TOPOLOGICAL FIELD THEORY OF THE INITIAL SINGULARITY OF SPACE-TIME

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Abstract

We suggest here a new solution of the initial space-time singularity. In this approach the initial singularity of space-time corresponds to a zero size singular gravitational instanton characterized by a Riemannian metric configuration (++++) in dimension D = 4. Connected with some unexpected topological datas corresponding to the zero scale of space-time, the initial singularity is thus not considered in terms of divergences of physical fields but can be resolved in the frame of topological field theory. Then it is suggested that the "zero scale singularity" can be understood in terms of topological invariants (in particular the first Donaldson invariant $\sum_i (-1)^{n_i}$). In this perspective we here introduce a new topological index, connected with 0 scale, of the form $Z_{ß=0}$ = Tr $(-1)^S$, which we call "singularity invariant". Interestingly, this invariant corresponds also to the invariant topological current yield by the hyperfinite II_∞ von Neumann algebra describing the zero scale of space-time. Then we suggest that the (pre)space-time is in thermodynamical equilibrium at the Planck scale state and is threrefore subject to the KMS condition. This might correspond to a unification phase between "physical state" (Planck scale) and "topological state" (zero scale). Then we conjecture that the transition from the topological phase of the space-time (around the scale zero) to the physical phase observed beyond the Planck scale should be deeply connected to the supersymmetry breaking of the N=2 supergravity.

0. INTRODUCTION

One of the limits of the standard space-time model remains its inability to provide a description of the singular origin of space-time. Here we suggest, in the context of N=2 supergravity, that the initial singularity, associated with zero scale of space-time, cannot be described by (perturbative) physical theory but might be resolved by a (non-perturbative) dual theory of topological type. Such an approach is based on our recent results [6-7] concerning the quantum fluctuations (or q-superposition) of the signature of the metric at the Planck scale. We have suggested that the signature of the space-time metric (+++–) is not anymore frozen at the Planck scale ℓ_p and presents quantum fluctuations (+++±) until zero scale where it becomes Euclidean (++++). Such a suggestion appears as a natural consequence of the non-commutativity of the space-time geometry at the Planck scale [11]. In this non-commutative setting, we have constructed (cf. 4.1) the "cocyle bicrossproduct" [6] :

$$U_q(so(4)^{op} \overset{\psi}{\triangleright\!\blacktriangleleft} U_q(so(3, 1)) \qquad (1)$$

where $U_q(so(4)^{op}$ and $U_q(so(3,1))$ are Hopf algebras (or "quantum groups"[16]), the symbol $\triangleright\!\blacktriangleleft$ a (bi)crossproduct and ψ a 2-cocycle of deformation (for more specific definitions, see ref[29]). The bicrossproduct (1) suggests an unexpected kind of "unification" between the Lorentzian and the Euclidean Hopf algebras at the Planck scale and yields the possibility of a "q-deformation" of the signature from the Lorentzian (physical) mode to the Euclidean (topological) mode [6-30]. Moreover equ.(1) defines implicitly a (semi)duality transformation between Lorentzian and Euclidean quantum groups (see equ.(42)). This is important insofar we consider that the Euclidean theory is the simplest topological field theory.

In other respect, it has been stated in string theory [25] that the behavior of string amplitudes at very high temperature (Hagedorn limit) reveals the existence of a possible phase transition and the restoration of large-scale symmetries of the system. In the context of this "unbroken phase", generally expected at the Planck scale, the theory is characterized by a general covariance preserving the exact symmetry of the system. The metric $g_{\mu\nu}$ is developed around zero and there exists at this level neither light cone, wave propagation, nor movement. The exploration of this unbroken (and non-physical) phase of the system is accessible only in the framework of a new kind of field theory proposed by E. Witten under the name "topological field theory" [37].

Topological field theory is usually defined as the quantization of zero, the Lagrangian of the theory being either (**i**) a zero mode or (**ii**) a characteristic class $c_n(V)$ of a vectorial bundle $V \xrightarrow{\pi} M$ built on space-time [31]. Starting from the Bianchi identity Tr$R \wedge R^* = \frac{1}{30}$ Tr$F \wedge F^*$, our approach of 4D supergravity leads us to describe the energy content of the pre-space-time system by the curvature $\boldsymbol{R}$. We therefore put $\mathcal{L} \sim R \wedge R^*$. The value of the topological action $S_{class} = \int_M \mathcal{L}_{class} = \int_M c_n(V) = k \in \mathbb{Z}$ is then either zero or corresponds to an integer. The topological limit of quantum field theory, described in particular by the Witten invariant Z = Tr $(-1)^n$ [36] is then given by the usual quantum statistical partition function taken over the (3+1) Minkowskian space-time

$$\mathbf{Z} = \mathrm{Tr}\,(-1)^n\; e^{-ßH} \tag{2}$$

with $ß = \frac{1}{kT}$ and n being the zero energy states number of the theory, for example the fermion number in supersymmetric theories [1]. Then **Z** describes all zero energy states for null values of the Hamiltonian H.

Now, we propose here (§(1.2)) a *new topological limit* of quantum field theory, non-trivial (i.e. corresponding to the non-trivial minimum of the action). Built from scale $ß \rightarrow 0$ and *independent* of H, this unexpected topological limit (in 4D dimensions) is then given by the temperature limit (Hagedorn temperature) of the physical system (3+1)D. In a way this can be derived from the "holographic conjecture" [42] following which the states of quantum gravity in d dimensions have a natural description in terms of a $(d-1)$-dimensional theory. In agreement with [4-34-39] and, in particular, the recent results of C. Kounnas *and al* [3-27], we argue in §(5.1.1) that on the hereabove limit (i.e. at the Planck scale), the "space-time system" is in a *thermodynamical equilibrium state* [34] and, therefore, is subject to the Kubo-Martin-Schwinger (KMS) condition [24]. A similar point of view has also been successfully developed in the context of thermal supersymmetry by Derendinger and Lucchesi in [13-28]. Surprisingly, the KMS and modular theories [11] might have dramatic consequences onto Planck scale physics. Indeed, when applied to quantum space-time, the KMS properties are such that the time-like direction of the system, within the limits of the "KMS strip" (i.e. between the zero scale and the Planck scale) should be considered as *complex* : $t \mapsto \tau = t_r + it_i$. In this case, on the $ß \rightarrow 0$ limit, the theory is projected onto the pure imaginary boundary $t \mapsto \tau = it_i$ of the KMS strip. Then the partition function (2) gives the pure topological state connected with the zero mode of the scale :

$$Z_{ß \to 0} = \mathrm{Tr}\,(-1)^{\mathrm{s}} \qquad (3)$$

where **s** represents the instantonic number. This new "singularity invariant" [6-7]), isomorphic to the Witten index Z = Tr $(-1)^F$, can be connected with the initial singularity of space-time, reached for $ß$ = 0 in the partition function **Z** = Tr $(-1)^{\mathrm{s}}\, e^{-ßH}$. According to sec. 3, when $ß \rightarrow 0$, the partition function **Z** gives the first Donaldson invariant

$$\mathrm{I} = \sum_i (-1)^{n_i} \qquad (4)$$

a non-polynomial topological invariant, reduced to an integer for $\dim \mathcal{M}_{\mathrm{mod}}^{(k)} = 0$ ($\dim \mathcal{M}_{\mathrm{mod}}^{(k)}$ being the dimension of the instanton moduli space). This suggests that the (topological) origin of space-time might be successfully represented by a singular zero size gravitational instanton [41]. A good image of this euclidean point-like object is the "transitive point", whose orbits under the action of $\mathbb{R}$ are dense everywhere from zero to infinity. Then at zero scale, the observables O_i should be replaced by the homology cycles $H_i \subset \mathcal{M}_{\mathrm{mod}}^{(k)}$ in the moduli space of gravitational instantons. We get then a deep correspondence -*a symmetry of duality*- [2-19-32], between physical theory and topological theory. More precisely, it may exist, at the Planck scale, a duality transformation (which we call "*i*-duality"[6]) between the BRST cohomology ring (physical mode) and the cohomology ring of instanton moduli space (topological mode) [19]. In the context of quantum groups [16-17], we have shown that transition from q-Euclidean to q-Lorentzian spaces [30-35] can also be viewed as a Hopf algebra duality [29]. Interestingly, the Hopf algebra duality has been recently connected to superstrings T-duality by C. Klimcik and P. Sevara [26].

The present article is organized as follows. In section 1 we define the topological field theory and suggest that there exists at the scale limit $ß \to 0$ a non-trivial topological limit of quantum field theory, *dual* to the topological limit associated with $ß \to \infty$. In section 2 we evidence that the $ß \to 0$ limit of some standard theories is topological. We give several examples of such a topological limit. In section 3, we show that the high temperature limit of quantum field theory corresponding to $ß \to 0$ should give the first Donaldson invariant. The signature of the metric of the underlying 4-dimensional manifold is therefore expected to be Euclidean (++++) at the scale zero. In section 4, we emphasize, in the quantum groups context, the existence of a symmetry of duality between the Planck scale (physical sector of the theory) and zero scale (topological sector). In section 5, we discuss in the framework of KMS state and von Neumann C^*- algebras a way to understand the transition from the topological (ultraviolet) phase of space-time to the standard physical (infrared) phase.

1. TOPOLOGICAL THEORY AT SCALE 0

1.1 Preliminaries

The field theory considered here is thermal supersymmetric [13-28] and in the context of D4 manifolds [40]. We have detailed the content of the (thermal) supermultiplet in a previous work [6]. The theory belongs to the class of N=2 supergravities [19], the Hamiltonian being given by the squared Dirac operator $\not{D}^2$ [11-31]. As such, the simplest bosonic multiplet reduces to a vector field plus two scalars exhibiting a special Kähler geometry. Rightly, N=2 is here of a particular interest, for two main reasons :

(**i**) the complex scalar fields of the theory (for example the dilaton S-field [32] or the T-field [2]) can be seen as "signatures" of the KMS condition [11-25] to which the space-time might be subject at the Planck scale. They might also be one of the best keys to understand the possible *duality* between physical observables (infrared) and topological states (ultraviolet) :

$$\textit{Topological vacuum } (ß = 0, \ \textit{instanton}) \xleftrightarrow{i-dualité} \textit{Physical vacuum } (ß = \ell_{Planck}, \ \textit{monopole})$$

This is based on the instantons / monopoles duality initially suggested by us in [6] and recently proved in the superstrings context by C.P. Bacchas, P. Bain and M.B. Green [5]. Moreover, in string theory again, has been conjectured a U=S⊗T-symmetry [25] from which we can infer the hereabove duality between (physical) observables and (topological) cycles on a four-manifold M :

$$\langle 0_1 0_2 \ \ldots \ 0_n \rangle \xleftrightarrow{U-duality} \chi(\gamma_1, \gamma_2, \ldots, \gamma_n)$$

Then the main contribution of the present article would be to emphasize that, as for conifolds cycles, a zero topological cycle might control the blow up of the space-time Initial Singularity.

(**ii**) From another point of view, the **S/T** fields are closely related to the existence, in the Lagrangian, of *non-linear terms.* As recalled by A. Gregori, C. Kounnas and P. M. Petropoulos [23], in the frame of N=2 supergravity, the theory is generally inducing some non perturbative corrections and a BPS-saturated coupling with higher derivative terms $R^2 + \ldots$ As our model is proposed in 4D dimensions, the development of higher derivative terms can be limited in a natural way to the R^2 term. Then the Lagrangian usually considered in supergravity is :

$$L = \int d^4x \ \sqrt{g} \left\{ l^2 (\alpha R_{\mu\nu} R^{\mu\nu} + \beta R^2) + R + \kappa L_M \right\} \tag{5}$$

from which we pull the simplified Lagrangian density that we use here :

$$\mathbf{L}_{supergravité} = \hat{\beta} R + \frac{1}{g^2} R^2 + \alpha R R^* \tag{6}$$

This type of Lagrangian density is coupling the physical component (the Einstein term $\hat{\beta} R$) with the topological term RR^*. This is of crucial interest since, as observed in ex.(2.1), when $ß \to 0$, we are only left with the topological term RR^*(decoupled, on this limit, of the axion field α).

Now, let's begin with a brief reminder of topological field theory as originally introduced by E. Witten in 1988 [37] :

Definition 1.1 *Topological field theory is defined by a cohomological field such that a correlation function of n physical observables* $\langle O_1 O_2 \ldots\ldots O_n \rangle$ *can be interpreted as the number of intersections*

$$\langle O_1 O_2 \ldots O_n \rangle = \#(H_1 \cap H_2 \cap \ldots \cap H_n)$$

of n cycles of homology $H_i \subset \mathcal{M}^{(k)}_{\text{mod}}$, *in moduli space* $\mathcal{M}^{(k)}_{\text{mod}}$ *of configurations of the instanton type* $[\phi(x)]$, *on the fields* ϕ *of the theory.*

The content of "cohomological fields" (for which the general covariance is exact) is given by the field variations (which induce a Fadeev-Popov ghost contribution and gauge fixing part). The point, however, is that the total gauge fixed action is a BRST commutator and the energy-momentum tensor is BRST invariant [19-37]. In other words, the correlation functions of cohomological fields are independent of the metric. Now, the topological field theory (for D = 4) is established when the Hamiltonian (or the Lagrangian) of the system is H=0, such as the theory is independent of the underlying metric. We propose to extend this definition, stating that a theory can also be topological *if it does not depend on the Hamiltonian* H (or the Lagrangian ***L***) of the system.

Definition 1.2 *A theory is topological if (the Lagrangian* ***L*** *being non-trivial) it does not depend on* ***L****.*

Def (1.2) means that ***L*** is a topological invariant of the form $\boldsymbol{L} = R \wedge R^*$. Based on this definition, we suggest that there exists a *second* topological limit of the theory, dual to that given by H = 0. In this case, we can have H $\neq$ 0, but the theory is taken at the limit of *scale zero* associated with ß $\rightarrow$ 0. Then the minimum of the action is not zero (as it is in the trivial case) but has a non-trivial (invariant) value.

We consider the possible existence of such a "topological field " at the high temperature limit of the system.

1.2 A new topological limit

Proposition 1.3 *There exists at the scale ß* $\rightarrow$ *0 a non-trivial topological limit of the theory, dual to the topological limit corresponding to ß* $\rightarrow \infty$.

Proof The (thermo)dynamical content of the quantum field theory can be described by the partition function :

$$Z = \mathrm{Tr}\,(-1)^{n}\, e^{-ßH} \tag{7}$$

where n is the "metric number" of the theory. When $ß \to 0$, the theory is no longer dependent on H. On this limit, such that the temperature $T \to T_{Hag}$(Hagedorn limit), equ.(7) becomes $Z_0 = \mathrm{Tr}(-1)^{n}$, H vanishing from the metric states partition function. $ß$ plays the role of a coupling constant, such that it exists an infinite number of states not interacting with each other and independent of H. The point is that for $ß = 0$, the action S is projected onto a non trivial minimum, corresponding to the self-duality condition $R = \pm\, R^{*}$. But in this case, the field configuration is necessarily *Euclidean* and defines a gravitational instanton, i.e. a topological configuration [6]. We are therefore confronted to a 4D pure *topological* theory , as described by the first Donaldson invariant [14] :

$$\mathrm{I} = \sum_{i} (-1)^{n_i}$$

n_i being the instanton number. The limit $ß = 0$ is here dual (in a sense precised in §(4)) to the usual topological limit $ß \to \infty$ given by H = 0. The density operator of the (pre)space-time system is written as :

$$\rho = e^{-\beta H + \lambda_0}$$

λ_0 being (classically) a factor of re-normalization of the system. When $ß = 0$, the density operator is thus reduced to $\rho = e^{\lambda_0}$, which is independent of $H \neq 0$, characteristic of a second topological limit of the theory.

Now we propose to show, through some very simple examples, that interesting contacts with topological field theory can be made in taking the $ß \to 0$ limit of some established standard results. To be as demonstrative as possible, we shall most often proceed in a heuristic way.

2. THE ß → 0 LIMIT OF SOME STANDARD THEORIES

To warm up, we first consider the ß → 0 topological limit of the standard (quantum) thermal field theory.

Example 2.1 *The topological 0-scale limit of the heat kernel*

(i) One famous mathematical proof of the Atiyah-Singer theorem (given, for example, by E.Getzler [20]) lies in the heat equation [21-22]. Considering the heat operator $e^{-ßH}$ acting on the differential forms on a closed, oriented manifold X, the ß → 0 limit of this operator corresponds to the local curvature invariants of the manifold [31].

Let's consider a (quantum) thermal field theory on a system defined by the first order elliptic differential operator P and it's adjoint P*. We put the laplacian Δ = PP* et Δ' = P*P. For any ß > 0, we can evaluate the partition function $K = \mathrm{Tr}\left(e^{-ß\Delta}\right)$ giving the states of the metric of the system. Now, to get the asymptotic ß → 0 limit, we take the symbol of $\mathrm{Tr}\left(e^{-ß\Delta}\right)$ (which can be expressed in terms of σ(Δ) and its derivatives) and we get :

$$\mathrm{Tr}\left(e^{-ß\Delta}\right) \cong \mathrm{Tr}\int_M \sigma(e^{-ß\Delta})\,dxdk \qquad (8)$$

For ß → 0, *K* degenerates on the Dirac mass and the right-hand side of (8) has an asymptotic expansion such that

$$\mathrm{Tr}\left(e^{-ß\Delta}\right) \cong \sum_0^{\infty} t^{i-n/2} B_i$$

and as a result, we get the well known ß-independent topological index (in the Atiyah-Singer sense [22]) :

$$Ind(P) = B_n[\Delta] - B_n[\Delta']$$

With this index we see in a simple way that the ß → 0 limit of thermal field theory is topological.

Another important argument lies in the fact that at ℓ_{Planck}, the (pre)space-time might enter a phase of thermodynamical equilibrium (§(5.1.1)). Consequently (§(5.1.2)) it should be subject to the KMS condition [24]. As evidenced in §(5.1.3) and in the ex.(5.2.1), this implies the holomorphicity of the time-like direction, the real time-like and the real space-like directions given by g_{44}^{c} being compactified on the two circles S^1_{t-like} and S^1_{s-like} [6]. But one can easily see that this configuration is equivalent to the dimensional reduction of the 4D Lorentzian theory onto a 3D theory. This type of reduction has been described by Seiberg and Witten [33]. We then are left with three-manifold invariants, in particular the Floer invariant of a supersymmetric non linear σ-model [18]. In this case, the three dimensional pseudo-gravity $\Gamma_{(3)}$ is coupled to the $\mathbf{S}, \mathbf{T}$ complex scalar fields :

(1) $\mathbf{S} = \frac{1}{g^2} \pm i.\alpha_i$ (axion) with $\mathbf{S}$ and $\bar{\mathbf{S}}$

(2) $\mathbf{T} = g_{44} \pm i g^{*}_{i4}$ with $\mathbf{T}$ and $\bar{\mathbf{T}}$

Those scalar fields are propagating. Then the coupling of the **S/T**-fields with the 3D pseudo-gravity is given by the extended σ-model :

$$\sum = SO(3) \times \frac{SL(2, \mathbb{R})}{SO(2)} \times \frac{SL(2, \mathbb{R})}{SO(1,1)} \tag{9}$$

As the theory is *independent* of g_{44}, the 2D field $\frac{SL(2, \mathbb{R})}{SO(1,1)}$ in the Lorentzian case and $\frac{SL(2, \mathbb{R})}{SO(2)}$ in the Euclidean case can be viewed as equivalent. Thus the corresponding "superposition state" of the signature $(+++\pm)$ is able to be described by the symmetric homogeneous space

$$\sum_h = \frac{SO(3,1) \otimes SO(4)}{SO(3)}$$

SO(3) being diagonally embedded in SO(3, 1) ⊗ SO(4). Next step, as suggested in [6], a "monopoles+instantons" configuration can be associated to this 5D metric configuration at the Planck scale. Instantons and monopoles are here connected by a S-field. The form of the 5D metric induced by the σ-model (9) and constructed in [6] is :

$$ds^2 = a(w)^2 d\Omega^2_{(3)} + \frac{dw^2}{g^2} - dt^2 \tag{10}$$

where the axion term is $a = f(w\ ,\ t)$, the 3-geometry being $d\Omega^2_{(3)} = f(x,y,z)$. Clearly the expected values of the running coupling constant (dilaton) $\varphi = \frac{1}{g^2}$ are giving the two 4D limits of the 5D metric of equ.(10). Thus we get: - ***Infrared*** : ß → ∞. In this strong coupling sector we have $\frac{dw^2}{g^2} \to 0$ and the w direction of Γ^5 is cancelled. So after dimensional reduction (D=5 → D= 4) the metric on Γ^5 becomes 4D Lorentzian :

$$ds^2 = a(w)^2 d\Omega^2_{(3)} - dt^2 \tag{11}$$

The σ-model (9) is reduced to the usual Lorentzian symmetry :

$$SO(3) \times \frac{SL(2,\ \mathbb{R})}{SO(1,1)} \xrightarrow{\ ß(g) \xrightarrow{infrared} \infty\ } SO(3,\ 1) \tag{12}$$

Likewise, when g → ∞, the R^2 term cancels in the 5D Lagrangian density $\mathbf{L}_{supergravit\acute{e}} = \hat{\beta} R + \frac{1}{g^2} R^2 + \alpha RR^*$, and, as $R = R^*$, the topological term αRR^* is also suppressed. So, we get $\mathbf{L} = \hat{\beta} R$.

Let's see now what happens on the (dual) ultraviolet limit, when ß → 0.

- ***Ultraviolet*** : ß → 0. We can construct a boundary of equ.(10), corresponding to the small coupling constant sector of the coupled theory and we get divergent values for the real dilaton field $\varphi = \frac{1}{g^2}$. Then naïvely, we can apply one of the results of [23] saying that the axion field is decoupled of the theory on this limit and we are left with the divergent dilaton field only. So, we have for the metric on Γ^5 the new *Euclidean* form :

$$ds^2 = a(w)^2 d\Omega^2_{(3)} + \frac{dw^2}{g^2} \tag{13}$$

Therefore, in the ultraviolet, the σ-model (9) is reduced to the four-dimensional target space :

$$\mathrm{SO}(3)\times\frac{\mathrm{SL}(2,\ \mathbb{R})}{\mathrm{SO}(2)}\ \xrightarrow{ß(g)\xrightarrow{ultraviolet}0}\ \mathrm{SO}(3)\times\mathrm{SO}(3)=\mathrm{SO}(4) \tag{14}$$

and on this small coupling limit, the reduced theory becomes *Euclidean*, i.e. topological.. Again, it appears reasonable to conclude that the ß → 0 limit has a pure topological content. Incidentally, this result could as well be understood in the frame of the isodimensional instanton-monopole duality proposed by us in [6] and proved in the string context by Bacchas *and al* [5]. Indeed, we have shown that the q-deformed 5D theory is dominated by the (3+1)D monopoles in the infrared ($ß \rightarrow \ell_{Planck}$) and by the 4D instantons in the ultraviolet ($ß \rightarrow 0$) [6]. In this sense, the Euclidean signature (++++) can be seen as *i*-dual to the Lorentzian one (+++−). Likewise, the topological limit $ß \rightarrow 0$ should be viewed as *i*-dual to the physical limit $ß \rightarrow \ell_{Planck}$. This might be an unexpected application of the Seiberg-Witten **S/T**-duality [32].

At present, let's explore the ultraviolet limit of another standard result, i.e. the Feynmann Path integral [39].

Example 2.2 *The topological 0-scale limit of the Feynmann (3+1) path-integral approach*

(i) It's well known that in quantized Minkowski space-time, the amplitude $(g_2,\phi_2,\sigma_2 \mid g_1,\phi_1,\sigma_1)$ is given by :

$$(g_2,\phi_2,\sigma_2 \mid g_1,\phi_1,\sigma_1) = \int \mathrm{D}[\phi]\ \exp\,[i\ \mathrm{S}(\phi\ )]$$

To include the point-like (0-modes) configurations of $g_{\mu\nu}$, we put $\mathrm{Tr}(-1)^n$ in the integral and we get :

$$(g_2,\phi_2,\sigma_2 \mid g_1,\phi_1,\sigma_1) = \int \mathrm{Tr}(-1)^n \mathrm{D}[\phi]\ \exp\,[i\ \mathrm{S}(\phi\ )] \tag{15}$$

So, the trivial {t=ß → 0, S=0} Lorentzian vacuum is distinct of the "topological vacuum" connected to the minimum of the *Euclidean* action $\mathrm{S_E} = \frac{8\pi^2}{g^2}$. But it has been shown [15-37] that the zero modes in the expansion about the minima of S are tangent to the instanton moduli space $\boldsymbol{M}_{\mathrm{k}}$, so the topological vacuum should be viewed as the "true vacuum" of the theory. Then equ.(15) becomes for ß → 0 :

$$(g_0, \phi_0, \sigma_0 \mid g_0, \phi_0, \sigma_0) = I_0 = \int \mathrm{Tr}(-1)^n \, D[\phi_0] \qquad (16)$$

To define I_0, one can assume that at zero scale, the measure $D[\phi_0]$ is concentrated on one *unique* point and becomes a pure state, i.e. a positive trace class operator with unit trace. Concerning ϕ, the field content can be given by the non linear term R^2, so that the *ß*-dependant typical form of the Lagrangian density is, as seen in [6] :

$$\mathbf{L}_{supergravité} = \hat{\beta} R + \frac{1}{g^2} R^2 + \alpha RR^* \qquad (17)$$

Now, for g= *ß* → 0, the Einstein term R is cancelled and as $R = R^*$, the only remaining term in equ.(17) is the topological invariant RR^*(itself decoupled from the axion field α). So, equ.(15) takes the new form :

$$(g_0, \phi_0, \sigma_0 \mid g_0, \phi_0, \sigma_0) \rightarrow \int \mathrm{Tr}\, R^2 = \int \mathrm{Tr} RR^* = I_0 \qquad (18)$$

and I_0 becomes a topological invariant. As

$$-\int_X \mathrm{Tr}(R(A)^2) = 8\pi^2 k(E)$$

and we apply the Gauss-Bonnet theorem to find :

$$\chi(M) = \frac{1}{32\pi^2} \int_X \varepsilon_{abcd} R_{ab} R_{cd} \qquad (19)$$

Therefore, the *ß* → 0 limit of the Feynmann path integral is giving the Euler Characteristic, i.e. the "true vaccum" mentioned hereabove and corresponding to the topological pole of the theory.

Next, we provide a new example showing that the *ß* → 0 limit of the N=2 supersymmetric theory is topological.

Example 2.3 *The topological 0-scale limit of the (supersymmetric) quantum field theory*

We apply here a well known quantum mechanical account of Morse theory due to Witten [40]. First, we start from the standard supersymmetry algebra $\left\{Q_i, Q_j\right\} = Q_i Q_j + Q_j Q_i = 0$. Next, we express this superalgebra in terms of data provided only by the space-time manifold ***M***. To do so, let's define a set of coboundary operators, the conjugation of *d* by $e^{ßH}$ being parametrised by $ß = \dfrac{1}{kT}$:

$$d_ß = e^{ßH} d\, e^{-ßH}$$
$$d^*_ß = e^{ßH} d^*\, e^{-ßH} \qquad (20)$$

for a Morse function *H*(x). Then the spectrum of the *ß*-dependant Hamiltonian is:

$$H_ß = d_ß d^*_ß + d^*_ß d_ß \qquad (21)$$

Now, let's send *ß* onto zero. We get for the Hamiltonian the invariant value :

$$H_0 = dd^* + d^*d = \bigoplus_{p \geq 0} \Delta_p \qquad (22)$$

But this invariant is nothing else than the Betti numbers of ***M***, given by $b_p = \dim ker\Delta_p$, which is a discrete function, independent of *ß*. Consequently, the space of zero energy states of *H* is given by the set of even (odd) harmonic forms on ***M*** and equals the Betti number of ***M***. So we have, for $ß \rightarrow 0$:

$$\mathrm{Tr}(-1)^{\mathrm{F}} = \sum_{k=0}^{4} (-1)^k b_i = \chi(\boldsymbol{M}) \qquad (24)$$

where b_i is the i^{th} Betti number and $\chi(\boldsymbol{M})$ the Euler-Poincaré characteristic of ***M***. Finally, on the zero scale limit, we recover the topological index [37] corresponding to any standard topological field theory.

To finish, we obtain in the last following example some analog results in the frame of full (N=2) supergravity.

Example 2.4 *The topological $ß \rightarrow 0$ limit of (N=2) supergravity*

As a matter of fact, for a spin manifold, we can express *H* in terms of the Dirac operator $\not{D}$. Then in dimension D=4, we can calculate on the $ß \rightarrow 0$ limit the index of the squared Dirac operator :

$$\mathrm{Ind}(\not{D}_+) = \lim_{ß \to 0} \mathrm{Str}\left[\left(e^{-ß\mathcal{D}_+^2}\right)\right] =$$

$$\frac{1}{(2\pi)^n}\int_{T^*M} \mathrm{Str}\left(e^{-|\xi|^2+\frac{1}{2}R\left(\xi,\frac{\partial}{\partial\xi}\right)+\frac{1}{16}(R\wedge R)\left(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi}\right)+B}\right) dx\, d\xi$$

By the Mehler formula, we find the Dirac index in function of the Dirac genus $\hat{A}(M)$:

$$\mathrm{ind}\,(\not{D}_+) = \int_M ch(B)\hat{A}(M) \tag{25}$$

ch being the Chern character, B the curvature and $\hat{A}(M)$ the Dirac genus of the auxillary fiber bundle. Since the spinors are interacting with Yang-Mills fields, the $\hat{A}(M)$ term is coming from the gravitational part whereas the rest of equ.(25) comes from the gauge part. As $\mathrm{Ch(B)} = \mathrm{Tr}\left(e^{-B/2i\pi}\right)$, we get :

$$\hat{A}(M) = \prod_{j=1}^{k}\frac{x_j/2}{\sinh(x_j/2)} \tag{26}$$

and we can express the complete Yang-Mills + gravity index through the following invariant :

$$\mathrm{Ind}(\not{D}_+) = \frac{\dim M}{8\pi^2}\int \mathrm{Tr}(R\wedge R) - \frac{1}{8\pi^2}\int \mathrm{Tr}(F\wedge F) \tag{27}$$

Finally, in Yang-Mills + gravity context, we obtain again a topological invariant on the *ß* → 0 limit.

Now, to go further, the next step consists to detect, on the *ß* → 0 limit, the nature of the topological invariant involved . We shall discover that Donaldson invariants are playing a very important role on this boundary.

3. *ß* → 0 SCALE AND DONALDSON INVARIANTS

From a topological point of view, Donaldson invariants are obtained from characteristic classes of an infinite dimensional bundle on the manifold equally infinite and canonically associated with a 4-dimensional manifold :

Definition 3.1 *Let **M** be a 4-dimensional manifold . The Donaldson invariant* $q_d(\boldsymbol{M})$ *is a symmetric integer polynomial of degree d in the 2-homology* $H_2(\boldsymbol{M};\boldsymbol{Z})$ *of **M***

$$q_d(\boldsymbol{M}) : H_2(\boldsymbol{M}) \times \ldots \times H_2(\boldsymbol{M}) \to \boldsymbol{Z}$$

$\mathcal{M}_{\text{mod}}^{(k)}$ *being the instanton moduli space of degree k, the Donaldson invariant is defined by the map* $m : H_2(\boldsymbol{M}) \rightarrow H^2(\boldsymbol{M})\mathcal{M}_{\text{mod}}^{(k)}$

Now, we suggest that on the *ß* → 0 limit, the 4D field theory is projected onto the first Donaldson invariant.

Proposition 3.2 *The high temperature limit of quantum field theory corresponding to ß → 0 in the partition function* Z = Tr (-1) ^{S} $e^{-ßH}$ *gives the first Donaldson invariant. The signature of the metric of the underlying 4-dimensional zero scale manifold is therefore Euclidean* (+ + + +).

Proof Let the partition function Z = Tr(-1)^{S} $e^{-ßH}$ connected with a set described by the density matrix :

$$Q = (-1)^{S} e^{-ßH} \tag{28}$$

According to standard arguments, we can write :

$$\text{Tr}\,(-1)^{S} e^{-ßH} = \int_{\text{CPB}} d\phi(t)\, d\psi(t) \exp - S_E(\phi,\ \psi) \tag{29}$$

It has been shown [1-36] that given a supersymmetric QFT, one can define the invariant I = Tr (-1)^{f}, f being the fermionic number. We propose to extend equ.(29) to supergravity and to define the topological invariant

$$\dot{\imath} = \text{Tr}\,(-1)^{S} \tag{30}$$

where S is the instanton number. So, the regularization of the trace (30) gives the index $\dot{\imath}$ of the Dirac operator :

$$\dot{\imath} = \text{Tr}\,\Gamma\, e^{-ß_c \not{D}^2} = \text{Tr}\,(-1)^{S}\, e^{-ß_c \not{D}^2} = \int_{cpl} [Dx]\,[D\psi]\, e^{-\int_0^{ß} dt\, L} \tag{31}$$

with $ß_c \in \mathbb{C}$. Then for $ß_c = 0$, the value of the partition function Z = Tr (-1) ^{S} $e^{-ß_c H}$ is :

$$Z_0 = \text{Tr}\,(-1)^{S} \tag{32}$$

and Tr $(-1)^S$ can be seen as the index of an operator acting on the Hilbert space $\mathscr{H}$. Dividing $\mathscr{H}$ in monopole and instanton sub-spaces $\mathscr{H} = \mathscr{H}_{\mathfrak{M}} + \mathscr{H}_{\mathrm{i}}$ and Q being a generator of supersymmetry, we get :

$$Q|\psi\rangle = 0\ ,\ \ Q^{*}|\psi\rangle = 0 \qquad (33)$$

So Tr $(-1)^S$ = Ker Q - Ker Q^{*} such that as topological index, Tr $(-1)^S$ is invariant under continuous deformations of parameters which do not modify the asymptotic behavior of the Hamiltonian H at high energy. H is given by H = dd* +d*d, the space of zero energy states corresponding to the set of even harmonic forms on M_n:

$$\mathrm{Tr}\,(-1)^S\, e^{-ßH} = \chi\,(M) = \sum_{k=0}^{n}(-1)^k b_k \qquad (34)$$

Δ = Tr $(-1)^S$ is independent of $ß$, the sole contributions to Δ coming from the topological sector of zero energy : $\Delta = n_i^{E=0} - n_m^{E=0}$. On formal basis, $n_i^{E=0} - n_m^{E=0}$ can be seen as the trace of the operator $(-1)^S$. Then Δ is a topological invariant, i.e. the first Donaldson invariant. The coupling constant g being dimensional, the limit $ß$ = 0 implies ρ = 0 and corresponds to the sector of zero size instantons [41]. So, Dim $\mathcal{M}^{(k)}_{\mathrm{mod}} = 0$. When Dim $\mathcal{M}^{(k)}_{\mathrm{mod}} \neq 0$, the Donaldson invariants are given by :

$$Z(\gamma_1\ \dots\ \gamma_r) = \int DX\, e^{-S} \prod_{i=1}^{r} \int_{\gamma_i} W_{k_1} = \left\langle \prod_{i=1}^{r} \int_{\gamma_i} W_{k_i} \right\rangle \quad (\mathrm{Dim}\ \mathcal{M}^{(k)}_{\mathrm{mod}} \neq 0) \qquad (35)$$

What happens when ? The solution is in the correspondence between the Donaldson invariants on 4D manifolds and the Floer homology groups [18] on 3D manifolds. Indeed, Donaldson invariants amount to the calculus of the partition function Z, expressed as an algebraic sum over the instantons [15]:

$$Z \mapsto \underset{\mathcal{M}^{(k)}_{\mathrm{mod}}=0}{Z} = \sum_i (-1)^{n_i} \qquad (36)$$

i indicating the i^{th} instanton and n_i = 0 or 1 determining the sign of its contribution to Z. Donaldson has shown on topological grounds [14] that when dim $\mathcal{M}^{(k)}_{\mathrm{mod}} = 0$, then $\sum_i (-1)^{n_i}$ is a non-polynomial topological invariant, reduced to an integer. We find the same result starting from $T_{\alpha\beta} = \left\{Q\ ,\ \lambda_{\alpha\beta}\right\}$. In fact, the partition function of the system at temperature $ß^{-1}$ has the general form Z_q = Tr $(-1)^S\, e^{-ßH}$

. For $ß = 0$, Z_q becomes $Z_{ß=0} = \mathrm{Tr}(-1)^S$, which is isomorphic to $\sum_i (-1)^{n_i}$, s and n_i giving in both cases the instanton number of the theory.

This result strongly suggests that on the high temperature limit $ß \to 0$ parameterizing the 0 scale of the theory, the partition function $Z_{\mathcal{M}^{(k)}_{\mathrm{mod}} = 0}$ projects the Lorentzian physical theory onto the Euclidean topological limit.

Now, starting from hereabove, we suggest the existence of a deep correspondence, of the duality symmetry type, between physical sector ($\lambda \geq$ Planck scale) and topological sector (0 scale) of the (pre)space-time.

4. DUALITY SYMMETRY BETWEEN PHYSICAL AND TOPOLOGICAL STATES

Ideally, the duality we are looking for (which we call "*i*-duality" $t \to \frac{1}{it}$ [5], of the type i = S⊗T) should exchange real time in strong coupling / large radius with imaginary time in weak coupling / small radius. In this sense, Planck (physical) scale should be *i*-dual to zero (topological) scale.
Let's first outline a few formal aspects of Lorentzian/Euclidean duality in terms of Hopf algebras.

4.1 Duality between q-Lorentzian and q-Euclidean Hopf algebras

Considering the non commutative constraints at the Planck scale, it appears interesting to adopt an approach in terms of "quantum groups" at this scale. So we have shown that in D=4, it should exist a *superposition* (+++±) between Lorentzian (physical) and Euclidean (topological) algebraic structures. Then we have constructed, in the enveloping algebras setting, the q-deformation of the cocyle bicrossproduct [6]:

$$\boldsymbol{M}_\chi(H) = H^{op} \,{}^{\psi}\!\triangleright\!\blacktriangleleft\, H_\chi \qquad (37)$$

where H is a Hopf algebra, $\triangleright\!\blacktriangleleft$ a bicrossproduct (i.e. a special type of crossproduct, defined in [29]) and χ a 2-cocycle or"twist" in the Drinfeld sense [16-17]. This is inspired by the idea to unify two different quantum groups within a *unique* algebraic structure. So, we propose the following :

Proposition 4.1 *The Euclidean and the Lorentzian Hopf algebras are related by the cocycle bicrossproduct*

$$U_q(so(4))^{op} \ {}^{\psi}\!\triangleright\!\blacktriangleleft \ U_q(so(3,1))$$

Proof Starting, in the setting of enveloping algebras, from the Euclidean Hopf algebra H = $U_q(so(4))$, we have the well known decomposition H = $U_q(su(2)) \otimes U_q(su(2))$ and the "opposite" $H^{op} = U_q(su(2))^{op} \otimes U_q(su(2))^{op}$, whereas the Lorentzian form is $A = H_\chi = U_q(su(2)) \blacktriangleright\!\blacktriangleleft U_q(su(2)) \cong U_q(so(3,1))$. As explained in [6], the cocycle of deformation is $\chi = \mathfrak{R}_{23}$. Then the action and the coaction are :

$$(a \otimes b) \triangleleft (h \otimes g) = h_{(1)} a S h_{(2)} \otimes g_{(1)} b S g_{(2)}$$

$$\begin{aligned} ß(h \otimes g) &= (h_{(1)} \otimes g_{(1)}).(Sh_{(3)} \otimes Sg_{(3)}) \otimes h_{(2)} \otimes g_{(2)} \\ &= h_{(1)} Sh_{(3)} \otimes g_{(1)} Sg_{(3)} \otimes h_{(2)} \otimes g_{(2)} \end{aligned} \qquad (38)$$

where we find the structure of tensor product of the action and the coaction for each $U_q(su(2))$ copy. On the other hand, the cocycle for $h,g \in U_q(su(2))$ is :

$$\begin{aligned} \psi(h \otimes g) = {} & (h_{(1)} \otimes g_{(1)})(1 \otimes \mathfrak{R}^{(1)})(Sh_{(4)} \otimes Sg_{(4)})(1 \otimes \mathfrak{R}^{-(1)}) \otimes \\ & (h_{(2)} \otimes g_{(2)})(\mathfrak{R}^{(2)} \otimes 1)(Sh_{(3)} \otimes Sg_{(3)})(\mathfrak{R}^{-(2)} \otimes 1) \end{aligned}$$

where the product is in H = $U_q(su(2)) \otimes U_q(su(2))$. This gives:

$$\begin{aligned} \psi(h \otimes g) &= h_{(1)} Sh_{(4)} \otimes g_{(1)} \mathfrak{R}^{(1)} Sg_{(4)} \mathfrak{R}^{-(1)} \otimes h_{(2)} \mathfrak{R}^{(2)} Sh_{(3)} \mathfrak{R}^{-(2)} \otimes g_{(2)} Sg_{(3)} \\ &= h_{(1)} Sh_{(4)} \otimes g_{(1)} \mathfrak{R}^{(1)} Sg_{(2)} \mathfrak{R}^{-(1)} \otimes h_{(2)} \mathfrak{R}^{(2)} Sh_{(3)} \mathfrak{R}^{-(2)} \otimes 1 \end{aligned} \qquad (39)$$

for the explicit bicrossproduct structures. **qed**

Clearly, prop.(4.1) proves the possible "unification" between the q-Lorentzian and the q-Euclidean Hopf algebras at the Planck scale.We give a detailed demonstration of this proposition in [6]. But also, the hereabove result suggests a certain type of "duality" between Lorentzian (physical) and Euclidean (topological) quantum groups. To see this, the next step consists in showing the existence of a very interesting "semidualisation" (proposed in the general case by S. Majid [29]) between Lorentzian and Euclidean Hopf algebras. Better still, such a duality allows a description of the *transition* from the q-Euclidean group to the q-Lorentzian group [30] :

Proposition 4.2 $U_{q^{-1}}(su(2)) \otimes U_q(su(2)) \cong U_q(su(2))^{op} \triangleright\!\blacktriangleleft U_q(su(2))$ *is connected by semidualisation to* $U_q(su(2)) \triangleright\!\triangleleft U_q(su(2))^{op*} \cong \mathfrak{D}(U_q(su(2)))$. *Then the semidualisation connects a version of* $U_q(so(4))$ *to a version of* $U_q(so(3, 1))$.

We have given in [6] a complete demonstration of prop. (4.2), based on the properties of the Drinfeld double $\mathfrak{D}(U_q(su(2)))$. Then, using our general cocycle construction $M_\chi(H)$, we get the interesting relation :

$$U_q(su(2)) \overset{\psi}{\triangleright\!\blacktriangleleft} U_q(su(2)) \cong U_q(so(4)) \xleftrightarrow{\textit{semidualisation}} U_q(su(2))^* \underset{\chi}{\triangleright\!\triangleleft} U_q(su(2)) \sim U_q(so(3, 1)) \tag{40}$$

The "q-deformation" from q-Euclidean to q-Lorentzian Hopf algebras corresponds to a duality transformation and induces the existence of a 2-cocycle of deformation. Likewise, the cocycle bicrossproduct

$$Uq(so(4)^{op} \overset{\psi}{\triangleright\!\blacktriangleleft} Uq(so(3, 1)) \tag{41}$$

defines implicitly the new (semi)duality transformation

$$U_q(so(4))^{op} \overset{\psi}{\triangleright\!\blacktriangleleft} U_q(so(3, 1)) \cong U_q(so(4)) \xleftrightarrow{\textit{semidualisation}} SO_q(3,1) \underset{\chi}{\triangleright\!\triangleleft} U_q(so(4))^{op}$$

where χ is constructed from ψ, this one being derived from the quasitriangular structure $\mathfrak{R}$ of $U_q(su(2))$ [5].

Now, an interesting consequence of those results concerns some duality characteristics at the level of q-deformation of space-time itself. We have shown [6] that the natural structures of the q-Euclidean space $\mathbb{R}_q^4$ and of the q-Lorentzian space $\mathbb{R}_q^{3,1}$, covariant under $U_q(so(4))$ and $U_q(so(3, 1))$ [8] are connected as follows :

$$\begin{array}{ccc} U_q(su(2)) & \xleftrightarrow{*\text{-Hopf algebras duality}} & SU_q(2) \sim \mathbb{R}_q^4 / \rho = 1 \\ \textbf{Transmutation} \updownarrow \approx & \approx & \updownarrow \textbf{q - signature change} \\ BU_q(su(2)) & \xleftrightarrow[\bigstar\text{- braided groups autoduality}]{} & BSU_q(2) = \mathbb{R}_q^{3,\,1} / \rho = 1 \end{array} \quad (42)$$

where we get a duality relation between $\mathbb{R}_q^4$ and $\mathbb{R}_q^{3,\,1}$ as a kind of T-duality [2]. This interpretation is possible only when $q \neq 1$ - i.e. at the Planck scale -. We can extend those results to q-Poincaré groups

$$\mathbb{R}_q^{3,\,1} \rtimes U_q(\widetilde{so}(3\,,\,1)) \quad (43)$$

seen as dual to the Euclidean q-Poincaré group

$$\mathbb{R}_q^4 \rtimes U_q(\widetilde{so}(4)) \quad (44)$$

Interestingly, the Hopf algebra duality has been recently related to superstrings T-duality by C.Klimcik and P.Sevara [26]. Such dualities in terms of quantum groups have also been proposed by S. Majid [29].

Now, we apply the hereabove results into a more physical context. So, we propose the following :

Proposition 4.3 *There exists, at the Planck scale, a symmetry of duality between the BRST cohomology ring (physical sector of the theory) and the cohomology ring of instanton moduli space (topological sector).*

Proof Let be, at the Planck scale, BRST cohomology groups, of which the generic form, reviewed in [37], is :

$$H_{BRST}^{(g)} = \frac{\ker Q_{BRST}^{(g)}}{\operatorname{im} Q_{BRST}^{(g-1)}} \quad (45)$$

where $Q_{BRST}^{(g)}$ is the BRST charge acting on operators of the ghost number g. From the theory of Donaldson [14-15], we conclude the existence, at 0 scale of space-time, of cohomology groups constructed by de Rham :

$$H^{(i)}(M^{(k)}_{\text{mod}}) = \frac{\ker d^{(i)}}{\text{im } d^{(i-1)}} \tag{46}$$

where $d^{(i)}$ represents the external derivative acting on the differential forms of degree i on $\mathcal{M}^{(k)}_{\text{mod}}$. Topological theory then brings about ring injection which follows:

$$H^{\bigstar}_{BRST} = \otimes^{\Delta U_k}_{g=0} \; H^{g}_{BRST} \xrightarrow{\iota} H^{\bigstar}\left(\mathcal{M}^{(k)}_{\text{mod}}\right) = \otimes^{d_k}_{i=0} H^{(i)}\left(\mathcal{M}^{(k)}_{\text{mod}}\right) \tag{47}$$

and which, according to conditions given in [19], becomes a ring isomorphism. There exists therefore an injective path from the physical mode to the topological mode. Now let 0_i be the physical observables considered, such that a correlation function of n observables is the number given by the matrix of intersections H_i :

$$\langle 0_1 \; 0_2 \; \ldots \; 0_n \rangle = \#(H_1 \cap H_2 \cap \ldots \cap H_n) \tag{48}$$

number associated with n cycles of homology $H_i \subset \mathrm{M}_{\text{mod}}$, in moduli space $\mathcal{M}^{(k)}_{\text{mod}}$ of configurations of the gravitational instanton type $\Im[\phi(x)]$, on the gravitational fields ϕ of the theory. The physical sector of the theory is described by the left hand side of equation (48) and the topological sector by the right hand side. One observes that $\langle 0_1 0_2 \; \ldots \; 0_n \rangle \neq 0$, i.e. the theory has a physical content if $\Delta U_k = \int \partial^{\mu} j_{\mu} \, d^4x$, with j_{μ} being the "ghost flow" of degree k, ΔU its integral anomaly and $d_i = gh \, [0_i]$ the ghost number of 0_i. Moreover

$$d_k = \dim_{\mathbb{R}} \mathcal{M}^{(k)}_{\text{mod}} \tag{49}$$

is the dimension of moduli space of degree k. Following the theorem of Atiyah-Singer [21], one can show that $\Delta U_k = d_k$. From this point of view, the correlation functions of a set of local observables

$$G(x_1 \ldots x_n) = \langle 0(x_1) \ldots 0(x_n) \rangle \tag{50}$$

amounts to the integral over moduli space of the number of cohomology classes of space. The associated BRST charge Q is of the form $Q = \sum (-1)^n$. When the divergence of the ghost flow is non-zero, i.e. $\partial^{\mu} \mathrm{j} \, \boldsymbol{M} \neq 0$, then the theory oscillates between (0_i) and (H_i) - i.e. between the Coulomb branch and the Higgs branch in metric superposition space - . For the 0 mode of the scale, $\partial^{\mu} j \boldsymbol{M} = 0$, then

$$\langle 0_1 0_2 \ \dots \ 0_n \rangle = 0 \qquad (51)$$

which suggests that on this limit, $\dim \mathcal{M}^{(k)}_{\mathrm{mod}} = 0$. In fact, after functional integration over the empty degrees of liberty of the theory, the physical observables are reduced to closed forms Ω_i of degree d_i, which signifies :

$$\Delta U = \dim \mathcal{M}^{(k)}_{\mathrm{mod}}$$

and when $\Delta U = 0$, there exists no embedding space for moduli space and the theory is projected into the Coulomb branch, at the origin of $\mathcal{M}^{(k)}_{\mathrm{mod}}$, on a singular instanton of zero size, identified to space-time at zero scale. The corresponding signature in this sector of the theory is therefore Euclidean (+ + + +). **qed**

This result suggests once more that at zero scale, the theory is no longer physical but purely topological.

Now, here is a critical question raised by this paper : how do we go from the topological state of the (pre)space-time around the origin to the usual physical state ? In the last section, we shall try to answer this question.

5. TRANSITION FROM INITIAL TOPOLOGICAL PHASE TO STANDARD PHYSICAL PHASE

Considering all the preceding developments, it's of crucial interest to worry about how the initial (generally covariant) topological phase possibly characterizing the (pre)space-time at the vicinity of the Initial Singularity does break down to the universe we observe to day. We then propose some (hopefully) stimulant tracks able to be worked out within some further researches.

On general basis, we claim hereafter that the transition *Topological phase*→ *Physical phase* might be deeply related to the breaking of the N = 2 supergravity at the Planck scale. In other words, supersymmetry breaking, as showed by C. Kounnas *and al* in superstrings context [3-27], is characterized by the loss of the thermodynamical equilibrium of the system. To sum up, the D-dimensional space-time supersymmetry is spontaneously broken in (D-1) dimensions by thermal effects. For this reason, supersymmetry breaking might bring about the *decoupling* of the topological and the physical states of the (pre)space-time system. How is it so ? To see it, according to [4-27], let's recall that at the Planck scale, the (pre)space-time is generally characterized by two fundamental properties : (i) the thermodynamical equilibrium state [34] and (ii) the non-commutativity of the underlying geometry [11]. Those two properties are very often considered, together or separately. However, it is critical to realize that for any system, properties (i) and (ii) are inducing the famous "Kubo-Martin-Schwinger"(KMS) condition [24]. Therefore, we propose now to consider that, most likely, space-time, as a thermodynamical system, is subject to the KMS condition at the Planck scale [6]. Consequently, in the interior of the "KMS strip", i.e. from $ß = 0$ to $ß = \ell_{Planck}$, the fourth coordinate g_{44} should be considered as *complex*, the two real poles being $ß = 0$ (topological pole) and $ß = \ell_{Planck}$ (physical pole). This is a direct (and standard) consequence of the KMS condition. So, we suggest [6] that within the KMS strip, the Lorentzian and the Euclidean metric are in a "quantum superposition state" (or coupled), this entailing a "unification" (or coupling) between the topological (Euclidean) and the physical (Lorentzian) states of space-time. Conversely, the transition from the topological state to the physical state of the space-time can be seen in terms of "KMS breaking" (cf. conj. (5.2.5)).

Now, let's begin with the hypothesis of global thermodynamical equilibrium at the Planck scale.

5.1 Thermodynamical equilibrium and KMS state of the space-time at the Planck scale

5.1.1 *Thermodynamical equilibrium of space-time*

From a thermodynamical point of view, it appears that the Planck temperature

$$ß^{-1}_{\text{planck}} \approx T_P \approx \frac{E_P}{k_B} \approx \left(\frac{\hbar c^5}{G}\right)^{1/2} k_B^{-1} \approx 1{,}4 \times 10^{32}\ \text{K}$$

represents the upper limit of the *physical* temperature of the system. Indeed, it is currently admitted that, before the inflationary phase, the ratio between the interaction rate (Γ) of the initial fields and the (pre)space-time expansion (H) is $\frac{\Gamma}{H}$ >> 1, so that the system can reasonably be considered in equilibrium state. This has been established a long time ago within some precursor works of S. Weinberg [34], E. Witten [4] and several others. It has recently been shown by C. Kounnas *and al* in the superstrings context [27]. However, this natural notion of equilibrium, when viewed as a global gauge condition, has dramatic consequences regarding physics at the Planck scale. Which kind of consequences? To answer, let's see on formal basis what an equilibrium state is.

Definition 5.1 *H being an autoadjoint operator and $\mathfrak{H}$ the Hilbert space of a finite system, the equilibrium state ω of this system is described by the Gibbs condition* $\varphi(A) = \frac{\mathrm{Tr}_{\mathfrak{h}}(e^{ßH}A)}{\mathrm{Tr}_{\mathfrak{h}}(e^{ßH})}$ *and satisfies the KMS condition.*

Here, Tr is the usual trace, $ß = \frac{1}{kT}$ is the inverse of the temperature, H the Hamiltonian, i.e. the generator of the one parameter group of the system. Of course, **A** is a von Neumann C^*- algebra (see §(5.1.4) for definitions). The equilibrium state implies that *ß* must be seen as a periodic (imaginary) time interval [0, $ß = \ell_{Planck}$]. Now, the famous Tomita-Takesaki modular theory [10-11] has established that to each state φ(*A*) of the system corresponds, in a unique manner, the strongly continuous one parameter $*$ - automorphisms group α_t :

$$\alpha_t(\boldsymbol{A}) = e^{iHt}\,\boldsymbol{A}\,e^{-iHt} \qquad (52)$$

with t $\in \mathbb{R}$. This one parameter group describes the time evolution of the observables and corresponds to the well known Heisenberg algebra. At this stage, we are brought to find the remarkable discovery of Takesaki and Winnink, connecting (**i**) the evolution group $\alpha_t\,(\boldsymbol{A})$ of a system (i.e. the modular group $\boldsymbol{M} = \Delta^{it}\,A\,\Delta^{-it}$) with (**ii**) its equilibrium state $\varphi(A) = \dfrac{\mathrm{Tr}(Ae^{-ßH})}{\mathrm{Tr}(e^{-ßH})}$ [11]. The famous "KMS condition" [24] is nothing else than this relation between $\alpha_t\,(\boldsymbol{A})$ and $\varphi(\boldsymbol{A})$, the content of this relation being precised in (i) and (ii) of §(5.1).

Then we claim in a natural way that the space-time, in equilibrium state at the Planck scale, is therefore subject to the KMS condition at this scale.

5.1.2 *The (pre)space-time in KMS state at the Planck scale*

When viewed as a hyperfinite system at the Planck scale, the (pre)space-time may be described by a von Neumann C^*-algebra **A** (a von Neumann algebra is *hyperfinite* if it is generated by an increasing sequence of finite dimensional sub-agebras). Now, let's see the essence of the KMS condition, given by the Haag-Hugenholtz-Winnink theorem [23] : a state ω on the C^*-algebra **A** and the continuous one parameter automorphisms group of **A** at the temperature $ß = 1 / k\,T$ verify the KMS condition if, for any pair A, B of the $*$ - sub-algebras of **A,** it exists a $f(t_C)$ function *holomorphic* in the strip $\{t_C = t + i\,ß \in \mathbb{C}\ ,\ \mathrm{Im}\ t_C \in [\,0\ ,\ ß\,]\ \}$ such that :

(i) $f\,(t) = \varphi(A\,(\,\alpha_t\,B))$,

(ii) $f\,(t + i\,ß\,) = \varphi(\,\alpha_t\,(B)A)$, $\qquad \forall\ t \in \mathbb{R}$. (53)

Then we observe with (i) and (ii), the two crucial properties of the KMS condition : the *holomorphicity* of the KMS strip and of course, due to the cyclicity of the trace, the *non commutativity* $\varphi((\alpha_t A)B) = \varphi(B(\alpha_{t+i\ ß}A))$ characterizing any "KMS space" (in fact, the two boundaries of the strip do not commute with each other).

Now, if we admit that around ℓ_{Planck}, the hyperfinite (pre)space-time system is in a thermal equilibrium state, then according to [24], we are also bound to admit that this system is in a KMS state. Incidentally, another good reason to apply the KMS condition to the space-time at ℓ_{Planck} is that at such a scale, the notion of commutative geometry vanishes and should be replaced by *non commutative* geometry [11]. In this new framework, the notion of "point" in the usual space collapses and is replaced by the "algebra of functions" defined on a non commutative manifold. Non commutative geometry and quantum groups theory [16-29] are addressing such non-commutative constraints. But the non-commutativity induced by the KMS state is in natural correspondence with the expected non commutativity of the space-time geometry at the Planck scale.

Next, let's push forwards the consequences raised by the holomorphicity of the KMS strip.

5.1.3 *Holomorphic time flow at the Planck scale*

As a consequence of the application of the KMS condition to space-time itself, we are induced to consider that the time-like coordinate g_{00} becomes *holomorphic* within the limits of the KMS strip. So we should have [11-24] :

$$t \rightarrow \tau = t_r + i\, t_i \qquad (54)$$

as showed in [6]. In the same way, the physical (real) temperature becomes also complex at the Planck scale :

$$T \rightarrow T_c = T_r + i T_i \qquad (55)$$

as proposed by Atick and Witten in another context [4]. So, the KMS condition suggests the existence at the Planck scale, of an effective one loop potential coupled, in N = 2 supergravity, to the complex dilaton + axion field $\varphi = \frac{1}{g^2} + i\alpha$ and yielding the following dynamical form of the metric

$$\eta_{\mu\nu} = \mathrm{diag}(1\ ,\ 1\ ,\ 1\ ,\ e^{i\theta}) \qquad (56)$$

The signature of (56) is Lorentzian (physical) for $\theta = \pm\, \pi$ and can become Euclidean (topological) for $\theta = 0$. This unexpected effect is simply due to the fact that, within the boundaries of the analytic KMS field -i.e. from the scale zero up to the Planck scale- the "time-like" direction is extended to the complex variable $t_C = t_r + i\,t_i \in \mathbb{C}$, Im $t_C \in [i\,t_i\ ,\ t_r]$, the function *f(t)* being analytic within the limits of the KMS field and continuous on the boundaries. What happens on the *ß* = 0 limit ? Applying the KMS properties, we find that the time like direction *t* becomes pure imaginary so that the signature is Euclidean (++++). Conversely, *t* is pure real for $ß \geq \ell_{Planck}$ (+++−). So, according to Tomita's modular theory [11], the KMS condition, when applied to the space-time, induces, within the KMS strip, the existence of the "extended" (holomorphic) automorphisms group :

$$\boldsymbol{M}_q \mapsto \sigma_{ß_C}(\boldsymbol{M}_q) = e^{\;Hß_C}\boldsymbol{M}_q\ e^{\;Hß_C} \qquad (57)$$

with the *ß* parameter being formally *complex* and able to be interpreted as a complex time *t* and / or temperature *T*. It is interesting to remark that in the totally different context of superstrings, J.J. Atick and E. Witten were the first to propose such an extension of the real temperature towards a complex domain [4]. Recently, in N=4 supersymmetric string theory, I. Antoniadis, J.P. Deredinger and C. Kounnas [3] have also suggested to shift the real temperature to imaginary one by identification with the inverse radius of a compactified Euclidean time on S^1, with $R = 1 / 2\pi T$. Consequently, one can introduce a complex temperature in the thermal moduli space, the imaginary part coming from the $B_{\mu\nu}$ antisymmetric field under type ***IIA*** $\xleftrightarrow{S/T/U}$ type ***IIB*** $\xleftrightarrow{S/T/U}$ ***Heterotic*** string-string dualities. More precisely, in Antoniadis *and al* approach, the field controlling the temperature comes from the product of the real parts of three complex fields : s= Re **S**, t= Re **T** and u= Re **U**. Within our KMS approach, the imaginary parts of the moduli **S**, **T**, **U** can be interpreted in term of Euclidean temperature. Indeed, from our point of view, a good reason to consider the temperature as complex at the Planck scale is that a system in thermodynamical equilibrium state must be considered as subject to the KMS condition [24].

Now, let's step forward a more algebraic comprehension of KMS state, in terms of von Neumann algebras.

5.1.4 *KMS state in terms of von Neumann algebras*

The von Neumann algebras are, naïvely speaking, the *non commutative* analogs of measure theory. They have a critical importance in our understanding of non commutativity of space-time around the Planck scale.

In the KMS state, the only von Neumann algebras involved are what is called "factors", i.e. a special type of von Neumann algebra, whose the center is reduced to the scalars $a \in \mathbb{C}$. There exists three types of factors : the type I and type II (in particular here II_∞) -which are commutative and endowed with a *trace*- and the type III, *non commutative* and traceless. A trace τ on a factor M is a linear form such that $\tau(AB) = \tau(BA),\ \forall A,B \in M$. In this case, any measure on M is invariant. When the measure on M is ill defined (which is the case of type III), the notion of trace vanishes and is replaced by the one of "weight", which is a linear map from M_+ to $\mathbb{R}_+ = [0, +\infty]$. The type III factors have no definite trace. They are very important hereafter as far as they are the only one involved in KMS states. We work here with "III_λ" factors, $\lambda \in]0, 1[$, characterized by the invariant $S(M) = \lambda^{\mathbb{Z}} \cup \{0\}$.

Rightly, the KMS condition, when applied to the (pre)space-time at the Planck scale, cuts up three different scales on the (pre)light cone, which can be described by three different types of von Neumann algebras (or "factors").

5.1.5 *From the topological scale to the physical scale of the space-time*

(i) ***the topological scale*** (ß = 0, signature {++++}) : this initial "topological" scale correspond to the imaginary vertex of the light cone, i.e. a zero-size gravitational instanton. All the measures performed on the Euclidean metric being ρ-equivalent up to infinity, the system is ergodic. As shown by A. Connes, any ergodic flow for an invariant measure in the Lebesgue measure class gives a unique type II_∞ hyperfinite factor [11]. This strongly suggests that the singular 0-scale should be described by a type II_∞ factor, endowed with a hyperfinite trace noted Tr_∞. By hyperfinite, we simply mean that the trace of the II_∞ factor is not finite. We call $\boldsymbol{M}^{0,1}_{Top}$ such a "topological" factor, which is an infinite tensor product $\otimes^\infty$ of matrices algebra (ITPFI) of the $R_{O,1}$ Araki-Woods type [11]. Now, the initial state on $\boldsymbol{M}^{0,1}_{Top}$, corresponding in ex. (2.1) to the divergent values of the dilaton field $\frac{1}{g^2}$, is given by :

$$\varphi(\boldsymbol{M}^{0,1}_{Top}) = \frac{\mathrm{Tr}_\infty(e^{-ßH}\boldsymbol{M}^{0,1}_{Top})}{\mathrm{Tr}_\infty(e^{-ßH})} \quad (58)$$

and, considering the hyperfinite characteristic of the trace Tr_∞, we have equivalently :

$$\varphi(\boldsymbol{M}^{0,1}_{Top}) = \mathrm{Tr}_\infty(e^{\ ßH}\boldsymbol{M}^{0,1}_{Top}\ e^{ßH}) \quad (59)$$

where $\varphi(\boldsymbol{M}^{0,1}_{Top})$ represents a very special type of "current", that we propose to call 'trace current" $\mathbf{T}^f$. Clearly, the invariant hyperfinite II_∞ trace current $\mathbf{T}^f$ is a pure topological amplitude [19-37] and, as such, "propagates" in imaginary time from zero to infinity. In this sense, $\varphi(\boldsymbol{M}^{0,1}_{Top})$ can be seen as a "zero topological cycle" which represents an intrinsic "Euclidean dynamic" controlling the blow up of the space-time Initial Singularity [6].

(ii) ***the quantum scale*** ($0 < ß < \ell_{Planck}$, signature {+++±}) : we reach the KMS domain [24]. Considering the quantum fluctuations of $g_{\mu\nu}$, there is no more invariant measure on the non commutative metric. Therefore, according to von Neumann algebra theory, the "good factor" addressing those constraints is uniquely a non commutative *traceless* algebra, i.e. a type III factor [9] (the only one able to be involved in KMS state). More precisely, it is a type III_λ that we call $\boldsymbol{M}_q$, with the period $\lambda \in]\,0, 1\,[$. Important, it has been demonstrated that any type III_λ factor can be canonically decomposed into the following way [9] :

$$\mathrm{III}_\lambda = \mathrm{II}_\infty \rtimes_\theta \mathbb{R}^*_+ \quad (60)$$

$\mathbb{R}^*_+$ (dual of $\mathbb{R}$) acting periodically on the "II_∞ factor". Then the *ß*-dependant *periodicity* of the action of $\mathbb{R}^*_+$ on $\boldsymbol{M}^{0,1}_{Top}$ takes the form :

$$\boldsymbol{M}\mathrm{q} = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathbb{R}^*_+/ß\mathbb{Z} \equiv \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta ß\, \mathrm{S}_1 \quad (61)$$

The relation between λ and ß is such that $\lambda = \frac{2\pi}{ß}$, so that when $ß \to \infty$, we get $\lambda \to 0$ (the periodicity is suppressed). Now, the theory being given on the infinite Hilbert space $\mathscr{L}(\mathfrak{h})$ = $\mathscr{L}\left[L^2\left(\mathbb{R}_+^*/ß\mathbb{Z}\right)\right]$, $\boldsymbol{M}$q becomes :

$$\boldsymbol{M}\text{q} = \boldsymbol{M}_{Top}^{0,1} \rtimes_{\theta} \mathscr{L}\left[L^2\left(\mathbb{R}_+^*/ß\mathbb{Z}\right)\right] \qquad (62)$$

The type I$_\infty$ factor $\mathscr{L}\left[L^2\left(\mathbb{R}_+^*/ß\mathbb{Z}\right)\right]$ yields the modular flow of (periodic) *evolution* of the system. So, the KMS type III$_\lambda$ factor $\boldsymbol{M}$q connects the "topological" type II$_\infty$ factor $\boldsymbol{M}_{Top}^{0,1}$ with the "physical" type I$_\infty$ factor $\boldsymbol{M}_{Phys}$:

$$\boldsymbol{M}\text{q} = \boldsymbol{M}_{Top}^{0,1} \rtimes_{\theta} \boldsymbol{M}_{Phys} \qquad (63)$$

In terms of "flows", Equ. (63) connects the topological flow of weights of $\boldsymbol{M}_{Top}^{0,1}$ with the physical modular flow raised by $\mathscr{L}\left[L^2\left(\mathbb{R}_+^*/ß\mathbb{Z}\right)\right]$. This furnishes a good image of the unification between topological and physical states, to be compared to the bicrossproduct (41) Uq(so(4)^{op} $\overset{\psi}{\triangleright\!\blacktriangleleft}$ Uq(so(3,1)) unifying Euclidean and Lorentzian q-groups. The quantum flow $\sigma_{ß_c}(\boldsymbol{M}_{q\ \ flow}) = e^{ß_c H}\boldsymbol{M}_{q\ \ flow}e^{\ ß_c H}$ is constructed in prop. (5.2).

(iii) ***the physical scale*** ($ß > \ell_{Planck}$, signature {+++−}) : this last scale represents the physical part of the light cone and, consequently, the notion of (Lebesgue) measure is fully defined. Therefore, the (commutative) algebra involved is endowed with a hyperfinite trace and is given on the infinite Hilbert space $\mathscr{L}(\mathfrak{h})$, with $\mathfrak{h} = L^2(\mathbb{R})$. Then $\mathscr{L}(L^2(\mathbb{R}))$ is a *type* I$_\infty$ *factor*, indexed by the real group $\mathbb{R}$, which we call $\boldsymbol{M}_{Phys}$. So, $\mathscr{L}(L^2(\mathbb{R}))$ = $\boldsymbol{M}_{Phys}$ and the flow raised by $\boldsymbol{M}_{Phys}$ is simply the (real) time evolution, given by the modular group :

$$\sigma_t(\boldsymbol{M}_{Phys}) = e^{\ iHt}\boldsymbol{M}_{Phys}\ \ e^{-\ iHt} \qquad (64)$$

In this case (type I$_\infty$ factor) all the automorphisms are inner automorphisms. We call "physical flow" $\mathbf{P}^f_{ß>0}$ this evolution flow in real time. Of course, $\sigma_t(\boldsymbol{M}_{Phys})$ is simply giving the usual algebra of observables [12].

At present, we shall evidence that the KMS state "unifies" the physical flow and the topological current.

Proposition 5.2 *At the KMS scale $0 < ß < l_{Planck}$, the two automorphisms groups $\sigma_t(M_{Phys})$ and $\sigma_ß(M^{0,1}_{Top})$ are coupled up to Planck scale within a unique IIIλ factor of the form* $\boldsymbol{M}$q = $\boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathscr{L}\left[L^2\left(\mathbb{R}^*_+ / ß\mathbb{Z}\right)\right]$. *The corresponding extended one (complex) automorphisms group describing the quantum evolution is*

$$M_q \mapsto \sigma_ß(M_q) = e^{Hß_C} M_q \ e^{-Hß_C}$$

Mq corresponds to the coupling between the one parameter automorphisms group giving the physical flow and the automorphisms semi-group giving the topological flow of the system.

Proof The KMS state of the (pre)space-time is yield by the unique IIIλ factor given by equ. (60) :

$$\mathbf{M}\text{q} = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \mathscr{L}\left[L^2\left(\mathbb{R}^*_+ / ß\mathbb{Z}\right)\right] = \boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \boldsymbol{M}_{Phys} \qquad (65)$$

which represents the KMS "unification" of the topological state and the physical state of the (pre)space-time at the Planck scale. Now, since it exists an operatorial weight of $\boldsymbol{M}$q on its sub-group $\boldsymbol{M}^{0,1}_{Top}$, the equilibrium state φ on $\boldsymbol{M}$q is given by the state on $\boldsymbol{M}^{0,1}_{Top}$. We express the state φ under the new form constructed in [6] :

$$\varphi(\boldsymbol{M}_{q\text{-}state}) = \mathrm{Tr}_\infty \left(e^{\ ßH} \boldsymbol{M}^{0,1}_{Top} \ e^{ßH}\right)$$

This represents what we have called in (5.1.5) the "trace current" of the "topological factor" $\boldsymbol{M}^{0,1}_{Top}$. However, Connes-Takesaki have shown [10] that the flow of weights on a factor is given by the flow of weights on the associated II$_\infty$ factor. For it exists an homomorphism OUT III$_\lambda$ → OUT II$_\infty$ such that the sequence (66) is exact :

$$\{\,1\,\} \to H^1(F) \xrightarrow{\bar{\partial} M} \mathrm{OUT}M \xrightarrow{\bar{\gamma}} OUT_{\theta,\tau}(N) \to \{\,1\,\} \qquad (66)$$

The multiplicative action of $\mathbb{R}$: $\tau \circ \theta_S = e^{-S}\tau, \;\; s \in \mathbb{R}$ on $\boldsymbol{M}_{Top}^{0,1}$ *translates* the trace τ of $\boldsymbol{M}_{Top}^{0,1}$, which generates the flow of weights on $\boldsymbol{M}_{Top}^{0,1}$ and $\boldsymbol{M}$q (cf.[10]). So, $\varphi(\boldsymbol{M}_{\boldsymbol{q\text{-}state}})$ becomes a *ß*-dependant automorphism (semi)group :

$$\sigma_{ß}\,(\boldsymbol{M}_{\boldsymbol{q\text{-}state}}) = \; e^{\;ßH}\boldsymbol{M}_{Top}^{0,1}\;\; e^{\,ßH} \tag{67}$$

Equ. (67) describes the *flow of weights* [10] of the type III$_\lambda$ factor $\boldsymbol{M}$q. But as pointed in [6], we can also interpret equ.(67) as a " modular flow in imaginary time" *it*, dual to the modular flow in real time given by :

$$\sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}})` = e^{\;iHt}\boldsymbol{M}_{\boldsymbol{Phys}}e^{\,-\,iHt} \quad , \;\; \mathrm{t} \in \mathbb{R}.$$

An interpretation of this type has also been proposed (in a different context, however) by Derendinger and Lucchesi in [13]. Finally, the KMS flow connects the flow of weights $\sigma_ß(\boldsymbol{M}_{\boldsymbol{q\text{-}state}})$ to the modular group $\sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}})$:

$$\begin{aligned}\sigma_{ß_C}(\boldsymbol{M}_{q\quad flow}) &= \sigma_ß(\boldsymbol{M}_{Top}^{0,1}) \otimes \sigma_t(\boldsymbol{M}_{Phys}) \\ &= e^{\;(ß+it)H}\boldsymbol{M}_{q\quad flow}\;\; e^{(ß+it)H} \\ &= e^{\,ß_C H}\boldsymbol{M}_{q\quad flow}e^{\;\;ß_C H}\end{aligned}$$

which is indexed by the complex time variable $ß_C$. Again, this flow is expressing the unification between the *physical* flow $\sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}}) = \sigma_t(\boldsymbol{M}_{Phys})$ and the *topological* flow $\sigma_ß(\boldsymbol{M}_{\boldsymbol{q\text{-}state}}) = \sigma_ß(\boldsymbol{M}_{Top}^{0,1})$ within the *unique* KMS (or quantum) flow $\underset{0<\mathbf{ß}<\ell_P}{\mathbf{Q}^f}$ given by the automorphisms group of $\boldsymbol{M}_q$:

$$\sigma_{ß_C}(\boldsymbol{M}_{q\quad flow}) = \sigma_ß(\boldsymbol{M}_{\boldsymbol{q\text{-}state}}) \;\bigoplus\; \sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}})$$

The (pre)space-time KMS strip has zero as infimum and the Planck scale as supremum. So between those bounds, the Euclidean topological flow and the Lorentzian physical flow are unified in a natural way within the holomorphic "quantum flow" $\underset{0<\mathbf{ß}<\ell_P}{\mathbf{Q}^f} \to \;\sigma_{ß_C}(\boldsymbol{M}_{q\quad flow}) = e^{\,ß_C H}\boldsymbol{M}_{q\quad flow}e^{\;\;ß_C H}$.

Another way to verify the coupling of M_{Phys} and $M_{Top}^{0,1}$ in the unique type III$_\lambda$ factor lies in the Conne's invariant

$$\delta : \; \mathbb{R} \to \mathrm{OUT\;M} = \frac{\mathrm{AUT\;M}}{\mathrm{INT\;M}} \tag{68}$$

(automorphisms of M quotiented by the inner automorphisms, necessarily present in the non commutative case). This M invariant represents an ergodic flow $\{W(M)\ ,\ W_{\lambda}\}$ where W_{λ} is a one parameter group of transformations - i.e. a flow - which admits a description in terms of class of weights and whose the natural parameter is $\mathbb{R}_{+}{}^{*}$. We consider now the type III_{λ} factor Mq of equ.(61). Starting from equ. (68), we can construct the extension *Ext* (noted $\top$) of OUT Mq by INT Mq in AUT Mq :

$$\text{AUT Mq} \equiv \text{ OUT Mq} \top \text{ INT Mq} \tag{69}$$

with $\{x,y\} \in$ OUT Mq and $\{x',y'\} \in$ INTMq. The inner automorphisms group INTMq is a normal sub-group of AUT Mq . Considering two weights φ et ψ of Mq, and applying the Radon-Nikodym theorem [10], it exists a unitary of Mq such that $\sigma_t^{\psi}(x) = u_t \sigma_t^{\varphi}(x) u_t^{*}$, with $u_t = (D\psi\ ;\ D\varphi)_t$ and $\sigma_t^{\psi}(x) \in$ INT Mq for a certain class of modular automorphisms. Considering the fact that under the trace of the factor II_{∞} involved in the crossproduct $\boldsymbol{M}q = M_{Top}^{0,1} \rtimes_{\theta} \mathbb{R}^{*}{}_{+}$ all the modular automorphisms are *inner* automorphisms, we restrict INTMq to the sub-group of the modular automorphisms, which we call INT_{mod} Mq. Then we look for the image of the inner modular group in OUTMq. Within a certain cohomology class $\{K\}$, the group $\sigma_t^{\psi}(x)$ is given by INT_{mod}Mq, whereas the non-unitary transformations $\sigma_{ß}(x)$ are given by OUT Mq. We get then for the "physical" flow :

$$\sigma_t^{\psi}(x) = e^{iHt} M_q\, e^{\ iHt} \in \text{INT}_{mod}\ \text{Mq} \tag{70}$$

whereas the "topological" flow of weights of *Mq* is given by :

$$\sigma_{ß}(x) = e^{\ ßH} M_q\, e^{ßH} \in \text{OUT Mq} \tag{71}$$

and the extension AUT Mq ≡ OUT Mq $\top$ INT_{mod} Mq yields :

$$\sigma_{(t\,\top\,ß)} = \sigma_t^{\psi}(x) \top \sigma_{ß}(x) \tag{72}$$

Within the general group of extensions $\{Ext\}$, we get the trivial holomorphic sub-group :

$\sigma_{ß+it}(M_q) = e^{\ (ß+it)H} M_q\ \ e^{(ß+it)H} = \sigma_{ß_C}(M_q) = e^{\ Hß_C} M_q\ \ e^{-Hß_C}$ or

which corresponds to the KMS state and"unifies" within the unique extended form $\sigma_{ß_C}(Mq)$ the physical flow $\sigma_t^{\psi}(x)$ and the topological current $\sigma_{ß}(x)$. Clearly, we get $\sigma_{ß_C}(Mq) \subset$ OUT Mq $\top$ INT_{mod} Mq. Again we find : $\sigma_{ß_C}(\boldsymbol{M}_{q\ \ flow}) = \sigma_{ß}(\boldsymbol{M}_{q\text{-}state}) \oplus \sigma_t(\boldsymbol{M}_{q\text{-}evolution})$ **qed**

Now, let's get over the last step. Our aim is to explain the transition from the topological state to the physical state (**TP** transition) of the space-time. We shall cope with this problem following two different ways :

(i) we conjecture that such a transition could be related to the N=2 supergravity breaking beyond ℓ_{Planck} ;

(ii) likewise, **TP** transition could be explained in terms of "decoupling", beyond the Planck scale, between the (Euclidean) "topological current" (raised by $\boldsymbol{M}^{0,1}_{Top}$) and the (Lorentzian) physical flow (yield by $\boldsymbol{M}_{Phys}$).

5.2 TP transition, supersymmetry breaking and flows decoupling

First of all, let's put in evidence the link between KMS state and supersymmetry. To do this, we propose hereafter a relevant example able to be seen as a good toy model expliciting the deep correspondence between thermal states, supersymmetry and extended space-time (i.e. extension of the time-like direction in the complex plane).

Example 5.2.1 *thermal states, supersymmetry and KMS condition*

In the following, we shall focus on some important results recently obtained by J.P. Derendinger and C. Lucchesi [13]. Interestingly, it has been demonstrated that thermal supersymmetry (as opposed to T=0 supersymmetry) must be considered in the context of thermal (i.e. KMS) superspace. We remark here that the authors apply the KMS condition to the thermal superspace (i.e. the thermal supersymmetric space) in a general setting. In our own approach, as suggested in ref. 6 and in the present paper as well, we apply the KMS condition to the *thermal (pre) space-time at the Planck scale*. Considering that in the standard “hot big-bang” theory the (pre) space-time is generally viewed as supersymmetric, such an identification is natural. Namely, the authors have established that the thermal supersymmetry parameters must be both *time dependant and (anti)periodic in imaginary time* on the interval [0, *ß*], where *ß = 1/T*. In other words, focusing on field representations of the thermal super-Poincaré algebra and on chiral supermultiplet, one can straightfully see that thermal superfields are characterized by their time/ temperature *periodicity* properties. To explicit this, let's simply recall that at zero temperature, supersymmetry can heuristically be represented as a set of "generalized translations", including Grassmann variables that are translated by the supersymmetry generators. Therefore, a "point" X in superspace has coordinates

$$X = (x^{\mu}, \theta^{\alpha}, \overline{\theta}^{\alpha}) \tag{73}$$

where θ and $\overline{\theta}$ are the usual Grassmannian objets. Since at zero temperature the parameters of supersymmetry transformations are constant, the zero-temperature superspace coordinates are also space-time constants. In fact, at T=0, the (anticommuting) Grassmann coordinates simply turn bosonic commutation relations into fermionic anticommutators and conversely. Now, what happens at finite temperature (i.e. the case of primordial universe investigated here)? As a matter of fact, the situation is not so simple, because fermion and boson statistics involve, in addition, the appropriate statistical weight in field theory Green's functions. In such a context, as pointed in refs [13] and [28], it is natural to require that the variables which are translated by the effect of thermal supersymmetry transformation express the same properties as the thermal supersymmetry parameters. Therefore, the construction of thermal supersymmetry requires that the Grassmann variables get promoted to be time-dependant and (anti)periodic in imaginary time. To see this, let's precise that the thermal average $\langle ... \rangle_{ß}$ of a field operator 0 is, as usual, given by

$$\langle 0 \rangle_{ß} \equiv \frac{1}{Z(ß)} \mathrm{Tr}\left(e^{-ßH} 0\right) \tag{74}$$

with the lowest energy state being $E_0 = 0$, so that we have on the zero temperature limit :

$$\langle 0 \rangle_{ß} \xrightarrow{\beta \to \infty} \langle 0 | 0 | 0 \rangle$$

Now, at finite temperature, the Green's functions are necessarily subject to periodicity constraints in imaginary time. However, as showed in [6], those constraints are *exactly defining the KMS condition*. To verify this important point, we now review those conditions for bosonic and fermionic fields. Let's first begin with a free scalar (i.e. bosonic) field φ at x = $(t$, $\mathbf{x})$ whose evolution is such that :

$$\phi(x) = e^{iHt}\phi(0,x)e^{-iHt} \tag{75}$$

where the time coordinate t is allowed to be *complex*. Then the n-point thermal Green's function G_{nC} of the system is :

$$G_{nC}(x_1,...,x_n) = \left\langle \mathrm{T}_C \phi(x_1)...\phi(x_n) \right\rangle_{ß} \tag{76}$$

T_C being the path-ordering operator and $\langle ... \rangle_{\beta}$ the canonical thermal average. Then the thermal path-ordered propagator takes the form (D_c being the thermal propagator of the theory) :

$$D_C(x_1,x_2) = \left\langle \theta_C(t_1 - t_2) D_C^{>}(x_1,x_2) + \theta_C(t_2 - t_1) D_C^{<}(x_1,x_2)...\phi(x_n) \right\rangle \tag{77}$$

where θ_c is the path Heaviside function. Then the thermal bosonic two-point functions $D_C^{>}$, $D_C^{<}$ are defined as :

$$\begin{aligned} D_C^{>}(x_1,x_2) &= \left\langle \phi(x_1)\phi(x_2) \right\rangle_{ß} \\ D_C^{<}(x_1,x_2) &= \left\langle \phi(x_2)\phi(x_1) \right\rangle_{ß} \end{aligned} \tag{78}$$

At this stage, as proposed in [6], the Boltzman weight $e^{-ßH}$ can be seen as an evolution operator in Euclidean time, so that after a translation in imaginary time we get the formula (79) :

$$e^{-ßH}\phi(t,\mathrm{x})e^{ßH} = \phi(t + iß,\mathrm{x}) \tag{79}$$

which is exactly the KMS condition formulated in equ. (53). Then $D_C^{>}(x_1,x_2)$ in equ.(78) becomes :

$$D_C^{>}(x_1,x_2) = \frac{1}{Z(ß)} \mathrm{Tr}\left[e^{-ßH}\phi(x_1)\phi(x_2) \right] \tag{80}$$

Likewise for $D_C^{<}(x_1,x_2)$. So using the cyclicity of the thermal trace and the notion of evolution in Euclidean time *it*, one can construct the "bosonic KMS condition" [13-28]. Interestingly, such a condition relates $D_C^{>}$ and $D_C^{<}$ by a translation in Euclidean (imaginary) time :

$$D_C^{>}(t_1;\mathrm{x}_1,t_2;\mathrm{x}_2) = D_C^{<}(t_1 + iß;\mathrm{x}_1,t_2;\mathrm{x}_2) \tag{81}$$

Of course the same construction holds for fermions. Indeed, defining the fermionic two-point function $S_{C_{ab}}^{>}$ and $S_{C_{ab}}^{<}$, (with a,b = 1 ...4 for Dirac four-components spinors) as

$$S_{C_{ab}}^{>}(x_1,x_2) = \langle \psi_a(x_1)\overline{\psi}_b(x_2)\rangle_ß$$

$$S_{C_{ab}}^{<}(x_1,x_2) = -\langle \overline{\psi}_b(x_2)\psi_a(x_1)\rangle_ß \tag{82}$$

and as in the bosonic case, the fermionic KMS condition takes the form :

$$S_{C_{ab}}^{>}(t_1;\mathrm{x}_1,t_2;\mathrm{x}_2) = -S_{C_{ab}}^{<}(t_1 + iß;\mathrm{x}_1,t_2;\mathrm{x}_2) \tag{83}$$

which differs from the bosonic condition only by a relative sign. From the structure of equ.(81) and equ.(83), we deduce that when the temperature of the supersymmetric system (here the (pre)space-time) is *not* zero, then bosonic fields are *periodic* in imaginary time whereas fermionic fields are *antiperiodic*. Let's remark that supersymmetry algebra is not sensible to this periodicity-antiperiodicity distinction. If (as pointed in [13-28]) it is true that the supersymmetry breaking is "encoded" in this difference, the breaking becomes effective only when the KMS state is cancelled. For this reason, as demonstrated in the hereabove refs., the KMS condition must be applied to the *superfields* of the theory. In [13-28], the superfields are superspace expansions which contain as components the bosonic and fermionic degrees of freedom of supermultiplets. Such superfields are usually formulated using two-component Weyl spinors ψ_α and $\overline{\psi}^{\dot{\alpha}}$, related to Dirac spinors through $\psi_a = \dfrac{\psi_\alpha}{\overline{\psi}^{\dot{\alpha}}}$. Then the KMS condition for Dirac spinors can be extended to Weyl spinors and, in the same way, to Majorana spinors. The fermionic KMS condition for majorana spinors takes the form :

$$\begin{aligned} S_{C_\alpha}^{>\,\dot{\beta}}(x_1,x_2) &= \left\langle \psi_\alpha(x_1)\overline{\psi}^{\dot{\beta}}(x_2)\right\rangle_ß \\ S_{C_\alpha}^{<\,\dot{\beta}}(x_1,x_2) &= -\left\langle \overline{\psi}^{\dot{\beta}}(x_2)\psi_\alpha(x_1)\right\rangle_ß \end{aligned} \tag{84}$$

Now, one can realize that imposing the KMS condition to superfields components implies that one must also allow Grassmann parameters to depend on imaginary time. In fact, in the context of supersymmetry, the main question is the following : under thermal constraints, how do we successfully achieve the transformation of periodic bosons into antiperiodic fermions and vice-versa? The answer, developed in [13-28], consists in constructing the *thermal superspace*, i.e. in introducing time dependant and antiperiodic space-time coordinates. Henceforth, a point in thermal superspace has "KMS coordinates", given by a new set of Grassmannian variables:

$$\hat{X} = (x^{\mu}, \hat{\theta}^{\alpha}(t), \hat{\bar{\theta}}^{\dot{\alpha}}(t)) \tag{85}$$

where the symbol "^" denotes the thermal quantities and $\hat{\theta}^{\alpha}(t), \hat{\bar{\theta}}^{\dot{\alpha}}(t)$ are subject to the antiperiodicity conditions

$$\hat{\theta}^{\alpha}(t + iß) = -\hat{\theta}^{\alpha}(t), \quad \hat{\bar{\theta}}^{\dot{\alpha}}(t + iß) = -\hat{\bar{\theta}}^{\dot{\alpha}}(t) \tag{86}$$

Consequently, the condition (86) induces a temperature-dependant constraint on the time-dependant superspace Grassmann coordinates $\hat{\theta}^{\alpha}(t)$ and $\hat{\bar{\theta}}^{\dot{\alpha}}(t)$. From equ. (85), we finally observe that the KMS condition must be applied to the space-time metric itself, as formulated in §(5.1.2). Among the consequences, we are therefore induced to consider that the time-like direction must be extended in the complex plan (see §(5.1.3)).

Now, what does these results mean in the context of our research? As a matter of fact, Derendinger and Lucchesi have clearly confirmed that there exists a deep relation between (thermal) supersymmetry and KMS condition. This relation is implemented at the level of thermal Grasmann coordinates, because of the (anti)periodicity conditions given by equ.(86). Indeed, it has been proved by the authors that the only way to preserve supersymmetry in the thermal context is to consider that the space-time metric itself must be subject to the KMS condition. Otherwise, the periodic bosons and antiperiodic fermions *could not* be related by supersymmetry. Now, let's put this simple question : what happens when the KMS state collapses? The analysis of the "KMS Grassmann coordinates", in particular the equ.(86), clearly show that supersymmetry cannot be implemented without applying the

KMS condition to space-time coordinates. The reason of this is that when the space-time system is *not* subject to the KMS state (e.g. non-equilibrium state), a point X of superspace is endowed again with the usual Grassmann coordinates

$$X = (x^{\mu}, \theta^{\alpha}, \overline{\theta}^{\alpha})$$

This is equivalent to T=0 supersymmetry, for which the parameters of transformation (i.e. the Grassmannians θ and $\overline{\theta}$) are space-time *constants*. But rightly, the main result of refs [13-28] establishes without ambiguity that at finite temperature, one *cannot* make use of constant parameters in supersymmetry transformations rules. The supersymmetry parameters must be time dependant *variables*, (anti)periodic in imaginary time. So, in a natural way, the thermal Grassmann coordinates $\hat{X} = (x^{\mu}, \hat{\theta}^{\alpha}(t), \hat{\overline{\theta}}^{\dot{\alpha}}(t))$ must be "translated" in imaginary time and are consequently subject to the antiperiodicity conditions $\hat{\theta}^{\alpha}(t + iß) = -\hat{\theta}^{\alpha}(t)$ and $\hat{\overline{\theta}}^{\dot{\alpha}}(t + iß) = -\hat{\overline{\theta}}^{\dot{\alpha}}(t)$ of equ.(86). Obviously, the only way to implement such a condition is to consider that globally, the space-time system is in KMS state at a given scale (i.e. in our case between the scale zero and the Planck scale). Incidentally, the hereabove approach can be seen as a confirmation that the ß $\rightarrow$ 0 limit is topological. As a matter of fact, the ß $\rightarrow$ 0 limit of equ.(79) is given by the scalar field $\phi(x)$, which, by construction, is a topological configuration marking the origin of the imaginary time direction of the theory.

From hereabove we can now conclude that (thermal) supersymmetry and KMS states are linked in such a manner that *the breaking of the KMS state beyond the Planck scale should induce the breaking of supersymmetry at the same scale*. Let's go further in the exploration of such a breaking. In a very stimulating way, Derendinger and Lucchesi have emphasized the fact that the thermal field boundary conditions characterizing KMS state carry information that is of global nature in space-time. By construction, the supersymmetry algebra being a *local* structure is insensitive to this global information. What is the nature of this "global information"? Indeed, the translation of Grassmannians variables into imaginary (topological) time clearly indicates that the natural state of such a global information is a topological state, correctly described by topological field theory (which is precisely a *non local* theory). More exactly, the boundary conditions characterizing the euclidean time dependence of the supersymmetry parameters can be seen as topological invariants. In this perspective, supersymmetry breaking can then be investigated in terms of cancellation of such topological invariants. Let's now explore this occurrence.

5.2.2 *Supersymmetry and topological invariants*

In a famous precursor paper [36], and in some others, E. Witten has clearly put in evidence that if we want supersymmetry breaking to occur, the various four-manifolds invariants (such that the Donaldson invariant, the Euler number, the Witten index etc..) must necessarily vanish. The outline of the argument is that the canceling of the supersymmetry index $Tr(-1)^F$ is canceling the zero energy modes, which consequently breaks the Bose-Fermi pairs [1]. At this stage, if we agree with supersymmetry theory, a reasonable conclusion is that (N=2) supergravity breaking could be viewed as related to the canceling of topological configurations. Let's now go further : can supersymmetry breaking explain the Topological → Physical transition? In a certain sense, the answer might be yes. In fact, since the context of the theory is supergravity N= 2, we may precise the conditions of topological modes canceling within supersymmetry breaking. So :

Conjecture 5.2.3 *On a D = 4 Riemannian (pre)space-time manifold, the N = 2 supergravity breaking at the Planck scale is related to the canceling of the Euler characteristic and of the topological mode of the manifold.*

Let ***M*** be the four dimensional Riemannian N=2 supersymmetric (pre)space-time. The Euler characteristic of ***M*** is

$$\chi(\boldsymbol{M}) = \frac{1}{32\pi}\int_{M}\varepsilon_{\mu\nu\rho\sigma}R_{\mu\nu}\wedge R_{\rho\sigma}$$

We have shown in prop. (3.2) that this invariant is given by $\mathrm{Tr}(-1)^{S}$. Now, according toWitten's results [36], a discontinuous change of $\mathrm{Tr}(-1)^{S}$ is possible, due to the asymptotic behavior of the manifold, allowing, for large field strengths, some energy states to "move in from infinity". For instance, let's consider the potential

$$V(\phi) = (m\phi - g\phi^2)$$

One can easily observe that arbitrarily small $g \neq 0$ induces the existence of extra low-energy states at $\phi \sim m/g$ which have no counterpart for the pure g = 0 value. Therefore, $\mathrm{Tr}(-1)^F$ will change discontinuously from g = 0 to $g \neq 0$. The same result can be extended to $\mathrm{Tr}(-1)^S$, when coupling the instanton radius to g. In this case, we meet again the conclusions of (ii) in example (2.1) (i.e. the instanton configuration is cancelled for large values of g).

Next, we have seen (5.1.2) that the (pre)space-time should be in KMS state at ℓ_{Planck}, so that the time like direction t becomes holomorphic within the KMS strip. The metric configuration is described by the symmetric homogeneous space

$$\sum_h = \frac{SO(3,1) \otimes SO(4)}{SO(3)} \tag{87}$$

SO(3) being diagonally embedded in SO(3, 1) ⊗ SO(4) [6]. To $\sum_h$ corresponds, at the level of the underlying spaces involved, the topological quotient space $\sum_{top} = \dfrac{\mathbb{R}^{3,1} \oplus \mathbb{R}^4}{SO(3)}$ from which, assuming that the compact part of the 3-geometry is a sphere S^3, the topology of the five dimensional (pre)space-time can be viewed as isomorphic to $S^3 \otimes \mathbb{R}^{\pm}$ ($\mathbb{R}^+$ being the space-like direction and $\mathbb{R}^-$ the time-like direction, out of the orbit of the action of SO(3) on $\mathbb{R}^{3,1} \oplus \mathbb{R}^4$). We then meet again the equivalent form $S^3 \otimes \mathbb{R}^+ \otimes \mathbb{R}^-$ of the five dimensional manifold described in (2.1). The point is that $\mathbb{R}^-$ allows us to define the boundary conditions of the (pre)space-time 5-geometry $\mathbf{\Gamma}^5$. Therefore, the form of the 5D metric is [6] :

$$ds^2 = a(\omega)^2\, d\Omega^2_{(3)} + \frac{d\omega^2}{g^2} - dt^2 \tag{88}$$

where the axion term is $a = f(\omega\,,\ t)$, the 3-geometry $d\Omega^2_{(3)} = f(x, y, z)$. Then, as showed in (2.1), on the (infrared) strong coupling bound (i.e. the Planck scale, in respect of the (ultraviolet) zero scale), condition (**i**) imply $\dfrac{1}{g^2} \to 0$ and the ω direction of $\mathbf{\Gamma}^5$ is cancelled. So, we get a dimensional reduction (D=4 → D=3) of the compact Riemannian 4-geometry embedded in the five dimensional (pre)space-time manifold $\mathbf{\Gamma}^5$. We have for the metric :

$$(++++-) \xrightarrow{w\ \ compactification\ on\ \ S^1 \to 0} (+++(0)-) \xrightarrow{Dimensional\ reduction} (+++-).$$

Obviously, the boundary condition $ß \rightarrow \infty$ gives rise to the asymptotic cancellation of the Γ^4 Euler characteristic:

$$\chi(M) = \frac{1}{32\pi}\int_M \varepsilon_{u\nu\rho\sigma} R_{u\nu} \wedge R_{\rho\sigma} = \mathrm{Tr}(-1)^F = 0 \tag{89}$$

Likewise, the asymptotic flatness condition [6] for $ß \rightarrow \infty$ gives $R_{u\nu} \wedge R_{\rho\sigma} \rightarrow 0$, which implies that the dimension D of the asymptotic manifold must be *odd*, so that, again, we get $\chi = 0$ for the (3+1) usual space-time. Therefore, according to ref. [36], the supersymmetry is broken. Simultaneously, the topological state, given by *even* values of the Euler number χ vanishes, implying the "TP transition" :

$$\textit{Topological mode} \xrightarrow{TP \ \ transition} \textit{Physical mode} .$$

To finish, we meet a novel problem : could TP transition be, in some way, related to the breaking of the KMS state described in (5.1)? This question is discussed in the last paragraph.

5.2.4 *TP transition and decoupling between topological flow and physical flow*

In answer to the hereabove question, we now conjecture that for $ß \geq \ell_{Planck}$, i.e. at the (semi-classical) scale where supersymmetry is being broken, the topological flow (evolution in imaginary time) corresponding to the zero topological pole of the theory is *decoupled* from the physical flow (evolution in real time).

According to most of the models, supergravity is considered as broken for scales greater than the Planck scale [25]. But thermal supersymmetry breaking is also closely connected to the cancellation of the thermodynamical equilibrium state [27-28]. Indeed, as already pointed in this paper, I. Antoniadis *and al* have recently demonstrated that a five-dimensional (N = 4) supersymmetry can effectively be described by a four-dimensional theory in which supersymmetry is spontaneously broken by finite thermal effects [3]. In a similar way, Derendinger and Lucchiesi have outlined the fact that thermal supersymmetry is a global (i.e. topological) property of the space-time in KMS state [13-28]. In this context, the cancellation of the thermodynamical equilibrium state necessarily cancels the KMS state and, consequently, breaks the supersymmetry [6]. This scenario is typically the one characterizing our setting. As a matter of fact, the five dimensional supersymmetric theory evoked hereabove correponds to the five dimensional supersymmetric (pre)space-time in KMS state. Then the (thermal) supersymmetry breaking is characterized in Kounnas approach, by a D=5 → D=4 dimensional reduction, which corresponds exactly, in our case, to the decoupling between imaginary time and real time. Indeed, we could have :

$$\mathbb{R}^3 \otimes \mathbb{C} \textit{ (five dimensional KMS space - time)} \xrightarrow{\text{ss breaking}} \begin{cases} \mathbb{R}^3 \otimes \mathbb{R}^+ \textit{ (four dimensional topological space - tim} \\ \mathbb{R}^3 \otimes \mathbb{R} \textit{ (four dimensional physical space - time} \end{cases}$$

So, supersymmetry breaking, KMS breaking and topological → physical transition appear as deeply connected. To see this, let's come back to the KMS state. We call "KMS breaking" the end of the KMS state beyond the Planck's scale. The observed cancellation of the thermodynamical equilibrium beyond the Planck scale (which gives the inflationary phase and the beginning of the cosmological expansion) is inducing KMS breaking (see ex. (5.2.1)). Such a breaking must be seen as the inverse of the KMS coupling between equilibrium state and physical evolution of the system. And logically, such a breaking should bring about the transition from the pure (non perturbative) topological phase around the Initial Singularity to the physical phase of the universe we can observe to day.

Now, here is our conjecture :

Conjecture 5.2.5 *In the infrared* $ß \geq \ell_{Planck}$ *scale, KMS breaking is inducing the decoupling between the topological flow and the physical flow of the theory.*

Considering the KMS state of space-time at the Planck scale, the KMS flow, as shown in prop. (5.2), is :

$$\sigma_{ß_c}(\boldsymbol{M}_q) = \sigma_ß(\boldsymbol{M}_{\boldsymbol{q\text{-}state}}) \oplus \sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}}) = e^{ß_c H}\boldsymbol{M}_q e^{\;ß_c H} \tag{90}$$

or

$$\sigma_{ß_c}(\boldsymbol{M}_q) = e^{\;(ß+it)H}\boldsymbol{M}\mathrm{q}\; e^{(ß+it)H} \tag{91}$$

Now, starting from $\boldsymbol{M}\mathrm{q} = \boldsymbol{M}_{Top}^{0,1} \rtimes_\theta \mathscr{L}\left[L^2\left(\mathbb{R}_+^*/ß\mathbb{Z}\right)\right] = \boldsymbol{M}_{Top}^{0,1} \rtimes_\theta \mathscr{L}[L^2(ß\mathrm{S}^1)]$, we can say that $ß > \ell_{Planck}$ is equivalent to $ß \to \infty$, with respect to the scale zero. So, when $ß > \ell_{Planck}$, the period of the system is so large that we can consider it as supressed from equ.(62), whereas the circle S^1 is decompactified on the straight line R. Moreover, this limit corresponds to $\lambda = \frac{2\pi}{ß} \rightarrow 0$. So, on this limit $\mathbb{R}^*_+ / ß\mathbb{Z} \rightarrow \mathbb{R}^*_+$. But the suppression of the period $\mathbb{R}^*_+ / ß\mathbb{Z}$ is equivalent to the cancellation of the equilibrium state and therefore induces the breaking of the KMS state. To see this, we can write the "extended" automorphisms group corresponding to the KMS state :

$$\sigma_{ß_c}(\boldsymbol{M}_q) = e^{ß_c H}[\boldsymbol{M}^{0,1}_{Top} \rtimes_\alpha \mathbb{R}^*_+] e^{ß_c H} = e^{(ß+it)H}[\boldsymbol{M}^{0,1}_{Top} \rtimes_\alpha \mathbb{R}^*_+] e^{(ß+it)H} \tag{92}$$

Then for $ß >> \ell_{Planck}$, we get $\mathbb{R}^*_+ \rightarrow \infty$ so the corresponding weight φ on $\boldsymbol{M}$q is such that $\varphi \rightarrow \infty$. But, according to Connes-Takesaki [10], the infinite dominant weight on $\boldsymbol{M}$q is dual to the hyperfinite *trace* on $\boldsymbol{M}^{0,1}_{Top}$. Therefore, the image of the "flow of infinite weights" on $\boldsymbol{M}$q becomes, under the ergodic action of $\mathbb{R}^*_+$:

$$\sigma_{\beta\rightarrow\infty}(\boldsymbol{M}_{\boldsymbol{q\text{-}state}}) = \mathrm{Tr}_\infty(e^{ßH}\boldsymbol{M}^{0,1}_{Top}\ e^{ßH}) \tag{93}$$

where we meet again the topological "trace current" $\mathbf{T}^f$ of $\boldsymbol{M}^{0,1}_{Top}$, independent of *ß*. But the independence of $\mathbf{T}^f$ with respect to *ß* implies in the same way that $\mathbf{T}^f$ is also independent of $\mathbb{R}^*_+$ on this limit. So, $\mathbb{R}^*_+$ must be decoupled of $\boldsymbol{M}^{0,1}_{Top}$, which means that the modular evolution group $\sigma_t(\boldsymbol{M}_{Phys}) = \Delta^{it}\boldsymbol{M}_{Phys}\Delta^{-it}$ is itself decoupled from the crossed product (65). Moreover, since the hyperfinite trace (93) is independent of *ß*, we are left with the "topological" state :

$$\sigma_{\beta\rightarrow\infty}(\boldsymbol{M}_{\boldsymbol{q\text{-}state}}) \equiv \mathrm{Tr}_\infty(\boldsymbol{M}^{0,1}_{Top})$$

which is equivalent to say that the only value of *ß* contributing to equ. (79) is $ß = 0$. So, on this boundary, (see equ.(64)), $\sigma_{ß_c}(\boldsymbol{M}_q)$ is reduced to the real pole, so that :

$$\sigma_{ß_c}(\boldsymbol{M}_q) \rightarrow \sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}}) = e^{iHt}\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}}e^{-iHt},$$

But of course, in this case **M**q, as type III algebra, is also suppressed. This is simply because, on the infinite limit of the action of $\mathbb{R}$ on $\boldsymbol{M}^{0,1}_{Top}$, the infinite trace Tr_∞ on $\boldsymbol{M}^{0,1}_{Top}$, dual to the dominant weight on $\boldsymbol{M}$q, is left invariant. Applying a result of [11] on infinite weights, one can find that the infinite weight φ_∞ on $\boldsymbol{M}$q is invariant under the inner automorphisms of $\boldsymbol{M}$q. Therefore, φ_∞ is a trace, which is a sufficient condition to cancel $\boldsymbol{M}$q as a III$_\lambda$ factor. But this is equivalent to say that on this limit, the action of $\mathbb{R}$ is *decoupled* of $\boldsymbol{M}^{0,1}_{Top}$. Therefore, the crossed product (65) is broken into its two subgroups $\boldsymbol{M}^{0,1}_{Top}$ and $\mathscr{L}(L^2(\mathbb{R}^*{}_+))$. This is as it should be, since beyond the Planck scale, i.e. at the classical scale, the KMS state is broken and the measure space on the metric is again well defined, so that the underlying algebra must be endowed with a trace. Consequently, it cannot be **M**q anymore. So, the new algebra involved should be a type I$_\infty$ sub-algebra of **M**q. Considering the decomposition $\boldsymbol{M}^{0,1}_{Top} \rtimes_\alpha \mathscr{L}(L^2(\mathbb{R}^*{}_+))$, this sub-algebra is necessarily $\mathscr{L}(L^2(\mathbb{R}^*{}_+)) = \boldsymbol{M}_{Phys}$. Then $\sigma_t(\boldsymbol{M}_{\boldsymbol{q\text{-}evolution}})$ becomes simply :

$$\sigma_t(\boldsymbol{M}_{Phys}) = e^{\,iHt}\boldsymbol{M}_{Phys}e^{\,-\,iHt} \qquad (94)$$

This corresponds to the usual modular group giving the physical evolution of the space-time. So the product $\boldsymbol{M}^{0,1}_{Top} \rtimes_\alpha \boldsymbol{M}_{Phys}$ shrinks onto $\boldsymbol{M}_{Phys}$ so that we finally get $\underset{ß > \ell_{Planck}}{\boldsymbol{M}_q} = \boldsymbol{M}_{Phys}$ in the infrared.

In the same way, applying the result of prop. 5.2, we see that the breaking of KMS state implies

AUT Mq ≡ OUT Mq ⊤ INT$_{mod}$ Mq

reduces to the well known case of a factor I, where all the automorphisms of the algebra are inner automorphisms:

AUT Mq ≡ INT Mq

So obviously, this transition causes the decoupling between OUTMq and INT$_{mod}$ Mq , i.e. between the topological current $\sigma_ß(x)$ and the physical flow $\sigma_t^\psi(x)$.

As a result of prop.(5.2.5), we finally can conclude that the breaking of the KMS state beyond the Planck scale induces the decoupling between the physical flow $\sigma_t(\boldsymbol{M}_{Phys}) = e^{\,iHt}\boldsymbol{M}_{Phys}e^{\,-\,iHt}$ and the zero topological current

$\sigma_ß(\boldsymbol{M}^{0,1}_{Top}) = \text{Tr}_\infty(e^{-ßH}\ \boldsymbol{M}^{0,1}_{Top}\ e^{ßH})$:

$$\sigma_{ß_c}(\boldsymbol{M}_q) = e^{ß_c H}\, \boldsymbol{M}^{0,1}_{Top}\; e^{\;ß_c H} \xrightarrow{KMS\ \ breaking} \begin{cases} \to \mathrm{Tr}_\infty(e^{-\,ßH} \boldsymbol{M}^{0,1}_{Top}\; e^{\,ßH}) \\ \to e^{\,iHt} \boldsymbol{M}_{Phys} e^{\,-\,iHt} \end{cases} \tag{95}$$

At the level of the von Neumann algebras, starting from the KMS algebra $\boldsymbol{M}$q = $\boldsymbol{M}^{0,1}_{Top} \rtimes_\theta \boldsymbol{M}_{Phys}$, the KMS breaking can be seen as the decoupling between $\boldsymbol{M}^{0,1}_{Top}$ and $\boldsymbol{M}_{Phys}$. This decoupling describes the transition from the topological phase (zero scale) to the physical phase (beyond the Planck scale).

6. CONCLUSION

Even though certain of the hereabove results might seem mysterious, their interest is to outline, through quantum groups theory and non commutative geometry, a possible phase transition from the topological zero scale to the physical Planck scale. We describe with more details in a forthcoming paper the unexpected "algebraic blow up" of the topological initial singularity. At this stage, we propose to draw the following main ideas :

(**i**) the metric, onto the zero scale, might be considered as Euclidean (++++) i.e. *topological* ;

(**ii**) the Initial Singularity of space-time could be understood as a 0-size singular gravitational instanton;

(**iii**) From (**i**) and (**ii**), we suggest the existence of a deep symmetry, of the duality type (*i* - duality), between physical state (Planck scale) and topological state (zero scale).

Then the possible resolution of the initial singularity in the framework of topological theory allows us to envisage the existence, before the Planck scale, of a purely topological first phase of expansion of space-time, parameterized by the growth of the dimension of moduli space dim$\boldsymbol{M}$ and described by the Euclidean "pseudo-dynamic" :

$$\sigma_ß\,(M^{0,1}_{Top}) = e^{-\,ß\,H}\;\; M^{0,1}_{Top}\;\; e^{\,ß\,H}$$

So, the chain of events able to explain the transition from the zero topological phase to the physical phase of the space-time might be the following :

$$\{Supersymmetry\ breaking\} \rightarrow \{thermodynamical\ equilibrium\ breaking\} \rightarrow \{KMS\ state\ breaking\} \rightarrow$$
$$\rightarrow \{imaginary\ time / real\ time\ decoupling\} \rightarrow \{topological\ state\ /\ physical\ state\ decoupling\}$$

In terms of C*-algebras, the hereabove transformations are given by :

$$II_{\infty} \otimes \mathbb{R}_{+}^{*} \xrightarrow[0<ß<\ell P]{\text{KMS flow}\quad \mathbf{Q}^{f}} \alpha_{ß_c}(\boldsymbol{M_q}) = e^{-ß_c H}\boldsymbol{M_q}\, e^{ß_c H} \begin{cases} \xrightarrow{\text{Topological flow}} \underset{ß=0}{\mathbf{T}^{f}} \rightarrow \alpha_{ß}(\boldsymbol{M}_{Top}^{0,1}) = e^{-ßH}\boldsymbol{M}_{Top}^{0,1} e^{ßH} \\ \xrightarrow{\text{Physical flow}} \underset{ß>0}{\mathbf{P}^{f}} \rightarrow \alpha_t(\boldsymbol{M}_{Phys}) = e^{iHt}\boldsymbol{M}_{Phys}\, e^{-iHt} \end{cases}$$

In a forthcoming article, we push forward the idea following which, that at 0 scale, the Lorentzian dynamic is replaced by an intrinsic "Euclidean dynamic". A first path to follow would be to investigate the zero limit of the The Euclidean dynamic engendered by the non-stellar automorphisms of the algebra $M_{Top}^{0,1}$ implies, following the results of [6], a "spectral increase" in the diameters of the space of states d(φ , ψ) in Euclidean time (dual to the space of observables in Lorentzian time). This Euclidean pseudo-dynamic, linked with semi-group automorphisms $\sigma_ß(M_{Top}^{0,1})$ is described in a natural way by the flow of weights (in the Connes-Takesaki [9] sense) of algebra Mq ; we suggest equally (**ii**) that the Euclidean modular flow representing the evolution of a system in imaginary time can be associated with an increase in the spectral distance separating the states of the system. Finally, it has been proposed by one of us [7] that the Euclidean dynamic raised above results from the existence of the topological amplitude yield by the topological charge Q = $\theta\int d^4x\ \mathrm{Tr}\ R_{\mu\nu}\tilde{R}^{\mu\nu}$ of the zero size singular gravitational instanton connected to the (topological) origin of space-time.

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