\beta R_\alpha R_\beta R_\alpha$ are $c'g$, $e'f$, $e'g$, $g'c$ and $g'f$.

The coefficients yielded by these are as follows:

$$\begin{aligned}
fe' &= e'f = E_\alpha E_\beta, \\
c'g &= -e'g = E_\alpha E_\beta, \\
g'c &= -g'f = E_\alpha E_\beta.
\end{aligned}$$

Hence, we have

$$X_{\beta\alpha\beta\alpha} = Y_{\beta\alpha\beta\alpha}.$$

We now do the same for $R_\beta R_\alpha R_\beta R_\alpha R_\beta$.

The only term involving $R_\beta R_\alpha R_\beta R_\alpha R_\beta$ in the expression of $p_{\alpha\beta}$ is fg'.

On the otherhand, the only term involving $R_\beta R_\alpha R_\beta R_\alpha R_\beta$ is $g'e$.

The coefficients yielded by these are

$$fg' = g'e = \Psi_\beta \Psi_\beta^s((\Psi_\beta)^s)^{s'}(\Psi_\alpha^{s'})^s \Psi_\alpha^{s'} E_\alpha.$$

□

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