

hep-ph/0011270  
CERN-TH/2000-252  
Edinburgh 2000-24  
RM3-TH/2000-13

# Small- $x$ Resummation and HERA Structure Function Data

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## Abstract

We apply our systematic NLO small  $x$  resummation of singlet splitting functions to the scaling violations of structure functions and compare the results with data. We develop various theoretical tools which are needed in order to relate resummed parton distributions to measurable structure functions, and we present results from a variety of fits to HERA data for the structure functions  $F_2$  and  $F_L$  using the resummation. The behaviour of the singlet splitting functions at small  $x$  and fixed  $Q^2$  is effectively parametrized as  $x^{-\lambda}$ . We find that, for  $\lambda$  small or negative, the resummed description of scaling violations may be phenomenologically as good as or even better than the standard next-to-leading order treatment. However, the best fit gluon density and value of  $\alpha_s$  can be significantly modified by the resummation.

CERN-TH/2000-252  
November 2000

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## 1. Introduction

A complete understanding of scaling violations at small  $x$  in deep inelastic structure functions within the framework of perturbative QCD remains elusive. Whereas techniques for the inclusion of small  $x$  contributions to leading twist evolution equations have been known for some time [1,2], their precise implications have been rather unclear. On the phenomenological side, while the corrections should be sizeable, the next-to-leading order (NLO) description of HERA structure function data is so successful [3] that there seems to be little room for improvement [4]. On the theoretical side, the evaluation of small  $x$  corrections to the singlet splitting function using the BFKL theory [5,6] appears to fail: the recent calculation [7–10] of the NLO contribution  $\chi_1$  to the kernel  $\chi = \alpha_s \chi_0 + \alpha_s^2 \chi_1 \dots$  shows that the expansion is very badly behaved, as the non leading term completely changes the qualitative features of the leading order. However, recent studies have led to considerable progress: in fact, it turns out that by assuming the simultaneous validity of leading twist  $Q^2$  evolution and the BFKL equation, most of the theoretical problems encountered in this approach can be overcome.

At large  $Q^2$  and not too small but fixed  $x$  the QCD evolution equations for parton densities [11] provide the basic framework for the description of scaling violations. The complete splitting functions have been computed in perturbation theory at order  $\alpha_s$  (LO approximation) and  $\alpha_s^2$  (NLO) [12]. For the first few moments the anomalous dimensions at order  $\alpha_s^3$  are also known [13]. However, at sufficiently small  $x$  the computation of the splitting functions based on only the first few terms of the expansion in powers of  $\alpha_s$  cannot in general be a good approximation. Even assuming that a leading twist description of scaling violations is still valid in some range of small  $x$ , as soon as  $x$  is small enough that  $\alpha_s \xi \sim 1$ , with  $\xi = \log 1/x$ , all terms of order  $\alpha_s (\alpha_s \xi)^n$  and  $\alpha_s^2 (\alpha_s \xi)^n$  which are present [6] in the splitting functions must be considered in order to achieve an accuracy up to terms of order  $\alpha_s^3$ .

In most of the kinematic region of HERA [3] the condition  $\alpha_s \xi \sim 1$  is indeed true. Since  $\alpha_s \xi \leq \alpha_s \log s/Q^2$ , and  $s \approx 10^5 \text{ GeV}^2$ ,  $\alpha_s(m_z^2) \approx 0.119$ , we see that at  $Q^2 = 3, 10, 10^2, 10^3 \text{ GeV}^2$ ,  $\alpha_s \xi$  can be as large as 4.3, 3.0, 1.2, 0.6, respectively. So one would expect many terms of the series to be important and, consequently, to see in the data indications of significant corrections to the approximation [14,15] in which only terms up to order  $\alpha_s^2$  are kept. In reality this appears not to be the case: the scaling violations in the data are in excellent agreement with the predictions of the leading twist evolution equations in the NLO approximation [3]. Of course it may be that some corrections exist but that they are hidden by a modification of the fitted gluon, which is the dominant parton density at small  $x$ , or by a change of the measured value of  $\alpha_s$ . However, a straightforward inclusion of small  $x$  corrections completely spoils [4] the agreement between the data and the NLO calculation.

From a theoretical viewpoint, the inclusion of contributions to perturbative QCD evolution to next-to-leading log  $x$  presents two main problems. The first is that the qualitative features of the BFKL evolution kernel at leading- and next-to-leading log  $x$  are completely different. In particular, leading log  $x$  evolution leads to a parton distribution which rises as a power when  $x \rightarrow 0$  at fixed  $Q^2$  [5], while at the next-to-leading level the structure function displays an unphysical oscillatory behaviour [16]. This is because at next-to-leading

order the function  $\chi(M)$  has a qualitatively different shape as a function of  $M$  (the variable which is conjugate to  $\log Q^2$  upon Mellin transformation): in particular near  $M = 0$   $\chi_0 \sim 1/M$ , while  $\chi_1 \sim -1/M^2$ . Furthermore, even if this problem is side-stepped by treating the subleading correction near  $M = \frac{1}{2}$  as a perturbation, there is a new difficulty, namely, that the ensuing correction to the asymptotic behaviour is very large [17,18].

The first problem is related to the presence of unresummed large logs of  $Q^2$  in the leading  $\ln x$  resummation [19–21]. It was recently shown by us [22] that it is possible to implement a reorganization of the perturbative expansion such that large logs of  $Q^2$  and  $\frac{1}{x}$  are both resummed simultaneously (double-leading expansion [1]). This can be achieved by exploiting the fact that when  $Q^2$  and  $\frac{1}{x}$  are both large, the perturbative evolution admits a dual description [22,19], either in terms of an evolution equation in  $t = \ln Q^2/\Lambda^2$  (the usual leading twist evolution equation [11]) or in terms of an equation in  $\xi = \ln 1/x$  (the BFKL equation [5]): leading and next-to-leading large logs can be determined from the knowledge of the fixed order kernels in both equations, and then used to construct double-leading kernels appropriate to either equation.

It should be clear that our function  $\chi(M)$  which corresponds by duality to the leading twist anomalous dimension order by order in perturbation theory is not precisely the same as the Mellin transform of the kernel of the  $\xi$  evolution equation: it cannot be since beyond LLx the latter is in general a differential operator. However the two are closely related. At LLx they are equal. If  $\alpha_s$  does not run, they are simply related to all orders through a change of scale [7]. When the coupling runs, duality still holds, however the relation becomes more complicated: it may be specified order by order at NLLx, NNLLx, etc. [23], but to all orders becomes ambiguous. What we do in the following is to extract unambiguous information from the NLLx BFKL kernel in order to learn about  $\chi(M)$  and thus by duality about the small  $x$  singularities of the leading twist anomalous dimension. Similarly, we use the NLO anomalous dimension to infer, by duality, information about the small  $M$  singularities of  $\chi(M)$ . Combining these two independent sources of information, we are able to construct the double-leading expansion in which all large leading and next-to-leading logarithms are correctly resummed at leading twist into an anomalous dimension: logarithms of  $x$  at NLLx, logarithms of  $Q^2$  at NLLQ through the usual renormalization group improvement.

Since logarithms of  $Q^2$  are resummed, the double-leading expansion turns out to be free of qualitative instabilities, thereby providing a complete solution [22] of the first problem mentioned above. The physical origin of this stability is the powerful constraint of momentum conservation, which is automatically taken into account in the double-leading expansion. By duality, this constraint fixes the all-order value of the kernel  $\chi(M)$  at  $M = 0$ , thereby stabilizing the expansion of the kernel in the neighbourhood of this point. At this stage one already understands why the corrections at small  $x$  are not catastrophic. It is important to notice that the possibility of matching evolution in  $\ln Q^2$  and evolution in  $\ln 1/x$  [22] shows that there is no breakdown of factorization at small  $x$  and that the assumption of the dominance of leading twist terms is tenable, in agreement with the conclusions of refs. [24], based on model calculations. Our solution of the first problem is similar to that of ref. [25] which was obtained from a resummation of the BFKL kernel, but we think that our approach, which also includes all the information

contained in the conventional NLO leading twist evolution equation, is more general and avoids model dependent assumptions. Alternative approaches to this problem have also been suggested [26].

The second difficulty mentioned above arises due to the fact that the asymptotic behaviour at small  $x$  and fixed  $Q^2$  (Regge limit) of the perturbative evolution is subject to large corrections because nonleading terms of order  $\alpha_s(Q^2)(\alpha_s(Q^2)\xi)^n$  are not asymptotically small for  $x \rightarrow 0$  at fixed  $Q^2$ . The effect of these terms is large and makes the small  $x$  expansion unstable. However it was recognized [20,27,23] that this instability is due to the fact that higher order contributions to the structure function must change the asymptotic small  $x$  behaviour from  $x^{-\lambda_0}$  to  $x^{-\lambda} = x^{-\lambda_0} e^{\Delta\lambda\xi} \approx x^{-\lambda_0} [1 + \Delta\lambda\xi + \dots]$ , with  $\lambda_0 = \alpha_s\chi_0(\frac{1}{2})$  and  $\Delta\lambda = \alpha_s^2\chi_1(\frac{1}{2}) + \dots$ . This problem can be cured by resumming the contribution to the asymptotic behaviour, i.e. effectively treating the whole of  $\lambda$  as a leading order term. Indeed, in ref. [23] it was proven that this can be done order by order, thereby removing the uncontrolled growth of the subleading small  $x$  contributions mentioned previously. This procedure requires knowledge of the asymptotic behaviour in the Regge limit to all orders. However the true behaviour is quite possibly nonperturbative and, in any case, certainly cannot be reliably evaluated by just a few perturbative terms (the NLO term is larger than the LO one for realistic values of  $\alpha_s(Q^2)$ ), even though it can of course be computed within specific models [24–26]. We prefer to take a more general approach, and thus, at fixed  $Q^2$ , we treat  $\lambda$  as a parameter to be fixed from the data.

For a given value of  $\lambda$  one obtains for the double-leading resummed splitting functions a well behaved perturbative expansion which resums all leading logs of  $x$  and  $Q^2$ , and should thus give an accurate description of scaling violations in a wide region of  $x$  and  $Q^2$ . There is still a residual ambiguity, due to the fact that, in order to avoid double counting, in the double-leading expansion one must subtract order by order terms which contain leading logs of both  $x$  and  $Q^2$ , and would thus otherwise be included twice. Since the small  $x$  part of the double-leading expansion has been stabilized by the resummation, there is an ambiguity in the treatment of these double-counting terms, in the sense that one can choose whether or not to include them in the resummation.

We will see that for reasonably small values of  $\lambda$  ( $\lambda \lesssim 0.2$ ) the resummed structure functions are quite close to the standard two-loop results, especially if the remaining ambiguity is solved in such a way that resummed splitting functions are as close as possible to the two-loop ones. Thus, the success of the two-loop description of structure function data from HERA can be understood. However, even in this case, we still find that the resummation can have a significant impact on the extraction of parton distributions and of the value of  $\alpha_s$  from the data. Thus one important practical conclusion of our work is that the gluon and the strong coupling extracted from the NLO fit to HERA data at small  $x$  are to some extent biased. The precise size and direction of this bias depends on the particular resummation adopted and on the value of  $\lambda$ . This introduces an extra source of uncertainty in parton distributions and on the value of  $\alpha_s$  extracted from scaling violations, which must be taken into account for a correct estimate of the associated theoretical error.

It is the purpose of this paper to discuss the phenomenology of structure functions at small  $x$  on the basis of the treatment of splitting functions developed in ref. [22]. Our

aim is to provide a comprehensive self-contained treatment, which on the one hand gives all the technical details which are needed for a practical implementation of the resummation procedure, and, on the other hand, discusses several theoretical issues raised by our resummation method. As discussed above, our resummation consists of two steps: first, one constructs the double-leading expansion of anomalous dimensions and their associated splitting functions, and then, the small  $x$  asymptotics is resummed thereby introducing a dependence on the parameter  $\lambda$ . Likewise, the resummation of structure function proceeds in two corresponding steps. First (sec. 2), we construct the double-leading expansion for the physically observable structure functions  $F_2$  and  $F_L$ . Then (sec. 3), we resum the ensuing expression of the structure function by extracting the asymptotic behaviour in the Regge limit. The choice of factorization scheme and the theoretical ambiguities which it induces in the resummation procedure and running of the coupling, are discussed in sec. 4. Readers who are only interested in the construction of the resummed structure functions in the DIS scheme can skip sec. 4, and turn to sec. 5, where we fit recent HERA data on the  $F_2$  and  $F_L$  structure functions within our framework, and show that the resummation somewhat improves the quality of the fit and its stability and has a significant impact on the determination of the gluon density and strong coupling constant.

## 2. Double-leading expansion

In our previous work [22] we have derived a resummed expression for the largest eigenvalue of the singlet anomalous dimension matrix and its associated splitting function in the double-leading expansion. In this section, we construct the double-leading expansion of the full matrix of anomalous dimensions and coefficient functions, and construct the resummed structure functions  $F_2$  and  $F_L$  in the DIS scheme, which we will later use for the fits to data. Other schemes will be discussed in sec. 4. We will henceforth adopt the notation, conventions and terminology of ref. [22]. Even though we will recall all results which are needed for the resummation of the structure function, we refer to ref. [22] for a more complete derivation of the resummation at the level of the largest eigenvector.

### 2.1. Double leading anomalous dimensions

First, we wish to discuss the full two by two matrix of anomalous dimensions. Recall that [6] only the gluon entries of the anomalous dimension matrix contain leading  $\log x$  (LLx) singularities, i.e.

$$\gamma_{\text{LLx}} = \begin{pmatrix} 0 & 0 \\ \gamma_s^{gq} \left(\frac{\alpha_s}{N}\right) & \gamma_s^{gg} \left(\frac{\alpha_s}{N}\right) \end{pmatrix}, \quad (2.1)$$

where the nonvanishing entries satisfy the color-charge relation

$$\gamma_s^{gq} = \frac{C_F}{C_A} \gamma_s^{gg}, \quad (2.2)$$

where the color factors are  $C_F = \frac{n_c^2 - 1}{2n_c}$  and  $C_A = n_c$ . This implies that at this order the eigenvalues of the anomalous dimension matrix are  $\gamma^+ = \gamma_s^{gg}$  and  $\gamma^- = 0$ . Because the

small eigenvalue vanishes at the LLx level, it is possible to set it to zero to all orders by choice of factorization scheme [28,19].

At the next-to-leading log  $x$  (NLLx) level the eigenvalues are given by [1]

$$\begin{aligned}\gamma_{\text{NLLx}}^+ &= \gamma_s^{gg} + \alpha_s \left( \gamma_{ss}^{gg} + \frac{\gamma_s^{gq}}{\gamma_s^{gg}} \gamma_{ss}^{gq} \right) + O(\alpha_s^2) \\ \gamma_{\text{NLLx}}^- &= \alpha_s \left( \gamma_{ss}^{qq} - \frac{\gamma_s^{gq}}{\gamma_s^{gg}} \gamma_{ss}^{gq} \right) + O(\alpha_s^2).\end{aligned}\tag{2.3}$$

while the corresponding eigenvectors are

$$\begin{aligned}Q^+ &= \alpha_s \frac{\gamma_{ss}^{gg}}{\gamma_s^{gg}} G^+ + O(\alpha_s^2), \\ Q^- &= -\frac{\gamma_s^{gq}}{\gamma_s^{gg}} G^- + O(\alpha_s).\end{aligned}\tag{2.4}$$

It is easy to see that a sufficient condition for singular contributions to  $\gamma^-$  to vanish at this order is that a color-charge relation also holds in the quark sector, i.e. that

$$\gamma_{ss}^{qq} = \frac{C_F}{C_A} (\gamma_{ss}^{gq} - e_0^q),\tag{2.5}$$

where  $e_0^q$  is a scheme dependent constant. This condition is respected in both the  $\overline{\text{MS}}$  and DIS schemes [29], and we will henceforth only consider factorization schemes in which it is satisfied.

Since in any such scheme the small eigenvalue  $\gamma^-$  is regular at  $N = 0$ , and therefore unaffected by the summation of leading log  $x$  singularities, it is possible [17,18] using eq. (2.3) to reconstruct the full matrix of anomalous dimensions to next-to-leading log  $x$  from the knowledge of the NLLx large eigenvalue [7], and of the NLLx quark anomalous dimension [29]. However here we are interested in the double-leading expansion of the evolution equations. In this expansion, the anomalous dimension is constructed by adding the usual one and two loop contributions on top of the leading and subleading singularities, and subtracting the double counting [22]:

$$\begin{aligned}\gamma_{\text{DL}}^+(N, \alpha_s) &= \left[ \alpha_s \gamma_0^+(N) + \gamma_s^+ \left( \frac{\alpha_s}{N} \right) - \frac{n_c \alpha_s}{\pi N} \right] \\ &+ \alpha_s \left[ \alpha_s \gamma_1^+(N) + \gamma_{ss}^+ \left( \frac{\alpha_s}{N} \right) - \alpha_s \frac{e_1^+}{N} - e_0^+ \right] + \dots,\end{aligned}\tag{2.6}$$

where  $\gamma_0^+$  and  $\gamma_1^+$  are the one and two loop contributions to the largest eigenvalue,  $\gamma_s^+$  and  $\gamma_{ss}^+$  are the leading and subleading singularities, and the double-counting subtractions in the DIS and  $\overline{\text{MS}}$  schemes are given by  $e_1^+ = n_f n_c (5 + 13/(2n_c^2))/(18\pi^2)$  and  $e_0^+ = -(\frac{11}{2}n_c^3 + n_f)/(6\pi n_c^2)$ . The small eigenvalue  $\gamma^-$  instead simply coincides with its two loop form:

$$\gamma_{\text{DL}}^-(N, \alpha_s) = \alpha_s \gamma_0^-(N) + \alpha_s^2 \gamma_1^-(N).\tag{2.7}$$

Likewise, we can construct a double-leading anomalous dimension matrix as

$$\gamma_{\text{DL}}^{ij}(N, \alpha_s) = \left[ \alpha_s \gamma_0^{ij}(N) + \gamma_s^{ij} \left( \frac{\alpha_s}{N} \right) - \text{d.c.} \right] + \alpha_s \left[ \alpha_s \gamma_1^{ij}(N) + \gamma_{ss}^{ij} \left( \frac{\alpha_s}{N} \right) - \text{d.c.} \right] + \dots,\tag{2.8}$$

where  $i, j = q, g$  and double counting term are subtracted as in (2.6). It is easy to prove that the eigenvalues of this double-leading anomalous dimension matrix are given order by order in perturbation theory by the double-leading sum of the eigenvalues  $\alpha_s \gamma_0^\pm + \alpha_s^2 \gamma_1^\pm + \dots$  of the large  $x$  evolution matrices  $\alpha_s \gamma_0^{ij} + \alpha_s^2 \gamma_1^{ij} + \dots$  and eigenvalues  $\gamma_s^\pm + \alpha_s \gamma_{ss}^\pm + \dots$  of the small  $x$  evolution matrices  $\gamma_s^{ij} + \alpha_s \gamma_{ss}^{ij} + \dots$ , after subtracting the double counting terms in each case. However because the eigenvalues depend nonlinearly on matrix elements, this linear relation between the double-leading eigenvalues and the eigenvalues of the small  $x$  and large  $x$  evolution matrices only holds order by order up to subleading corrections. Therefore, it is in practice more convenient to use as primary quantities the eigenvalues (2.6),(2.7) and two matrix elements, and determine the other two matrix elements from them.

We thus choose to adopt the eigenvalues and quark sector matrix elements  $\gamma^{qq}$  and  $\gamma^{qg}$  as primary quantities. This choice is motivated by the fact that the large eigenvector is the primary quantity in the small  $x$  evolution equation (BFKL equation) whose kernel determines the small  $x$  contributions to the anomalous dimension  $\gamma^+$  by duality [22]; while the quark sector small  $x$  anomalous dimensions are directly determined through all-order small  $x$  factorization ( $k_T$  factorization) of the deep-inelastic structure functions  $F_2$  and  $F_L$  [29]. In double leading expansion

$$\begin{aligned}\gamma_{\text{DL}}^{gg}(N, \alpha_s) &= \alpha_s \gamma_0^{gg}(N) + \alpha_s \left[ \alpha_s \gamma_1^{gg}(N) + \gamma_{ss}^{gg} \left( \frac{\alpha_s}{N} \right) - \alpha_s \frac{e_1^q}{N} - e_0^q \right] + \dots \\ \gamma_{\text{DL}}^{qq}(N, \alpha_s) &= \alpha_s \gamma_0^{qq}(N) + \alpha_s \left[ \alpha_s \gamma_1^{qq}(N) + \frac{C_F}{C_A} \left( \gamma_{ss}^{qq} \left( \frac{\alpha_s}{N} \right) - \alpha_s \frac{e_1^q}{N} - e_0^q \right) \right] + \dots\end{aligned}\quad (2.9)$$

where (in DIS scheme)  $e_0^q = n_f/6\pi$ ,  $e_1^q = 13n_c n_f/36\pi^2$ , and in the second expression we have used the color-charge relation (2.5). Given the eigenvalues and quark entries we can then determine  $\gamma^{gg}$  from the trace condition, and  $\gamma^{qg}$  from the determinant:

$$\gamma^+ + \gamma^- = \gamma^{gg} + \gamma^{qq}, \quad \gamma^+ \gamma^- = \gamma^{gg} \gamma^{qq} - \gamma^{qg} \gamma^{gq}. \quad (2.10)$$

Note however that while to determine  $\gamma^{gg}$  at NLLx it is sufficient to know the eigenvalues and  $\gamma^{qq}$  at NLLx, to fix  $\gamma^{qg}$  from the determinant condition would require  $\gamma^{qq}$  and  $\gamma^{qg}$  at NNLLx. It follows that the value of  $\gamma^{qg}$  at NLLx is of no consequence for a NLLx calculation.

As already discussed in ref. [22], at next-to-leading order in the double leading expansion, momentum conservation will be violated by next-to-next-to leading terms, and can thus be restored by adding a subleading correction. Indeed, momentum conservation implies that, at  $N = 1$ ,  $\gamma^{qq}(1, \alpha_s) + \gamma^{gq}(1, \alpha_s) = 0$  and  $\gamma^{qg}(1, \alpha_s) + \gamma^{gg}(1, \alpha_s) = 0$ , so

$$\gamma^+(1, \alpha_s) = 0, \quad \gamma^-(1, \alpha_s) = \gamma^{qq}(1, \alpha_s) - \gamma^{qg}(1, \alpha_s). \quad (2.11)$$

While in DIS or  $\overline{\text{MS}}$  the one and two loop contributions to the large eigenvalue  $\gamma_{\text{DL}}^+$  eq. (2.6) already vanish at  $N = 1$ , the singular contributions  $\gamma_s$  and  $\gamma_{ss}$  give a non-vanishing contribution of  $O(\alpha_s^3)$ . Since this violation is sub-subleading we are free to remove it by subtraction, i.e. by setting

$$\gamma_{\text{DL}}^+(N, \alpha_s)|_{\text{mom}} = \gamma_{\text{DL}}^+(N, \alpha_s) - \gamma_{\text{DL}}^+(1, \alpha_s). \quad (2.12)$$

Likewise, the condition on the small eigenvalue  $\gamma^-$  is automatically respected by the one and two loop contributions to  $\gamma_{\text{DL}}^{gg}$  and  $\gamma_{\text{DL}}^{qq}$  eq. (2.9), but is violated by the singular terms  $\gamma_{ss}^{gg}$ : recalling that  $\gamma^-$  is free of  $N = 0$  singularities, one sees that the condition on  $\gamma^-$  eq. (2.11) would require that, when  $N = 1$ ,  $\gamma_{ss}^{gg}(\alpha_s) = \alpha_s e_1^q + e_0^q$ . So the small eigenvalue condition is also violated by terms of  $O(\alpha_s^3)$ , and can be restored by a sub-subleading subtraction:

$$\begin{aligned}\gamma_{\text{DL}}^{gg}(N, \alpha_s)|_{\text{mom}} &= \gamma_{\text{DL}}^{gg}(N, \alpha_s) - \alpha_s(\gamma_{ss}^{gg}(\alpha_s) - \alpha_s e_1^q - e_0^q) \\ \gamma_{\text{DL}}^{qq}(N, \alpha_s)|_{\text{mom}} &= \gamma_{\text{DL}}^{qq}(N, \alpha_s) - \frac{C_F}{C_A} \alpha_s(\gamma_{ss}^{gg}(\alpha_s) - \alpha_s e_1^q - e_0^q),\end{aligned}\tag{2.13}$$

consistent with the color-charge relation (2.5).

The anomalous dimension matrix to NLO in the double-leading expansion is fully determined by eq. (2.10), using the double-leading eigenvalues eqs. (2.12),(2.7) and the quark matrix elements eq. (2.13). All the quantities which are needed for a NLO computation are known. Specifically, the NLO singularities of the large eigenvector  $\gamma_{ss}^+$  can be determined using the NLO duality relations

$$\chi_0(\gamma_s^+(\frac{\alpha_s}{N})) = \frac{N}{\alpha_s},\tag{2.14}$$

$$\gamma_{ss}^+(\frac{\alpha_s}{N}) = -\frac{\chi_1(\gamma_s^+(\frac{\alpha_s}{N}))}{\chi_0'(\gamma_s^+(\frac{\alpha_s}{N}))},\tag{2.15}$$

in terms of the well-known BFKL [5] kernel

$$\chi_0(M) = -\frac{n_c}{\pi} [\psi(M) + \psi(1-M) - 2\psi(1)]\tag{2.16}$$

and the NLO kernel  $\chi_1$ , which was determined in refs.[7–10] in a scheme which is closely related to  $\overline{\text{MS}}$ , and in the  $\overline{\text{MS}}$  scheme is given by [23]

$$\chi_1(M) = \frac{1}{4\pi^2} n_c^2 \tilde{\delta}(M) + \frac{1}{8\pi^2} \beta_0 n_c (2\psi'(1) - \psi'(M) - \psi'(1-M)) + \frac{1}{4n_c^2} \chi_0(M)^2,\tag{2.17}$$

where the function  $\tilde{\delta}$  is defined in the first of ref. [7], and  $\beta_0 = \frac{11}{3}n_c - \frac{2}{3}n_f$ . As we will show below, the same expression also holds in the DIS scheme. The quark singular terms  $\gamma_{ss}^{gg}$ ,  $\gamma_{ss}^{qq}$  have been computed in the  $\overline{\text{MS}}$  and DIS scheme in ref. [29]: in DIS

$$\alpha_s \gamma_{qg}^{ss}(\frac{\alpha_s}{N}) = h_2(\gamma_s(\frac{\alpha_s}{N})) R(\gamma_s(\frac{\alpha_s}{N})),\tag{2.18}$$

where  $h_2(M)$  is a process dependent contribution (given by eq. (5.20) of ref. [29]), while  $R(M)$  is the process independent gluon normalization factor (given in ref. [29] as the solution (3.17) of the differential equation (B.18)).

In practice the anomalous dimensions are evaluated by first computing the coefficients in their expansions in powers of  $\frac{\alpha_s}{N}$ :

$$\begin{aligned}\gamma_s(\frac{\alpha_s}{N}) &= \frac{1}{4\ln 2} \sum_{n=1}^{\infty} a_n \left(\frac{\bar{\alpha}_s}{N}\right)^n, \\ \gamma_{ss}(\frac{\alpha_s}{N}) &= -\frac{11n_c}{12\pi} \left(1 + \sum_{n=1}^{\infty} b_n^0 \left(\frac{\bar{\alpha}_s}{N}\right)^n\right) - \frac{n_f}{54\pi} \left(1 + \sum_{n=1}^{\infty} b_n^f \left(\frac{\bar{\alpha}_s}{N}\right)^n\right) \\ \gamma_{ss}^{gg}(\frac{\alpha_s}{N}) &= \frac{n_f}{6\pi} \left(1 + \sum_{n=1}^{\infty} c_n \left(\frac{\bar{\alpha}_s}{N}\right)^n\right),\end{aligned}\tag{2.19}$$



where  $\bar{\alpha}_s \equiv 4 \ln 2n_c \alpha_s / \pi$ , and the coefficients are thus normalised so that each series has radius of convergence unity. The corresponding series for the splitting functions are uniformly convergent for all  $x > 0$  [1]: enough coefficients for sufficiently precise calculations in the HERA kinematic region are given in tables 1 and 2.

To determine the evolution of the singlet quark and gluon by solution of the singlet evolution equations an explicit determination of the complete two by two matrix of anomalous dimensions is in fact not necessary. Rather, we can conveniently construct the solution from the eigenvalues and quark-sector entries by means of a projector formalism [30]. The anomalous dimension matrix is decomposed as

$$\gamma = M_+ \gamma^+ + M_- \gamma^- \quad (2.20)$$

where  $\gamma^\pm$  are the eigenvalues of the matrix  $\gamma$ , and the projectors satisfy

$$M_+ + M_- = \mathbb{1}; \quad M_\pm M_\pm = M_\pm; \quad M_+ M_- = 0. \quad (2.21)$$

Explicitly, the projectors are given in terms of the eigenvalues and quark-sector entries of the anomalous dimension matrix by

$$\begin{aligned} M_+ &= \frac{1}{\gamma^+ - \gamma^-} \begin{pmatrix} \gamma_{qq} - \gamma^- & \gamma_{qg} \\ X & \gamma^+ - \gamma_{qq} \end{pmatrix}; \\ M_- &= \frac{1}{\gamma^+ - \gamma^-} \begin{pmatrix} \gamma^+ - \gamma_{qq} & -\gamma_{qg} \\ -X & \gamma_{qq} - \gamma^- \end{pmatrix}, \end{aligned} \quad (2.22)$$

where  $X = (\gamma^+ - \gamma_{qq})(\gamma_{qq} - \gamma^-) / \gamma_{qg}$ . The evolution equation  $\frac{d}{dt} f = \gamma(N, \alpha_s(t)) f$  for the vector  $f(N, t) = \begin{pmatrix} q(N, t) \\ g(N, t) \end{pmatrix}$  is then immediately solved in terms of the path-ordered exponential

$$f(N, t) = \mathcal{P} \exp \int_{t_0}^t dt' [M_+(N, \alpha_s(t')) \gamma^+(N, \alpha_s(t')) + M_-(N, \alpha_s(t')) \gamma^-(N, \alpha_s(t'))] f(N, t_0). \quad (2.23)$$

Introducing the double-leading expansions (2.6),(2.7),(2.9) of  $\gamma^\pm$ ,  $\gamma^{qq}$  and  $\gamma^{qg}$  into the eq. (2.22) we thus obtain a double-leading expansion of the projector itself, which in turn, once substituted in eq. (2.23) gives the standard next-to-leading solution [30], but now in the double-leading expansion. Indeed, for numerical computations the solution eq. (2.23) can be used directly. This completes the construction of the double-leading approximation to the evolution equations for parton distributions, since the nonsinglet and valence quark distributions are free of small  $x$  logs and can be treated at two loops in the usual way.

## 2.2. Double leading structure functions

We can thus proceed to the determination of the structure functions. In the DIS scheme [31], the structure function  $F_2$  simply coincides with the quark distribution:

$$F_2(x, t) = \langle e^2 \rangle 2n_f Q(x, t) + F_2^{\text{NS}}(x, t), \quad (2.24)$$

where  $\langle e^2 \rangle \equiv \frac{1}{2n_f} \sum_{i=1}^{n_f} e_i^2$ ,  $Q(x, t) = x \sum_{i=1}^{n_f} (q_i(x, t) + \bar{q}_i(x, t))$  and  $F_2^{\text{NS}}(x, t)$  is the non-singlet component. However, the identification of  $F_2$  with the quark distribution still leaves some freedom in the definition of the gluon distribution. Introducing the general scheme change matrix  $U_{ij}(N, \alpha_s)$ , with  $i, j = q, g$ , such that (in matrix notation)

$$f'(N, \alpha_s) = U(N, \alpha_s) f(N, \alpha_s), \quad (2.25)$$

then if  $f' = \left( \begin{smallmatrix} Q \\ G \end{smallmatrix}^{\text{DIS}} \right)$  and  $f = \left( \begin{smallmatrix} Q \\ G \end{smallmatrix}^{\text{MS}} \right)$ , the condition  $F_2 = Q^{\text{DIS}}$  only determines the quark-sector matrix elements  $U_{qq} = C_2^q(N, \alpha_s)$  and  $U_{gg} = C_2^g(N, \alpha_s)$ , where  $C_2^i$  are the quark and gluon  $F_2$  coefficient function in the  $\overline{\text{MS}}$  scheme, leaving the gluon sector matrix elements  $U_{gq}$  and  $U_{gg}$  undetermined. This freedom can be fixed by assuming [32] the validity for all moments of the relations (2.11) which momentum conservation imposes on second moments, i.e.  $U_{gg}(N, \alpha_s) = 1 - C_2^g(N, \alpha_s)$ ,  $U_{gq}(N, \alpha_s) = 1 - C_2^q(N, \alpha_s)$ . Because the  $\overline{\text{MS}}$  coefficient functions are free of leading singularities, with this choice the singular contributions to the eigenvalues of the anomalous dimension matrix up to NLLx are the same in the DIS scheme as in the  $\overline{\text{MS}}$  scheme. Furthermore, the  $F_L$  coefficient functions, which start at next-to-leading  $\ln x$ , are also the same in the  $\overline{\text{MS}}$  and DIS scheme. Since the eigenvalues and quark entries of the anomalous dimension matrix fully determine the structure function, the scheme is fully determined by this somewhat weaker assumption, which we adopt as a definition of the DIS scheme in the double leading expansion. We will come back to a fuller discussion of the relation between double-leading  $\overline{\text{MS}}$  and DIS schemes in sec. 4.

We can thus determine easily the double-leading expansion of the structure functions in terms of the double-leading parton distributions:  $F_2$  is just given by eq. (2.24), while  $F_L$  is constructed from  $Q$  and  $G$  using double-leading coefficient functions. These are in turn constructed as the sum of two loops and leading singular contributions, minus the double counting:

$$C_{L, \text{DL}}^i(N, \alpha_s) = \alpha_s \left[ C_{L, 1}^i(N) + C_{L, ss}^i\left(\frac{\alpha_s}{N}\right) - e_L^i \right], \quad (2.26)$$

where  $i = q, g$ . The two loop coefficients  $C_{L, 1}^i(N)$  were computed in ref. [31], and the singular terms  $C_{L, ss}^i\left(\frac{\alpha_s}{N}\right)$  in ref. [29]:

$$\alpha_s C_{L, ss}^g\left(\frac{\alpha_s}{N}\right) = h_L(\gamma_s(\frac{\alpha_s}{N})) R(\gamma_s(\frac{\alpha_s}{N})), \quad (2.27)$$

where the process dependent piece  $h_L(M)$  is given by eq. (5.14) in ref. [29], and  $C_L^q$  is given by the color-charge relation  $C_{L, ss}^q = \frac{C_F}{C_A} (C_{L, ss}^g - e_L^g)$ . The double-counting subtractions are  $e_L^g = \frac{n_f}{3\pi}$ ,  $e_L^q = 0$ , and the coefficient functions are again best evaluated through their series expansion

$$C_{L, ss}^g\left(\frac{\alpha_s}{N}\right) = \frac{n_f}{3\pi} \left( 1 + \sum_{n=1}^{\infty} d_n^L \left(\frac{\alpha_s}{N}\right)^n \right), \quad (2.28)$$

where the coefficients  $d_n^L$  are listed in table 2.

### 3. Resummation of the structure function

A resummation of the expansion of the splitting function at small  $x$  is required [22,23] in order to obtain a stable expansion in powers of  $\alpha_s$ . The resummation affects the small  $N$  behaviour of the anomalous dimension, and specifically the expansion of the anomalous dimensions in leading, subleading, ... singularities: [1,22]:  $\gamma = [\gamma_s + \alpha_s \gamma_{ss} + \dots]$ . In this section we will discuss how this resummation affects the determination of the structure functions. The resummed version of the small  $x$  expansion will then finally be combined with the usual loop expansion in order to construct a resummed double-leading expansion.

#### 3.1. Resummed anomalous dimensions

The resummation is based on treating the asymptotic small  $x$  behaviour of the splitting functions as effectively leading order. This means that a constant is subtracted from the contribution to the BFKL kernel at each perturbative order, and added to the leading order:

$$\chi(M, \alpha_s) = \alpha_s \chi_0(M) + \alpha_s^2 \chi_1(M) + \dots \quad (3.1)$$

$$= \alpha_s \tilde{\chi}_0(M) + \alpha_s^2 \tilde{\chi}_1(M) + \dots, \quad (3.2)$$

where

$$\alpha_s \tilde{\chi}_0(M, \alpha_s) \equiv \alpha_s \chi_0(M) + \Delta\lambda(\alpha_s), \quad \tilde{\chi}_i(M) \equiv \chi_i(M) - c_i \quad (3.3)$$

for  $i = 1, 2, \dots$ , and thus

$$\Delta\lambda(\alpha_s) \equiv \sum_{n=1}^{\infty} \alpha_s^{n+1} c_n. \quad (3.4)$$

The constants  $c_i$  are uniquely fixed [23] order by order by the requirement that the associated splitting functions  $P_s^+$ ,  $P_{ss}^+$ , ..., define a stable expansion, in the sense that

$$\lim_{x \rightarrow 0} P_{ss}(x, \alpha_s) / P_s(x, \alpha_s) = f(\alpha_s), \quad (3.5)$$

where  $f(\alpha_s)$  does not depend on  $x$ , and thus in particular does not grow when  $x$  decreases. At next-to-leading order the constant is simply equal to the value of the subleading correction to the BFKL kernel evaluated at the leading-order minimum:

$$c_1 = \chi_1\left(\frac{1}{2}\right). \quad (3.6)$$

More generally, if  $\chi(M)$  has a minimum then  $c_i$  are such that the minimum of  $\tilde{\chi}_0(M)$  coincides with the minimum of  $\chi$ . For convenience we define  $c_0 = \chi_0(\frac{1}{2})$ , so that if  $\chi(M, \alpha_s)$  has a minimum,

$$\lambda(\alpha_s) = \alpha_s \sum_{n=0}^{\infty} \alpha_s^n c_n \quad (3.7)$$

is its value at that minimum.

The sum in the definitions of  $\lambda$  and  $\Delta\lambda$  is to be understood as a symbolic indication that even though  $\Delta\lambda$  can be formally expanded order by order in perturbation theory, it

is its all-order value which determines the behaviour of the splitting function in the Regge limit, and is thus relevant for phenomenology at small  $x$ . It is important to notice that this is an inevitable consequence of the assumption that the coupling runs with  $Q^2$ , and it is thus common to any perturbative computation based on this assumption, such as those of ref. [25]. Of course, a fixed order computation may turn out to provide a good approximation to the all-order value of  $\lambda$ . However, the known terms suggest at best a slow convergence of the expansion:  $c_1/c_0 \approx -6.2$ , so the NLO term is as large as the LO for realistic values of  $\alpha_s$ . It has been argued [25,24] on the basis of various model calculations that the perturbative expansion of  $\chi$  can be improved, using nonperturbative information, in such a way that the perturbative expansion of  $\lambda$  makes more sense. Here we prefer to treat  $\lambda$  as an unknown free parameter.

Since  $\lambda$  is defined only through the formal all-order resummation eq. (3.7), its scale dependence is presumably non-perturbative. If  $\tilde{\chi}_0$  is treated on the same footing as  $\chi_0$  then, given that  $\chi_0$  is scale-independent,  $\lambda$  is effectively treated as being proportional to  $\alpha_s$ . In other words, even though the expansion  $\Delta\lambda$  eq. (3.4) starts at  $O(\alpha_s^2)$ ,  $\Delta\lambda$  is treated as being effectively order  $\alpha_s$ , so that  $\tilde{\chi}_0$  is also scale independent. This approximation to the unknown non-perturbative scaling of  $\lambda$  need not be correct. An alternative simple option consists of assuming that the value of  $\lambda$  is scale-independent. Such an assumption can however only be valid as an approximation in a limited kinematical region, because asymptotic freedom implies that if we take the limit  $Q^2 \rightarrow \infty$  at fixed  $x$ , then at some sufficiently large scale the low-order perturbative behaviour must be recovered. If  $\lambda$  were strictly constant this requirement could only be satisfied if it also vanished: the  $O(\alpha_s)$  contribution to  $\gamma_s^+$  is  $\frac{n_c \alpha_s}{\pi(N-\Delta\lambda)}$ , and  $\Delta\lambda = \lambda - \alpha_s \lambda_0$  which only vanishes as  $Q^2 \rightarrow \infty$  if  $\lambda$  also vanishes. However,  $\lambda$  approximately constant might be a reasonable approximation over a limited range of  $Q^2$ , and indeed in some explicit models [20], where the exact asymptotic small  $x$  behaviour can be computed, the value of  $\lambda$  turns out to be reasonably well approximated by a constant [33] in the HERA region. We will thus consider both these options, and compare their phenomenological viability in sec. 5.

The subtracted  $\tilde{\chi}_i$  can then be used to compute the resummed leading, next-to-leading,... singularities  $\tilde{\gamma}_s^+$ ,  $\tilde{\gamma}_{ss}^+$ , ..., of the anomalous dimension  $\gamma^+$  by means of resummed versions of the usual duality relations [23,22]: instead of (2.14),(2.15), we now have

$$\tilde{\chi}_0(\tilde{\gamma}_s^+(\frac{\alpha_s}{N})) = \frac{N}{\alpha_s}, \quad \tilde{\gamma}_{ss}^+(\frac{\alpha_s}{N}) = -\frac{\tilde{\chi}_1(\tilde{\gamma}_s^+(\frac{\alpha_s}{N}))}{\tilde{\chi}'_0(\tilde{\gamma}_s^+(\frac{\alpha_s}{N}))}, \quad (3.8)$$

Therefore, this resummation of the anomalous dimension amounts to a reorganization of the small  $x$  expansion of the anomalous dimension, i.e. the expansion of  $\gamma(N, \alpha_s)$  in powers of  $\alpha_s$  at fixed  $\frac{\alpha_s}{N}$ : formally

$$\gamma^+(N, \alpha_s) = \gamma_s^+(\frac{\alpha_s}{N}) + \alpha_s \gamma_{ss}^+(\frac{\alpha_s}{N}) + O(\alpha_s^2) = \tilde{\gamma}_s^+(\frac{\alpha_s}{N}) + \alpha_s \tilde{\gamma}_{ss}^+(\frac{\alpha_s}{N}) + O(\alpha_s^2). \quad (3.9)$$

It is easy to see from (3.8) and (3.3) that to NLLx

$$\begin{aligned} \tilde{\gamma}_s^+(\frac{\alpha_s}{N}) &= \gamma_s^{gg}(\frac{\alpha_s}{N-\Delta\lambda}), \\ \tilde{\gamma}_{ss}^+(\frac{\alpha_s}{N}) &= -\frac{\chi_1(\gamma_s^+(\frac{\alpha_s}{N-\Delta\lambda})) - \chi_1(\frac{1}{2})}{\chi'_0(\gamma_s^+(\frac{\alpha_s}{N-\Delta\lambda}))}. \end{aligned} \quad (3.10)$$

at LLx one simply lets  $N \rightarrow N - \Delta\lambda$  in the unresummed anomalous dimension, while at NLLx one also lets  $\chi_1 \rightarrow \tilde{\chi}_1$  in (2.15). One can see explicitly that, because  $\Delta\lambda$  is formally of order  $\alpha_s^2$ , the resummed and unresummed expansions eq. (3.9) are equivalent at NLLx, and differ only by terms of order NNLLx.

### 3.2. Resummed structure functions

Having discussed the resummation of the eigenvectors of the anomalous dimension matrix, we now turn to the determination of resummed structure functions. To this purpose, we need first to construct resummed expression for the individual parton distributions, and then resummed coefficient functions. In general, parton distributions can be determined by decomposing  $Q$  and  $G$  in terms of large and small eigenvector components,  $Q = Q^+ + Q^-$ ,  $G = G^+ + G^-$ , with

$$Q^\pm(N, t) = K_{qg}^\pm G^\pm(N, t), \quad (3.11)$$

which can be effectively done by means of the projectors  $M^\pm$  eq. (2.22). Because the coefficients  $K_{qg}$  in general depend on  $t \equiv \ln Q^2$ , they also contribute to the scale dependence of the parton distributions. The coefficients  $K_{qg}^\pm$  were given to leading nontrivial order in eqs. (2.4), and are explicitly determined using the color-charge relation eq. (2.2) and the quark sector anomalous dimensions, given in the DIS scheme in eq. (2.18). In fact, to NLLx eq. (2.4) implies that

$$\frac{d}{dt}G^+(x, t) = \gamma_{\text{NLLx}}^+(N, \alpha_s(t))G^+(x, t) + O(\alpha_s^2), \quad (3.12)$$

so the determination of the quark and gluon distribution is straightforward.

Now, the quark anomalous dimensions  $\gamma_{qg}$ ,  $\gamma_{qq}$  eq. (2.18), and the coefficient  $K_{qg}^+$  eq. (3.11), and indeed the longitudinal coefficient functions eq. (2.27), as well as all the  $\overline{\text{MS}}$  coefficient functions, are all determined [29] as functionals of the leading log  $x$  anomalous dimension  $\gamma_s^+(\frac{\alpha_s}{N})$ . It is easy to understand the reason for this as a further consequence of the  $k_T$  factorization [29] which gave the duality [19,23,22] equations (2.14),(2.15) relating  $\gamma^+$  to  $\chi$ . We thus quickly review this derivation [23], after which the natural extension of our resummation of  $\gamma^+$  to the quark sector will become clear. At small  $x$  and large  $Q^2$ ,  $k_T$  factorization implies that we can write the large component  $Q^+$  of the quark distribution in the factorized form

$$Q^+(N, t) = \int_{c-i\infty}^{c+i\infty} \frac{dM}{2\pi i} e^{Mt} K_{qg}^+(M, N) G^+(M, N), \quad (3.13)$$

where the Mellin transform has been defined by

$$Q^+(N, M) = \int_{-\infty}^{\infty} dt e^{-Mt} Q^+(N, t), \quad (3.14)$$

with inverse

$$Q^+(N, t) = \int_{c-i\infty}^{c+i\infty} \frac{dM}{2\pi i} e^{Mt} Q^+(M, N), \quad (3.15)$$

where the contour passes to the right of any perturbative singularities near  $M = 0$ , but to the left of  $M = \frac{1}{2}$ . The leading twist behaviour of parton distributions is then given by closing the contour to the left and picking up the contribution of the rightmost singularity of the integrand in eq. (3.15) (see e.g. refs. [19,22]). The large component of the gluon distribution  $G^+(M, N)$  satisfies the BFKL equation, with kernel  $\chi(M, \alpha_s)$  eq. (3.1) and LLx solution

$$G^+(M, N) = \frac{G_0^+(M)}{N - \alpha_s \chi_0(M)}. \quad (3.16)$$

It thus has a simple pole at  $M = \gamma_s^+(N, \alpha_s)$ , where  $\gamma^+$  is given by the duality relation eq. (2.14). Solving the BFKL equation at higher orders gives corrections to the location of the pole as a series in  $\alpha_s$  at fixed  $\frac{\alpha_s}{N}$ , which correspond to the duality relations eq. (2.15), whereby all anomalous dimensions are expressed as functions of the position of the LLx pole.

Because  $K_{qg}^+(M, N)$  eq. (3.13) is free of collinear singularities [29], and is therefore regular in the neighbourhood of  $M = 0$ , the same pole which gives the leading twist small  $x$  behaviour of  $G^+(N, t)$  also gives that of  $Q^+(N, t)$ . It follows that

$$\begin{aligned} Q^+(N, t) &= K_{qg}^+(\gamma^+(N, \alpha_s), N)G^+(N, t) \\ &= K_{qg}^+(\gamma_s^+(\frac{\alpha_s}{N}))G^+(N, t) + O(\alpha_s), \end{aligned} \quad (3.17)$$

where we have replaced  $\gamma^+(N, \alpha_s)$  by  $\gamma_s^+(\frac{\alpha_s}{N})$ , and  $K_{qg}^+(M, N)$  by  $K_{qg}^+(M, 0) \equiv K_{qg}^+(M)$  since the explicit  $N$  dependence only generates subleading singularities. Thus the coefficient  $K_{qg}^+$  at leading nontrivial order only depends on  $\frac{\alpha_s}{N}$  through  $\gamma_s^+$ . Explicit computation of the cross-section for off-shell photon-gluon scattering [29] in  $\overline{\text{MS}}$  and DIS schemes gives (in DIS)  $K_{qg}^+(M) = h_2(M)R(M)/M$ , which when combined with (3.17) lead to the expression (2.18) for the quark anomalous dimension  $\gamma_{ss}^{qg}$ . A similar argument shows that all the leading  $\overline{\text{MS}}$  coefficient functions and anomalous dimensions have the same property, i.e. they may be determined as functions of  $M$  which is then at leading twist set equal to  $\gamma_s^+(\frac{\alpha_s}{N})$  by the duality relation at LLx.

After the resummation eq. (3.9), the LLx anomalous dimension becomes  $\tilde{\gamma}_s^+$  eq. (3.10). The resummed expression of parton distributions to NLLx are then simply found by using the resummed anomalous dimension (3.10) not only for the evolution of the large eigenvectors, but also in the expression eq. (2.18) for the quark anomalous dimension, and thus the coefficient  $K_{qg}^+$  eq. (3.11), (2.4). The resummed quark anomalous dimensions are thus given by

$$\alpha_s \tilde{\gamma}_{ss}^{qg}(\frac{\alpha_s}{N}) = h_2(\tilde{\gamma}_s^+(\frac{\alpha_s}{N}))R_N(\tilde{\gamma}_s^+(\frac{\alpha_s}{N})) = \alpha_s \gamma_{ss}^{qg}(\frac{\alpha_s}{N - \Delta\lambda}), \quad (3.18)$$

since the pole in (3.16) shifts from  $\gamma_s^+(\frac{\alpha_s}{N})$  to  $\tilde{\gamma}_s^+(\frac{\alpha_s}{N}) = \gamma_s^+(\frac{\alpha_s}{N - \Delta\lambda})$ . Thus the resummation in the quark sector leads simply to a shift  $N \rightarrow N - \Delta\lambda$  in the argument of the quark anomalous dimensions.

The resummed expression of the structure function  $F_2$  in the DIS scheme is given by the resummed quark distribution. Likewise, we can obtain a resummed expression for the longitudinal structure function, by using  $k_T$  factorization to express, at small  $x$  and large

$Q^2$  to NLLx the large component of  $F_L(N, t)$  as

$$F_L^+(N, t) = \int_{c-i\infty}^{c+i\infty} \frac{dM}{2\pi i} e^{Mt} (K_{L,ss}^q(M)Q^+(M, N) + K_{L,ss}^g(M)G^+(M, N)), \quad (3.19)$$

where the coefficient functions are free of collinear poles and given by [29]  $K_{L,ss}^g(M) = h_L(M)R(M)$  and a color-charge relation. Taking the residue of the pole then gives (2.27) for the coefficient functions in the unresummed case, whereas after resummation

$$\alpha_s \tilde{C}_{L,ss}^g\left(\frac{\alpha_s}{N}\right) = h_L(\gamma_s^+\left(\frac{\alpha_s}{N}\right))R_N(\gamma_s^+\left(\frac{\alpha_s}{N}\right)) = \alpha_s C_{L,ss}^g\left(\frac{\alpha_s}{N-\Delta\lambda}\right). \quad (3.20)$$

Thus the resummation again amounts to a shift  $N \rightarrow N - \Delta\lambda$  in the argument of the coefficient function. Clearly the same holds true for  $\overline{\text{MS}}$   $F_2$  coefficient functions, and indeed for any hard cross-section which is free of collinear singularities.

### 3.3. Double leading resummation

Having constructed resummed expressions for anomalous dimensions, coefficient functions and structure functions to NLLx, we can now combine them with the two loop results according to the lines discussed in the previous section to construct a resummed double-leading expansion. As discussed in ref. [22], the resummed double-leading expansion of the large eigenvalue  $\gamma_{\text{DL,R}}^+$  is constructed by replacing the singular terms  $\gamma_s, \gamma_{ss}, \dots$  in eq. (2.6) with their resummed expressions eq. (3.9), i.e., in practice, by letting  $N \rightarrow N - \Delta\lambda$  in the corresponding contributions:

$$\begin{aligned} \gamma_{\text{DL,R}}^+(N, \alpha_s) = & \left[ \alpha_s \gamma_0^+(N) + \gamma_s^+ \left( \frac{\alpha_s}{N-\Delta\lambda} \right) - \frac{n_c \alpha_s}{\pi N} \right] \\ & + \alpha_s \left[ \alpha_s \gamma_1^+(N) + \gamma_{ss}^+ \left( \frac{\alpha_s}{N-\Delta\lambda} \right) - \alpha_s \frac{e_1^+}{N} - e_0^+ \right] + \dots \end{aligned} \quad (3.21)$$

Likewise, we can construct the quark sector double-leading anomalous dimensions and coefficient functions by performing the same replacement in eqs. (2.9), (2.26): for example the resummed double leading anomalous dimension  $\gamma_{\text{DL,R}}^{gg}$  is given at NLO by

$$\gamma_{\text{DL,R}}^{gg}(N, \alpha_s) = \alpha_s \gamma_0^{gg}(N) + \alpha_s \left[ \alpha_s \gamma_1^{gg}(N) + \gamma_{ss}^{gg} \left( \frac{\alpha_s}{N-\Delta\lambda} \right) - \alpha_s \frac{e_1^q}{N} - e_0^q \right] + \dots \quad (3.22)$$

and similarly for  $\gamma_{\text{DL,R}}^{qg}$ , while

$$C_{L, \text{DL,R}}^i(N, \alpha_s) = \alpha_s \left[ C_{L,1}^i(N) + C_{L,ss}^i \left( \frac{\alpha_s}{N-\Delta\lambda} \right) - e_L^i \right]. \quad (3.23)$$

Notice that the momentum subtractions eq. (2.12), (2.13) are affected by the resummation and therefore must be recomputed at the resummed level.

As already discussed in ref. [22], this procedure generates an ambiguity in the treatment of the double-counting terms in  $\gamma_{\text{DL}}^+$ : because these terms are common to the loop expansion  $\gamma_0, \gamma_1, \dots$  and the small  $x$  expansion  $\gamma_s, \gamma_{ss}, \dots$ , we are free to decide whether

to leave them unaffected by the replacement  $N \rightarrow N - \Delta\lambda$  (as in eq. (2.6)) or not. The difference between the two procedures is formally sub-subleading, provided we suitably readjust the subleading double-counting subtractions. An equally acceptable alternative to the ‘R-resummation’ eq. (3.21) is thus the ‘S-resummation’

$$\begin{aligned} \gamma_{\text{DL,S}}^+(N, \alpha_s) &= \left[ \alpha_s \gamma_0(N) + \gamma_s \left( \frac{\alpha_s}{N - \Delta\lambda} \right) - \alpha_s \frac{n_c}{\pi(N - \Delta\lambda)} \right] \\ &+ \alpha_s \left[ \alpha_s \gamma_1(N) + \tilde{\gamma}_{ss} \left( \frac{\alpha_s}{N - \Delta\lambda} \right) + \frac{n_c \Delta\lambda}{\pi(N - \Delta\lambda)^2} - \alpha_s \frac{e_1^+}{N - \Delta\lambda} - e_0^+ \right] + \dots, \end{aligned} \quad (3.24)$$

where the double-counting subtraction is now also resummed. Clearly, a variety of intermediate alternatives would also be possible. If the S-prescription eq. (3.24) is used, then the double counting terms in the quark sector anomalous dimensions are also affected by the replacement  $N \rightarrow N - \Delta\lambda$ . Since in these terms all singularities start at the NLLx level, no further readjustment is necessary:

$$\gamma_{\text{DL,S}}^{gg}(N, \alpha_s) = \alpha_s \gamma_0^{gg}(N) + \alpha_s \left[ \alpha_s \gamma_1^{gg}(N) + \gamma_{ss}^{gg} \left( \frac{\alpha_s}{N - \Delta\lambda} \right) - \alpha_s \frac{e_1^g}{N - \Delta\lambda} - e_0^g \right] + \dots \quad (3.25)$$

The resummed longitudinal coefficient functions (3.23) are unchanged in the S-resummation, since there the double counting terms are independent of  $N$ .

The main difference between the two resummed expressions eq. (3.21) and (3.24) is the nature of the small- $x$  singularities of  $\gamma^+$ : the resummed anomalous dimension always has a cut starting at  $N = \lambda$ , which corresponds [23] to an  $x^{-\lambda}$  behaviour of splitting functions at small  $x$ . However, if the S-prescription eq. (3.24) is adopted, the anomalous dimension also has a simple pole at  $N = 0$ , which leads to a “double-scaling” [14,15] rise at small  $x$ . If  $\lambda$  is positive, then the power rise will dominate the asymptotic behaviour, and the two resummations give similar results [22], but if  $\lambda \leq 0$  the double scaling rise is dominant. In the latter case, the S-resummation eq. (3.21) will give results at small  $x$  which are very close to those obtained in a fixed leading or next-to-leading order computation, which are dominated by double scaling, unlike the R-resummation eq. (3.24), which instead would give a valencelike drop of the splitting functions at small  $x$ . Indeed, even when  $\lambda = 0$  the R-resummed splitting function and structure function drop logarithmically [23].

It turns out that different prescriptions for the all-order running of the coupling, to be discussed in sec. 4 below, can lead to anomalous dimensions whose small  $N$  behaviour is characterized by either a pole or a cut; we can therefore take our R vs. S ambiguity in the resummed NLLx double-leading result as an indication of this further ambiguity, which could only be resolved by arguments that go beyond perturbation theory. The phenomenological implications of these different options will all be discussed in detail in sec. 5.

#### 4. Scheme dependence and resummation ambiguities

In this section, we discuss the dependence of the resummed structure function on the choice of factorization scheme and its interplay with the resummation procedure. At the double-leading level, a factorization prescription must be specified both in the treatment



of the leading  $\ln Q^2$  and of the leading  $\ln x$  terms. The first choice allows one to switch, for instance, from the DIS to the  $\overline{\text{MS}}$  scheme, while the latter choice relates different small  $x$  resummations such as DIS and  $Q_0\text{DIS}$  [34]. This latter choice interferes with the resummation prescription, thereby raising theoretical issues such as the uniqueness of the resummation procedure, and the process-independence of the resummed results. This will also lead us to discuss some ambiguities which are related to the way the running of the coupling is treated in the Regge limit. These issues are not directly relevant for phenomenology, but they are useful in order to clarify the theoretical underpinnings and limitations of our approach. Readers who are only interested in the construction of the standard  $\overline{\text{MS}}$  scheme can skip this somewhat technical discussion (contained in sec. 4.2-4.4), while those who are interested only in the resummed DIS results should skip this section altogether.

#### 4.1. NLO scheme changes

In general, to next-to-leading order in the double-leading expansion, we can define a double-leading scheme change matrix eq. (2.25) as (in matrix notation)

$$U(N, \alpha_s) = \mathbb{1} + \alpha_s z_{DL}(N, \alpha_s), \quad z_{DL}(N, \alpha_s) = z_1(N) + z_{ss} \left( \frac{\alpha_s}{N} \right) - z_1^0, \quad (4.1)$$

where  $z_1(N)$  is regular at  $N = 0$ ,  $z_{ss} \left( \frac{\alpha_s}{N} \right) = \sum_{k=0}^{\infty} z_{ss}^k \left( \frac{\alpha_s}{N} \right)^k$ , and  $z_1^0 \equiv z_1(0) = z_{ss}^0$ . The change in the double-leading anomalous dimension matrix (2.8) induced by using this form of the scheme change matrix in eq. (2.25) is then

$$\begin{aligned} \gamma_1' &= \gamma_1 + [z_1, \gamma_0] - \frac{\beta_0}{4\pi} z_1 \\ \gamma_{ss}' &= \gamma_{ss} + [z_{ss}, \gamma_s], \end{aligned} \quad (4.2)$$

with a corresponding change in the subleading double counting term. The scheme change separates into a standard NLO loop scheme change, and a small  $x$  scheme change [28] ( $z_{ss}$  being the  $V$  matrix of ref. [28]) because all cross terms are formally subleading: while the constant contribution  $z_1^0$  contributes both to the  $\gamma_1$  and  $\gamma_{ss}$  scheme changes,  $z_1(N) - z_1^0$  introduces extra factors of  $N$  and thus cannot contribute to a change in  $\gamma_{ss}$  while  $z_{ss} \left( \frac{\alpha_s}{N} \right) - z_1^0$  introduces extra factors of  $\alpha_s$  and thus cannot contribute to a change in  $\gamma_1$ . Furthermore, the  $\beta_0$  term only contributes to the two-loop part of the scheme change eq. (4.2), because the running of the coupling is a  $\text{NLL}_X$  effect.

Let us now consider the specific case of the  $\text{DIS} \rightarrow \overline{\text{MS}}$  scheme change. The  $\overline{\text{MS}}$  scheme is defined by the computation of the corresponding collinear finite coefficient functions, whose singular terms have been determined in ref. [29]. The  $F_2$  double-leading coefficient functions are then

$$C_{2, \text{DL}}^i(N, \alpha_s) = \delta^{iq} + \alpha_s [C_{2, 1}^i(N) + C_{2, ss}^i \left( \frac{\alpha_s}{N} \right) - e_2^i], \quad (4.3)$$

where  $i = q, g$ , and there is a color-charge relation  $C_{2, ss}^q = \frac{C_F}{C_A} (C_{L, ss}^g - e_2^g)$  just as in the longitudinal case, eq. (2.26). The double-counting subtractions are now  $e_2^g = \frac{n_f}{6\pi}$ ,  $e_2^q = 0$ ,

and the coefficient functions may be evaluated through a series expansion analogous to (2.28):

$$C_{2,ss}^g\left(\frac{\alpha_s}{N}\right) = \frac{n_f}{6\pi} \left(1 + \sum_{n=1}^{\infty} d_n^{\text{MS}} \left(\frac{\alpha_s}{N}\right)^n\right), \quad (4.4)$$

where the coefficients  $d_n^{\text{MS}}$  are again listed in table 2. Unlike the longitudinal coefficient functions, which start at next-to-leading order in the double-leading expansion, and therefore are unaffected by scheme change at this order, the  $F_2$  coefficient functions begin at leading order and thus depend on the choice of scheme.

Demanding invariance of  $F_2$  implies that the scheme change matrix which takes us from the DIS scheme eq. (2.24) in which  $C_{2,DL}^i = \delta^{iq}$  to the  $\overline{\text{MS}}$  scheme in which eq. (4.3) holds must have

$$U^{qg} = -C_{2,DL}^g \quad U^{gg} = C_{2,DL}^q. \quad (4.5)$$

Because the  $F_2$  coefficient functions have the form eq. (4.3), and in particular they do not contain any leading singularity, i.e.  $C_{2,s} = 0$ , the quark entries of the scheme change have the form of eq. (4.1). These entries are sufficient to determine [28] the singular contributions to the  $\overline{\text{MS}}$  quark anomalous dimensions eq. (2.8):

$$\gamma_{ss}^{qg'} = \gamma_{ss}^{qg} + z_{ss}^{qg} \gamma_s^{gg}, \quad \gamma_{ss}^{qq'} = \gamma_{ss}^{qq} + \frac{C_F}{C_A} z_{ss}^{qg} \gamma_s^{gg}. \quad (4.6)$$

These are the same as the leading singularities of the  $\overline{\text{MS}}$  quark anomalous dimension computed in ref. [29]. In other words within the double leading prescription the quark entries of the double-leading anomalous dimension matrix eq. (2.9) transform consistently: the DIS and  $\overline{\text{MS}}$  double-leading expressions are found by combining the respective DIS and  $\overline{\text{MS}}$  large  $x$  and singular terms.

Since the  $\overline{\text{MS}}$  quark anomalous dimensions still satisfy the color-charge relation (2.5), it follows that the singular terms in the small eigenvalue remain zero in  $\overline{\text{MS}}$ . Moreover

$$\gamma_{ss}^{gg'} = \gamma_{ss}^{gg} - \frac{C_F}{C_A} z_{ss}^{qg} \gamma_s^{gg}, \quad (4.7)$$

so it follows from eq. (2.3) that the singular part of the large eigenvalue remains unchanged too. This latter conclusion may also be seen directly from (4.2): since the scheme change of  $\gamma_s + \alpha_s \gamma_{ss}$  is a commutator, whose trace vanishes, the sum of the eigenvalues must remain unchanged, so  $\gamma_{ss}^{+'} + \gamma_{ss}^{-'} = \gamma_{ss}^+ + \gamma_{ss}^-$ . Since the singular parts of  $\gamma^-$  remain unchanged, those of  $\gamma^+$  must too.

In order for the full scheme change to be consistent, it is sufficient to require that  $\gamma_1'$  coincides with the usual  $\overline{\text{MS}}$  matrix of NLO anomalous dimensions, and that the scheme change matrix be of the form eq. (4.1). Indeed, it then follows that the eigenvalues of  $\gamma_1$  transform in the standard way, while the eigenvalues of  $\gamma_{ss}$  do not change at all. Because, as discussed in sec. 2, the eigenvalues of the double-leading  $\gamma_{DL}'$  are the sum of the eigenvalues of their small  $x$  and large  $x$  components, this ensures that the double-leading eigenvalues eq. (2.6),(2.7) also transform consistently, i.e. they are the sum of the large and small  $x$  contributions in the corresponding scheme. This is sufficient to fix the scheme, since, given eigenvalues and quark entries the evolution equations and their solutions are completely determined.

The remaining two entries of the scheme change matrix may be fixed entirely by a natural extension of the method employed at two loops [32]: this gives

$$U^{qq} = C_{2, \text{DL}}^q - 1 \quad U^{gg} = 1 + C_{2, \text{DL}}^g. \quad (4.8)$$

In fact the only effect of these entries beyond two loops is to change  $\gamma_{ss}^{gg}$ :

$$\gamma_{ss}^{gg'} = \gamma_{ss}^{gg} + \gamma_s^{gg} \left[ \frac{C_F}{C_A} (z_{ss}^{gg} - z_{ss}^{qq}) - z_{ss}^{gg} \right]. \quad (4.9)$$

However, as we explained in sec. 2,  $\gamma_{ss}^{gg}$  is of no consequence at NLLx.

#### 4.2. LLx scheme changes

On top of the ‘standard’ double leading scheme changes eq. (4.1), if we allow any factorization scheme change of the form eq. (2.25) provided only that the leading log  $x$  anomalous dimensions be unchanged, there is the freedom to perform an extra ‘leading log  $x$ ’ scheme change. This is a consequence of the fact that the leading small  $x$  singularities only appear in the gluon sector of the anomalous dimension matrix eq. (2.1). Indeed, this requirement is satisfied [28] by any scheme change of the form

$$U = \begin{pmatrix} 1 & 0 \\ \frac{C_F}{C_A} z_s^{gg}(\alpha_s/N) & z_s^{gg}(\alpha_s/N) \end{pmatrix}, \quad (4.10)$$

where  $z_s^{gg}(\alpha_s/N) = 1 + z_{s,1}^{gg} \frac{\alpha_s}{N} + \dots$ . This scheme change amounts to a LLx modification of the normalization of the gluon distribution, so the the DIS identification of  $F_2$  with the quark distribution remains unaffected.

Upon this scheme change  $\gamma^{gg}$  and  $\gamma^+$  change according to

$$\begin{aligned} \gamma_{ss}^{gg'} &= \gamma_{ss}^{gg} / u(\gamma_s^+) \\ \gamma_{ss}^{+'} &= \gamma_{ss}^+ + \frac{\beta_0}{4\pi} \frac{\chi_0(\gamma_s^+)}{\chi_0'(\gamma_s^+)} \frac{d \ln u}{dM} \Big|_{M=\gamma_s^+}, \end{aligned} \quad (4.11)$$

where we have for convenience defined  $u(M)$  by the implicit equation  $z_s^{gg}(\alpha_s/N) = u(\gamma_s(\frac{\alpha_s}{N}))$ , and made use of the fact that (differentiating (2.14))

$$\frac{\partial \gamma_s^+}{\partial \ln \alpha_s} = -\frac{\chi_0(\gamma_s^+)}{\chi_0'(\gamma_s^+)}. \quad (4.12)$$

The anomalous dimension  $\gamma_{ss}^{qq}$  changes in such a way as to preserve the colour–charge relation (2.5), and thus  $\gamma^-$  is unaltered. The scheme change of the NLLx anomalous dimension eq. (4.11) corresponds to a dual scheme change [23] of the next-to-leading BFKL function

$$\chi_1'(M) = \chi_1(M) - \frac{\beta_0}{4\pi} \chi_0(M) \frac{d \ln u}{dM}. \quad (4.13)$$

Since these scheme changes only affect the singular contributions to the anomalous dimensions, for the remainder of this section we will only discuss the small  $x$  expansion

of the structure function. In fact, they generate a class of schemes which only differ in the way the small  $x$  resummation is factorized, and specifically, in the factorization of the expression eq. (3.13) of  $Q^+$  in terms of the solution to the large eigenvector evolution equation and the coefficient  $K_{qg}^+$ . For definiteness, let us assume that we start from the DIS scheme, where  $F_2$  and  $Q$  coincide: the LLx scheme change then moves within the class of DIS schemes.

Upon resummation in any of these schemes, the expansion of  $\chi$  gets rearranged according to eq. (3.2), and the subleading anomalous dimensions eq. (3.10) change accordingly. However, the scheme changes do not in general commute with the resummation procedure, in the sense that a scheme change characterized by a given function  $u(M)$  does not have the same effect if performed before or after resummation. To see this, consider a pair of schemes connected by a scheme change function  $u(M)$ , whose NLLx anomalous dimensions are thus related by eq. (4.11). We can now determine the resummed anomalous dimension in the primed scheme in two different ways. One possibility is to first perform the resummation eq. (3.10) in the unprimed scheme, then transform the result according to eq. (4.11), thereby obtaining a NLLx anomalous dimension  $[\tilde{\gamma}_{ss}^+]'$ :

$$[\tilde{\gamma}_{ss}^+]' = -\frac{\tilde{\chi}_1(\tilde{\gamma}_s^+)}{\chi_0'(\tilde{\gamma}_s^+)} + \frac{\beta_0}{4\pi} \frac{1}{\chi_0'(\tilde{\gamma}_s^+)} \frac{\partial[\alpha_s \tilde{\chi}_0(M)]}{\partial \alpha_s} \frac{d \ln u}{dM} \Big|_{M=\tilde{\gamma}_s^+}, \quad (4.14)$$

where  $\tilde{\chi}_0(M)$  eq. (3.3) depends on  $\alpha_s$  through  $\Delta\lambda$ .

Alternatively, we first perform the scheme change eq. (4.11) of the unresummed anomalous dimension, and then resum the result according to eq. (3.10), thereby obtaining a NLLx anomalous dimension (stable in the sense of eq. (3.5))  $[\widetilde{\gamma}_{ss}^+]$ :

$$[\widetilde{\gamma}_{ss}^+] = -\frac{\tilde{\chi}_1(\tilde{\gamma}_s^+)}{\chi_0'(\tilde{\gamma}_s^+)} + \frac{\beta_0}{4\pi} \frac{1}{\chi_0'(\tilde{\gamma}_s^+)} \left[ \chi_0(\tilde{\gamma}_s^+) \frac{d \ln u}{dM} \Big|_{M=\tilde{\gamma}_s^+} - \chi_0\left(\frac{1}{2}\right) \frac{d \ln u}{dM} \Big|_{M=\frac{1}{2}} \right]. \quad (4.15)$$

The results obtained by the two procedures (4.14) and (4.15) thus differ by

$$\begin{aligned} [\widetilde{\gamma}_{ss}^+] - [\tilde{\gamma}_{ss}^+]' &= -\frac{\beta_0}{4\pi} \frac{1}{\chi_0'(\tilde{\gamma}_s^+)} \left[ \frac{d\Delta\lambda}{d\alpha_s} \frac{d \ln u}{dM} \Big|_{M=\tilde{\gamma}_s^+} + \chi_0\left(\frac{1}{2}\right) \frac{d \ln u}{dM} \Big|_{M=\frac{1}{2}} \right] \\ &= -\frac{\beta_0}{4\pi} \frac{1}{\chi_0'(\tilde{\gamma}_s^+)} \left[ \frac{d\Delta\lambda}{d\alpha_s} \left( \frac{d \ln u}{dM} \Big|_{M=\tilde{\gamma}_s^+} - \frac{d \ln u}{dM} \Big|_{M=\frac{1}{2}} \right) + \frac{d\lambda}{d\alpha_s} \frac{d \ln u}{dM} \Big|_{M=\frac{1}{2}} \right]. \end{aligned} \quad (4.16)$$

Because  $\lambda$  is formally of order  $\alpha_s$  and  $\Delta\lambda$  of order  $\alpha_s^2$ , the non-commutativity given by the first term on the r.h.s. of eq. (4.16) is formally subleading, whereas in the second term it is leading (i.e. the same order as  $\gamma_{ss}^+$ ).

To understand this, notice that at  $\tilde{\gamma}_s^+ = \frac{1}{2}$  the second term in eq. (4.14) will in general be singular, so  $[\tilde{\gamma}_{ss}^+]'$  eq. (4.14) will be unstable, unless

$$\frac{d\lambda}{d\alpha_s} \frac{d \ln u}{dM} \Big|_{M=\frac{1}{2}} = 0. \quad (4.17)$$

By contrast,  $[\widetilde{\gamma_{ss}^+}]$  eq. (4.15) is stable by construction. This means that if we have resummed the anomalous dimension to give a stable expansion, after a further scheme change the expansion will still be stable only if the condition eq. (4.17) is satisfied. Otherwise stated, the subtraction coefficient  $c_1$  eq. (3.4) required to stabilize the primed and unprimed schemes is not the same. The difference between these stable and unstable prescriptions is the leading order second term on the r.h.s. of eq. (4.16). The condition (4.17) for the vanishing of this term may thus be seen as a condition on the further scheme changes which are permissible after resummation has been performed.

After resummation, as discussed in sec. 3.1, the scale dependence of  $\lambda$  is unknown: we will model it by taking  $\lambda$  to be either proportional to  $\alpha_s$ , or scale-independent, even though, as discussed above, the latter option can only be true as an approximation valid in a limited kinematical range, or if  $\lambda = 0$  identically. In both cases  $\Delta\lambda$  contains a contribution of order  $\alpha_s$ : if  $\lambda \propto \alpha_s$  then  $\frac{d\Delta\lambda}{d\alpha_s} = \frac{\lambda - \lambda_0}{\alpha_s}$ , while if  $\frac{d\lambda}{d\alpha_s} = 0$ , then  $\frac{d\Delta\lambda}{d\alpha_s} = -\frac{\lambda_0}{\alpha_s}$ . Note that this implies that the size of this contribution is comparable in the two cases whenever  $|\lambda| \ll \lambda_0$ : this will turn out to be the case in actual phenomenology (sec. 5). In both cases, the first term on the r.h.s. of eq. (4.16) leads thus to a potentially significant ambiguity in the scale dependence of resummed parton distributions upon LLx scheme changes eq. (4.10). We have however checked that for common scheme choices this ambiguity is significantly smaller than that which we explore by switching from the S- to the R-resummations discussed in sect. 3.2, and concentrated in the same region. On the other hand, the second term on the r.h.s. of eq. (4.16) is proportional to  $\frac{d\lambda}{d\alpha_s} = \frac{\lambda}{\alpha_s}$ , if  $\lambda \propto \alpha_s$ , while it vanishes if  $\lambda$  is independent of  $\alpha_s$  (and thus in particular if  $\lambda = 0$ ). Hence, the stability condition eq. (4.17) is automatically satisfied in any scheme if  $\lambda$  doesn't depend on  $\alpha_s$ . Furthermore, the condition is approximately satisfied even when  $\lambda$  is proportional to  $\alpha_s$ , provided it is small enough: in such case, the ratio eq. (3.5) rises linearly  $P_{ss}(x, \alpha_s)/P_s(x, \alpha_s) \underset{x \rightarrow 0}{\sim} \ln 1/x$ , but the slope of the rise is proportional to  $\lambda$  [23].

### 4.3. Coefficient function resummation

So far we discussed the implications of LLx scheme changes for the stability of the resummed process-independent large anomalous dimension  $\gamma^+$  in DIS (or  $\overline{\text{MS}}$ ). However, as discussed in Sect. 3.2, the coefficient  $K_{qg}^+$  eq. (3.11) and the coefficient functions also contribute to the scale-dependence of the structure function. In order to achieve perturbative stability of the structure function, therefore, we must consider the resummation of these process-dependent quantities as well. To do this, we first define an 'effective' NLLx anomalous dimension

$$\gamma_{\text{NLLx, eff}}^+(N, \alpha_s(t)) \equiv \frac{d}{dt} \ln F_2^+(x, Q^2), \quad (4.18)$$

where for definiteness we consider first the structure function  $F_2^+ \equiv Q^+$  in DIS schemes. The effective anomalous dimension is by definition scheme-independent, but process dependent: it coincides with  $\gamma_{\text{NLLx}}^+$  only in schemes where not only the coefficient functions are trivial (such as DIS), but also the coefficient  $K_{qg}^+$  eq. (2.18) is trivial, in the sense that neither contributes to the scale dependence of  $F_2(x, Q^2)$ . It follows from eqs. (3.12), (3.11)

that we can always go from the resummed DIS anomalous dimension to the effective anomalous dimension by means of a LLx scheme change (4.11), with

$$u(M) = \frac{\kappa_{gg} h_2(M) R(M)}{\alpha_s M \chi_0(M)}, \quad (4.19)$$

where we have used the fact that  $\alpha_s \tilde{\chi}_0(\tilde{\gamma}_s^+) = N$  independent of  $t$ , and the normalization constant  $\kappa_{gg} = \frac{n_c}{\pi e_0^g}$  (with  $e_0^g$  defined in eq. (2.9)) is to ensure that the scheme change takes the required form (4.10), with requires  $u(0) = 1$  [28]. After resummation in DIS

$$\gamma_{\text{NLLx, eff}}^+(N, \alpha_s) = \tilde{\gamma}_{\text{NLLx}}^+(N, \alpha_s) + \frac{\beta_0}{4\pi} \left[ 1 + \frac{1}{\chi_0'(\tilde{\gamma}_s^+)} \frac{\partial(\alpha_s \tilde{\chi}_0)}{\partial \alpha_s} \frac{d \ln(h_2(M) R(M)/M)}{dM} \Big|_{M=\tilde{\gamma}_s^+} \right]. \quad (4.20)$$

At  $\tilde{\gamma}_s^+ = \frac{1}{2}$  the second term is singular, and the effective anomalous dimension unstable, unless

$$\frac{d\lambda}{d\alpha_s} \left[ 2 + \frac{d \ln R}{dM} \right] = 0 \quad (4.21)$$

in the limit  $M \rightarrow \frac{1}{2}$ . Notice that, because  $h_2(M)/M^2$  is symmetrical about  $M = \frac{1}{2}$ , the stability condition (4.21) only involves the process independent function  $R(M)$ . Similar results may be derived for the unstable contribution of the coefficient function to the scale dependence of other physical observables: that for  $F_L$  turns out to be the same as (4.21), whereas for other quantities, such as for instance heavy quark production cross-sections, they would be different.

One might think therefore that the effective anomalous dimension (and thus the resummed expression for, say  $F_2$ ) may be stabilized by resumming the process-dependent coefficients as well, i.e. by a further subtraction analogous to  $c_1$  eq. (3.3). If we try to do that, however, we have a problem:  $R(M) \sim (\frac{1}{2} - M)^{-1/2}$  when  $M \sim \frac{1}{2}$  [29], so the required subtraction diverges in the  $M \rightarrow \frac{1}{2}$  limit. This is to be contrasted to the case discussed in sect. 4.2, eq. (4.16), where even though  $[\tilde{\gamma}_{ss}^+]'$  was singular in the limit  $M \rightarrow \frac{1}{2}$ , the subtraction needed to stabilize it is finite.

There are several ways of dealing with this issue:

(i) We may argue that it should be ignored, on the grounds that the process-dependent  $F_2$  coefficient is actually leading: after all, without it  $F_2$  would vanish. Hence, the effective anomalous dimension argument is not relevant: we have two contributions to scaling violations of physical observables, one from the evolution of the parton distributions (which should be stable), but another from the hard cross-section, which may be relatively large. Stability of the latter becomes a relevant issue only when the next-to-leading correction to it is computed [35], included in it (at the next order), and compared to the leading order.

(ii) We may ignore it, on the grounds that the effect is in practice rather small. Indeed, if  $\lambda$  is taken to be proportional to  $\alpha_s$ , when the stability criterion eq. (4.21) is not met, so for the effective splitting function  $P_{ss}^{\text{eff}}(x, \alpha_s)/P_s^{\text{eff}}(x, \alpha_s) \underset{x \rightarrow 0}{\sim} (\ln 1/x)^{3/2}$  [23] the slope of the rise is proportional to  $\lambda$ . Now, it turns out (sec. 5) that the actual phenomenological value of  $\lambda$  is generally close to zero, so the slope of the rise is rather

small. Of course, the rise goes away altogether if  $\lambda$  is taken to be independent of  $\alpha_s$ . In fact, one can check explicitly that this rise is undetectable for all practical purposes in the HERA region provided  $|\lambda| \lesssim 0.2$ . Therefore, for phenomenological applications we may stick to the resummation procedure discussed in sec. 3, and simply check a posteriori that no significant instabilities are present.

(iii) If  $\lambda \equiv 0$ , it makes no difference whether the subtraction is determined from the effective anomalous dimension or in  $\overline{\text{MS}}$ , since then the stability criterion (4.21) is always met. Various theoretical arguments [19] suggest that  $\lambda = 0$ . Furthermore, in the S-resummation,  $\lambda \simeq 0$  is consistent with the HERA data, as we will see below.

(iv) One should trace the origin and meaning of the singularity in  $R(M)$ , and try to deal with it through a further resummation. This will be discussed in sect. 4.4, where we will show that the source of the singularity appears to be related to the NLLx truncation of running coupling effects in the BFKL kernel. Therefore, a model-independent resummation of the singularity does not seem possible at present, though we will argue that some effects of it can be modeled by comparing the S- and R-resummations of sec. 3.3.

#### 4.4. Regge limit and the running of the coupling

The singularity in the process-independent function  $R(M)$  is related to some limitations of the NLLx approach, and thus warrants further discussion. The origin of this singularity is exposed by noting that  $R(M) = r(M)/\sqrt{-\chi_0'(M)}$  where the function  $r(M)$  is regular if  $0 < M < 1$ . Then, the function  $\chi_{\text{eff}}$  which is dual to the effective anomalous dimension (4.18) contains a term (see eq. (4.13))

$$\chi_1^{rc} = \frac{1}{2} \frac{\beta_0}{4\pi} \frac{\chi_0 \chi_0''}{\chi_0'}. \quad (4.22)$$

This contribution to  $\chi_{\text{eff}}$  has a simple pole at  $M = \frac{1}{2}$ . As discussed in sec. 4.3, the presence of a pole in  $\chi_1$  at  $M = \frac{1}{2}$  is a problem in view of the fact that the  $\chi_1(\frac{1}{2})$  fixes the value of the NLO subtraction  $c_1$  eq. (3.6) which would be required to stabilize  $\chi_{\text{eff}}$ . This problem will be present in schemes (such as the  $\text{Q}_0\text{DIS}$  scheme [34]) where the contribution eq. (4.22) is included in the anomalous dimension, and indeed in any scheme if we attempt to implement a resummation prescription at the level of effective anomalous dimensions, as discussed above.

The origin of such a singular contribution to  $\chi$  can be traced to the perturbative treatment of the running of the coupling to NLLx. Indeed, the effects of the running of the coupling can be viewed as the addition of higher order terms to the anomalous dimension. It turns out [23,22] that these terms can be always expressed as an order-by-order modification of the duality relations: the form of the duality relation remains unchanged, but the function  $\chi(M)$  no longer coincides with the BFKL kernel, rather, it differs by it by the addition of contributions of order  $(\beta_0 \alpha_s)^k$ . To NLLx, there is thus a running coupling  $O(\beta_0)$  contribution to  $\chi_1$ , which turns out [23] to be precisely of the form (4.22). The singularity can be moved from the anomalous dimension to the coefficient by a LLx scheme change but it is of course always present in the effective anomalous dimension. In fact, this is what happens in the DIS (or  $\overline{\text{MS}}$ ) scheme, where the singularity

in the running coupling contribution to  $\chi_1$  is cancelled by a similar contribution from the normalization of the LLx unintegrated gluon distribution, but then re-appears as the square root singularity in the  $R(M)$  contribution to the coefficient function  $K_{qg}$  eq. (3.17). All this suggests that the running coupling singularity eq. (4.22) and the singularity in the DIS stability condition eq. (4.21) are one and the same.

The running coupling singularity eq. (4.22) signals a failure of the NLLx treatment of the running of the coupling in the vicinity of the point  $M = \frac{1}{2}$ . In this respect, it is analogous to the singularity in the NLLx anomalous dimension  $\gamma_{ss}^+(\gamma_s^+)$  eq. (2.15) as  $\gamma_s^+ \rightarrow \frac{1}{2}$  which leads [23] to the instability which is removed by the resummation of  $\chi$  eq. (3.2). This analogy suggests that the running coupling singularity (4.22) might be understood, and possibly removed, by treating the running of the coupling to all orders. Although interesting models for the running of the coupling in the BFKL equation beyond NLLx have been discussed [24], a model-independent treatment does not appear to be possible at present. An investigation of models is beyond the scope of this paper, where we only wish to make use, as far as possible, of model-independent results. However, it is worth noticing that the inclusion of running coupling terms beyond NLLx leads to a series of contributions to the duality relation to all orders in  $\beta_0$ , of which  $\chi^{rc}$  eq. (4.22) is the leading term. Resumming this series to all orders may change the qualitative structure of  $\chi$  in the vicinity of the leading order minimum.

For instance, if we assume that we can let the coupling run by setting

$$\alpha_s \rightarrow \frac{\alpha_s}{1 + \frac{\beta_0}{4\pi} \alpha_s \frac{d}{dM}} \quad (4.23)$$

it is possible to determine in closed form [36] the solution  $G(M, N)$  to the BFKL equation. If one further approximates the kernel  $\chi$  with a parabolic form  $\chi = \lambda + k(M - \frac{1}{2})^2$ , which is accurate close enough to the minimum, it is possible to determine analytically the solution  $G(N, t)$  in terms of the Airy function [37], and thus the associated anomalous dimension. One can then show that whenever  $\lambda > 0$ , the anomalous dimension has a simple pole in the range  $0 < N < \lambda$ . This changes qualitatively the perturbative behaviour of the small  $x$  terms  $\gamma_s, \gamma_{ss}, \dots$ : for all  $\lambda > 0$ , if  $\chi$  is parabolic then  $\gamma$  which is dual to it has a branch point at  $N = \lambda$ , however the branch point turns into a simple pole when running coupling effects are resummed according to eq. (4.23). Now, recall that the simple one-loop anomalous dimension is characterized by a simple pole at  $N = 0$ . It follows that if  $\lambda$  is close enough to zero, the small  $x$  behaviour induced by this resummation of running coupling effects, which also generates a pole in the anomalous dimension, will be quite close to that of the one-loop result. This is also the small  $x$  behaviour which is found in the S-double leading expansion eq. (3.24), while in the R-double leading expansion, as discussed at the end of sec. 3, there is no pole but only a cut. Therefore, switching between the two resummed forms of the double-leading expansion discussed in sec. 3 changes the form of the dominant small  $N$  singularity in a way which can give a feeling for the kind of uncertainty related to the resummation of running coupling effects in the Regge limit.

## 5. Comparison of the resummed structure functions with the data

In the previous sections we have developed a formalism to resum a sequence of all



order small  $x$  corrections to splitting functions and coefficient functions. The resulting expressions for structure functions should be valid in an extended range of  $x$  down to much smaller values than would be the case in the usual two loop approximation. We now compare our theoretical description of structure functions with the data, with particular attention to the HERA data which probe the small  $x$  region. When compared to the two loop results, the resummed calculation has several important new features. First, in our formalism, in addition to the input parton densities at some initial scale  $Q_0^2$  and the value of  $\alpha_s(m_z^2)$ , we also have to extract from the data the value of the exponent  $\lambda$  that fixes the asymptotic behaviour of the resummed splitting functions in the Regge limit. Second, our resummation procedure has a number of further ambiguities which must either be resolved on phenomenological grounds, or else considered to be part of the theoretical errors on the fitted quantities. Specifically, as discussed in sec. 3, there is some uncertainty in deciding the scale dependence of  $\lambda$ . Furthermore, there is an important ambiguity in the treatment of the double-counting terms in the resummed double-leading expansion, which gives rise to the pair of alternative prescriptions, denoted by R and S, and discussed at the end of sec. 3, eqs. (3.21), (3.24).

Since this work is focussed on small  $x$  data, we want to start from input parton densities that already embody the information contained in the large  $x$  data so that we can study the details of the small  $x$  behaviour without having to perform a global fit. Therefore, our procedure for analysing the small  $x$  data is to start from a set of globally fitted parton distributions, at a starting scale  $Q_0$ , and treat as free parameters the exponents  $\lambda_q$  and  $\lambda_g$  which characterize the small  $x$  behaviour of the singlet quark and gluon as  $Q = xq \sim x^{-\lambda_q}$ ,  $G = xg \sim x^{-\lambda_g}$ . Of course for each new choice of  $\lambda_q$  and  $\lambda_g$  all the other parameters must be readjusted in order to maintain the large  $x$  shape (say for  $x \gtrsim 0.01$ ) of the parton distributions and preserve the momentum sum rule. These new distributions, together with the valence and nonsinglet densities, which are left unchanged, are then used as input distributions, evolved up to the HERA data, and  $\lambda_q$  and  $\lambda_g$  are tuned to obtain a best fit. Of course,  $\lambda_q$  and  $\lambda_g$  should not be confused with  $\lambda$ : the former are parameters in the input parton distributions at  $Q_0^2$ , while the latter is a property of the resummed splitting functions and thus of the parton evolution.

The data set that we use consists of H1 data collected in 1995–1997 [38]. The H1 data have been recently shown [39] to be consistent with large- $x$  data as well as with the LEP value of  $\alpha_s$ , which dominates the world average [40]. We only include neutral current data with  $x < 0.01$  and  $5 \text{ GeV}^2 < Q^2 < 1000 \text{ GeV}^2$  in order to be above the charm threshold and well below the  $Z$ . We fit directly to the ‘reduced cross-section’

$$\sigma_{red}(x, y, Q^2) = F_2(x, Q^2) + \frac{y^2}{2(1-y) + y^2} F_L(x, Q^2). \quad (5.1)$$

The  $\chi^2$  is calculated with statistical and systematic errors added in quadrature, except for the overall correlated normalization uncertainties which are treated separately. As input partons we use MRSA4 [41], which have  $\alpha_s(m_z^2) = 0.120$ . We checked that essentially the same results are obtained with the CTEQ4A4 set [42].

### 5.1. Fixed $\alpha_s(m_z^2)$

In a first round of fits we fix  $\alpha_s(m_z^2) = 0.119$ , a value close to the current world

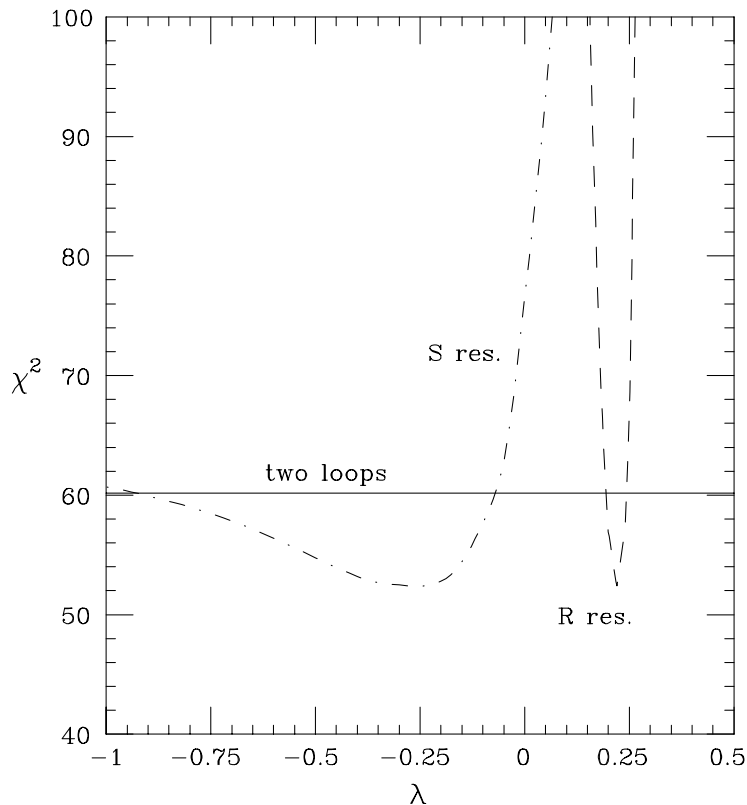


Figure 1:  $\chi^2$  for the fit to the 95 H1 data [38] of the reduced cross section eq. (6.1) at two loops (solid), and in the double-leading expansion, S-resummation (dotdash) and R-resummation (dashes) as a function of  $\lambda$ . Here  $\alpha_s(m_z^2) = 0.119$ .

average [40], and proceed to fit  $\lambda_{q,g}$  at  $Q_0 = 2$  GeV for different values of  $\lambda$ . The number of degrees of freedom is in this case  $n_{dof} = 93$ . In fig. 1 we present the behaviour of the best-fit  $\chi^2$  as a function of  $\lambda$  for different variants of our approach. In fact, we show curves for  $\chi^2$  computed in the two resummed expansions discussed at the end of sec. 3, as a function of  $\lambda$ , and compared with the two loop value. In the resummed fits, we have taken  $\lambda(Q^2) = \alpha_s(Q^2)c$ , where  $c$  is a constant independent of  $Q^2$ . The value given on the abscissa in fig. 1 is (for definiteness)  $\lambda = 0.2c$ ; 0.2 is taken as a representative value of  $\alpha_s(Q^2)$  in the HERA data region.

We see that in the case of the S-resummation a fit at least as good as the NLO fit is obtained over a wide range of negative values of  $\lambda$ . In particular the resummed fit has the lowest  $\chi^2$  for  $\lambda \approx -0.25$ . On the contrary the R-resummation only approaches the NLO result in a very narrow range around  $\lambda \approx 0.2$ . Thus, it turns out that fitting the data with this resummation prescription requires a substantial fine tuning in  $\lambda$ , which is however not necessary when using the S-resummation. This can be understood on the basis of the results of ref. [22], where it was shown that the S-resummation gives a small  $x$  splitting function which is extremely close to the two loop one whenever  $\lambda \leq 0$ , whereas the R-resummation only gives a result close to the two loops one for a fine-tuned value of  $\lambda \approx 0.2$ . Comparison with the data indicates that only very small deviations from two

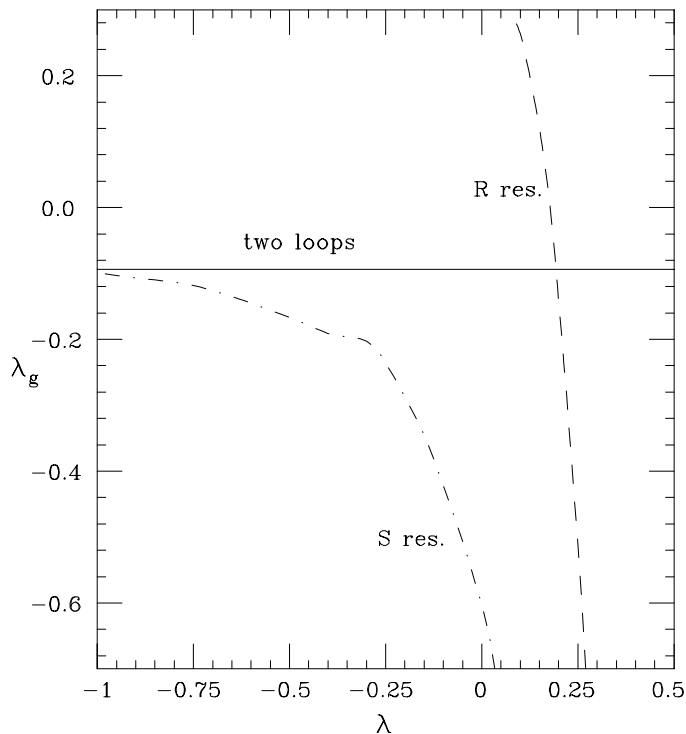


Figure 2: Starting gluon slope,  $G(x, 4 \text{ GeV}^2) \sim x^{\lambda_g}$  for the fit to the 95 H1 data [38] of the reduced cross section eq. (6.1) at two loops (solid), and in the double-leading expansion, S-resummation (dotdash) and R-resummation (dashes) as a function of  $\lambda$ . Here  $\alpha_s(m_z^2) = 0.119$ .

loops are tolerable.

The behaviour of the best-fit gluon exponent  $\lambda_g$  at the initial scale  $Q = Q_0$  is shown in fig. 2 as a function of  $\lambda$ . In general,  $\lambda_g$  increases with decreasing  $\lambda$ : the input gluon becomes more valencelike if the dynamical exponent from the evolution becomes larger. The fall of  $\lambda_g$  in the R-resummation is extremely steep: the input gluon changes rapidly in an attempt to compensate for the variation with  $\lambda$  of the singlet evolution. This is again a sign that this expansion requires fine tuning. However, for the fine tuned best-fit  $\lambda$ , the resummed value of  $\lambda_g$  is close to that obtained in the two loop fit. The results for  $\lambda_g$  found in the S-resummation are much more stable. In particular, we find that the input density is rather more valence-like than the unresummed two loop one throughout the region of small  $\lambda$  where the  $\chi^2$  is near the minimum. In fact, *all* the fits with a reasonably good  $\chi^2$  have a value of  $\lambda_g$  such that the input gluon density is flat or valencelike, and thus leads to double scaling behaviour (faster than any log but slower than any power) [14,15] when evolved into the HERA region.

Note that at negative values of  $\lambda$  the S-resummation inevitably approaches the two loop fit. This is because the resummation suppresses the high order terms when  $\lambda \rightarrow -\infty$ , so that the unmodified one and two loop terms dominate at small  $x$ , and the two loop results are recovered. Of course, we do not consider very large negative values of  $\lambda$  to be realistic: the first and second terms of the series for  $\lambda$  suggest that its value should be perhaps  $\lambda \gtrsim -0.5$ . Finally, note that at large values of  $\lambda$ , the S-resummation and the

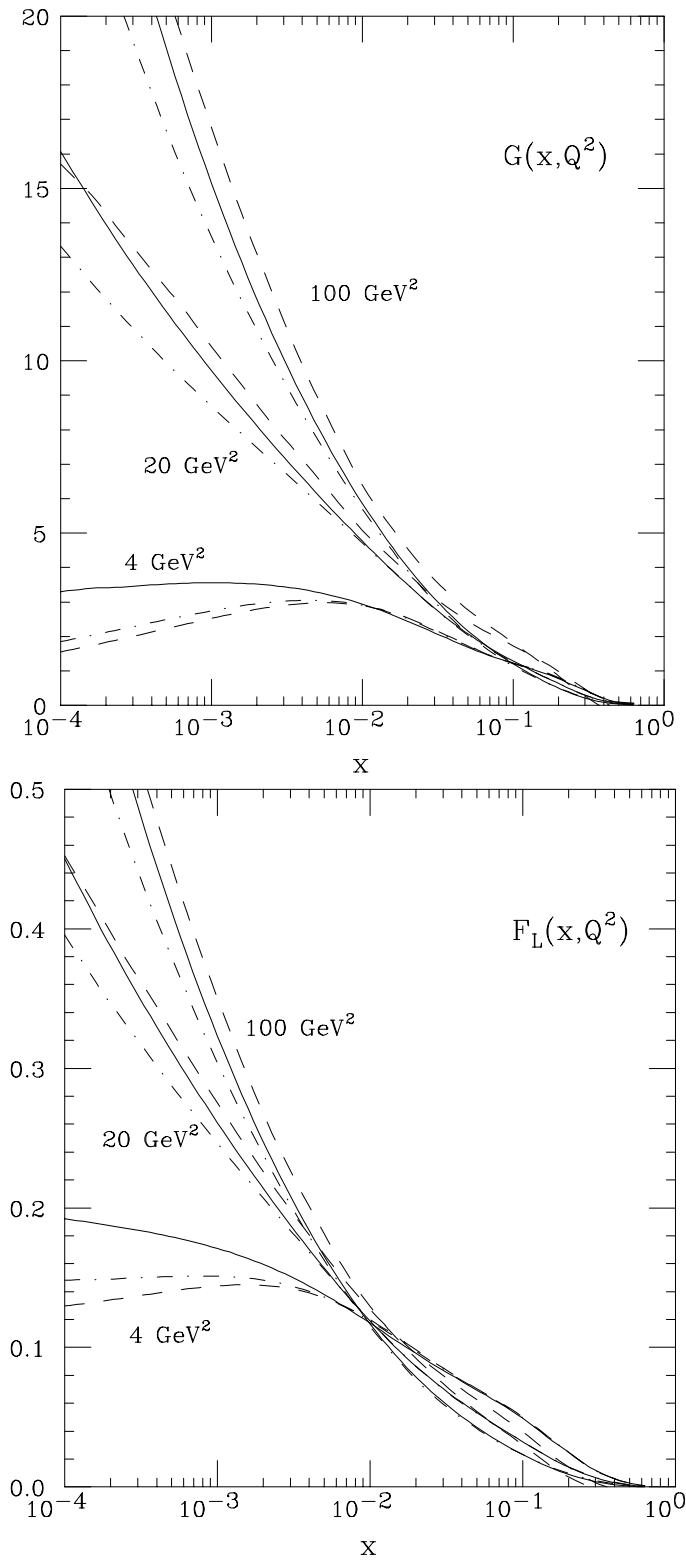


Figure 3: The gluon distribution and longitudinal structure function corresponding to the best-fit value of  $\lambda_g$  of fig. 2 at  $Q^2 = 4, 20$  and  $100 \text{ GeV}^2$  at two loops (solid), and in the double-leading expansion, S-resummation (dot-dash) and R-resummation (dashes).

R-resummation come together because the more singular behaviour of the higher order resummed terms makes the ambiguous terms negligible at small  $x$ , as discussed in the end of sec. 3.

The resummed gluon density and longitudinal structure function obtained in these fits are displayed in Fig. 3a,b as functions of  $x$  for three different values of  $Q^2$ . In particular, we compare results obtained in the two loop fit with the best fits in the S- and R-resummation. We see that even though at the starting scale  $Q_0^2 = 4 \text{ GeV}^2$  both the resummed gluon densities are more valencelike than the two loop one, when evolved to higher  $Q^2$  the R-resummation quickly becomes more singular. The same features can be seen reflected in  $F_L$ . This shows that the gluon extracted from a conventional unresummed fit could be sizeably different from the resummed one, which is sensitive to the small  $x$  corrections to the evolution equations. It also suggests that more precise data on  $F_L$  (or rather on the reduced cross-section at high  $y$ ) could serve to discriminate more precisely between the various alternatives.

## 5.2. Varying $\alpha_s(m_z^2)$

In order to explore the sensitivity of these results to  $\alpha_s$ , we first repeat the analysis with  $\alpha_s$  at the extremes of a range which is representative of the current uncertainty [40]:  $\alpha_s(m_z^2) = 0.116$  and  $\alpha_s(m_z^2) = 0.122$ . The  $\chi^2$  values are shown in Figs. 4a,b while the corresponding values of  $\lambda_g$  are in Figs. 5a,b. For low values of  $\alpha_s$  both resummations give significantly better fits than two loops, while for high values all the best fits are of similar quality. We already see at this stage that at two loops the best  $\chi^2$  is obtained when  $\alpha_s(m_z^2) = 0.122$ , while in both S and R resummations good fits may be obtained for each of the chosen values of  $\alpha_s$ . The qualitative behaviour of the gluon distribution as a function of  $\lambda$  is relatively insensitive to the value of  $\alpha_s$ ; however, as  $\alpha_s$  increases, all gluon distributions become more valencelike to compensate.

We now describe the results of a second round of fits where also the value of  $\alpha_s(m_z^2)$  is left free. The number of degrees of freedom is now  $n_{dof} = 92$ . In fig. 6 we show the resulting  $\chi^2$  as a function of  $\lambda$ . Some interesting new features are evident. For most values of  $\lambda$ , the minimum value of the  $\chi^2$  generally decreases when compared to the fits with fixed  $\alpha_s$  because we have one more parameter to adjust. The result in the R-resummation is similar to that shown in the fixed  $\alpha_s$  fits, but the range of  $\lambda$  where the quality of the fit is comparable with that of the unresummed two loop one is now somewhat wider. In the S-resummation, the best-fit  $\chi^2$  is now always close to the two loop value throughout the range of  $\lambda \lesssim 0.2$ . Thus, if  $\alpha_s(m_z^2)$  is left as a free parameter, the  $\chi^2$  is no longer improved by implementing the small  $x$  corrections. However, if we look at the fitted value of  $\alpha_s(m_z^2)$ , plotted in fig. 7, we see that even when the quality of the resummed and unresummed fits are essentially the same, the resummed best-fit value of  $\alpha_s(m_z^2)$  is generally lower than the two-loop one. Furthermore, as a function of  $\lambda$ , we see that  $\alpha_s(m_z^2)$  decreases as  $\lambda$  increases. Indeed, the value  $\alpha_s(m_z^2) \approx 0.122$  for the NLO fit is somewhat large in comparison with the world average central value  $\alpha_s(m_z^2) \approx 0.119$  (as it was in previous fits to older small  $x$  HERA data [43]). However the resummation tends to bring this value down: in the S-resummation in the range  $\lambda \lesssim 0$ ,  $\alpha_s(m_z^2)$  drops from 0.122 down to a value that can be as low as 0.115. For larger values of  $\lambda$  the value of  $\alpha_s$  decreases further but the quality of

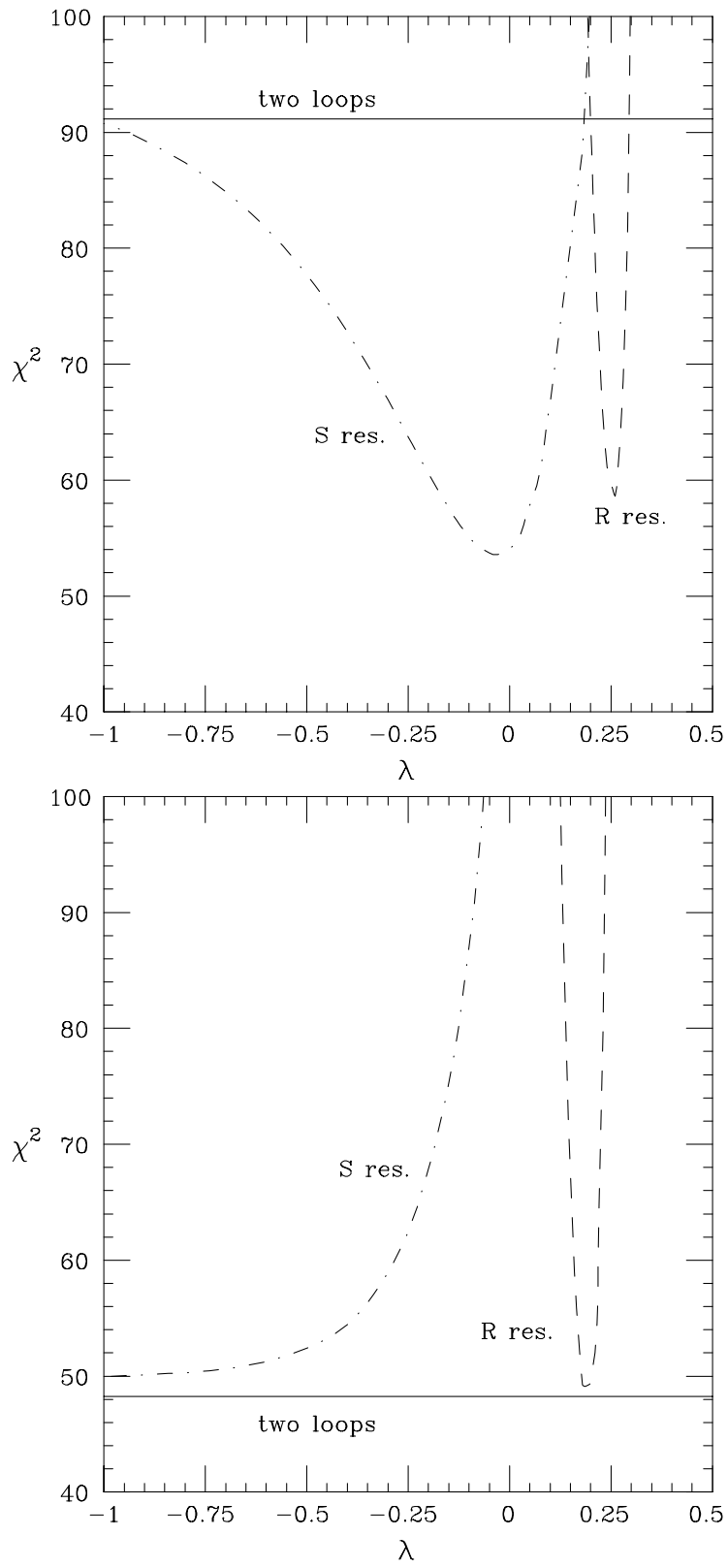


Figure 4: Same as Fig. 1, but with  $\alpha_s(m_z^2) = 0.116$  (above) or  $\alpha_s(m_z^2) = 0.122$  (below).

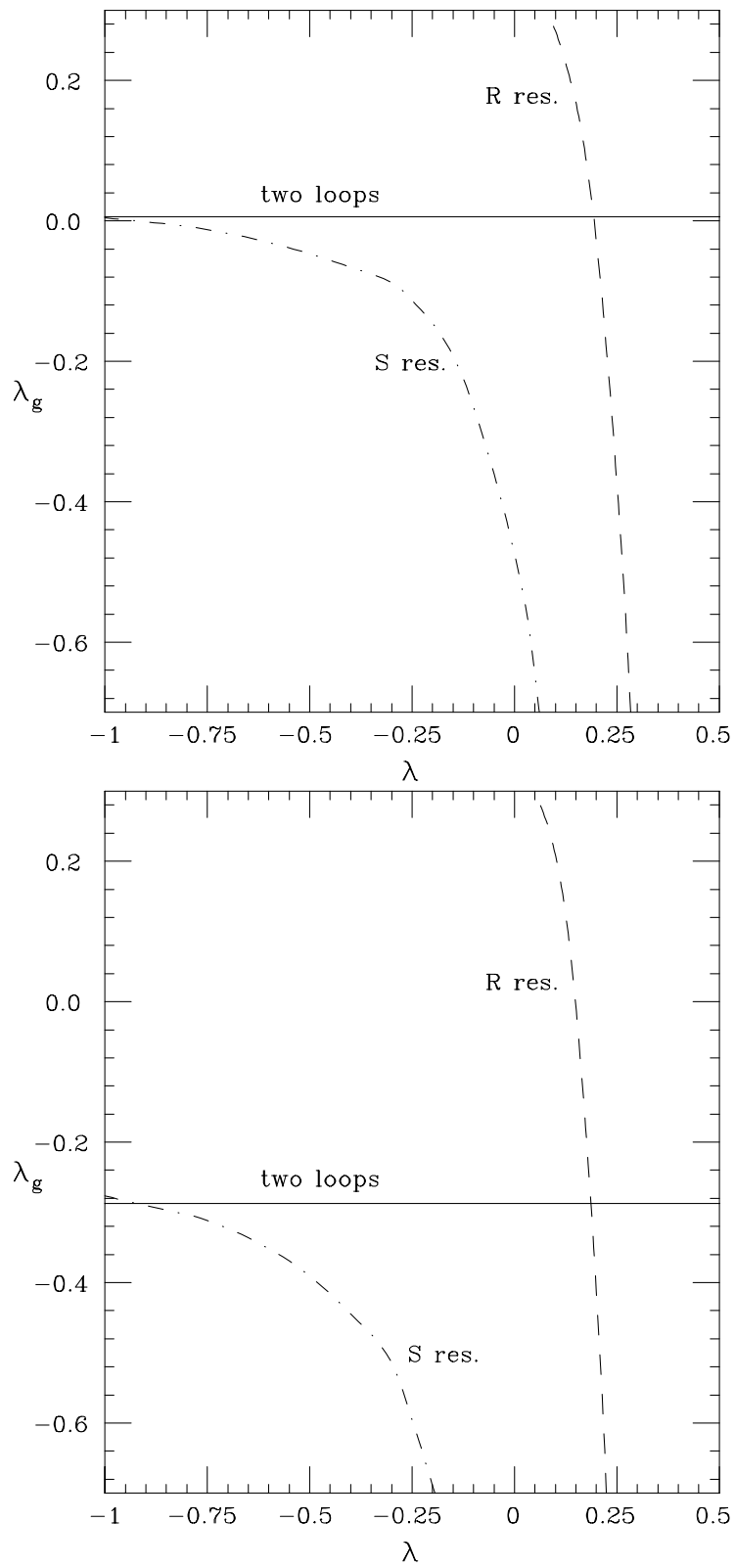


Figure 5: Same as Fig. 2, but with  $\alpha_s(m_z^2) = 0.116$  (above) or  $\alpha_s(m_z^2) = 0.122$  (below).

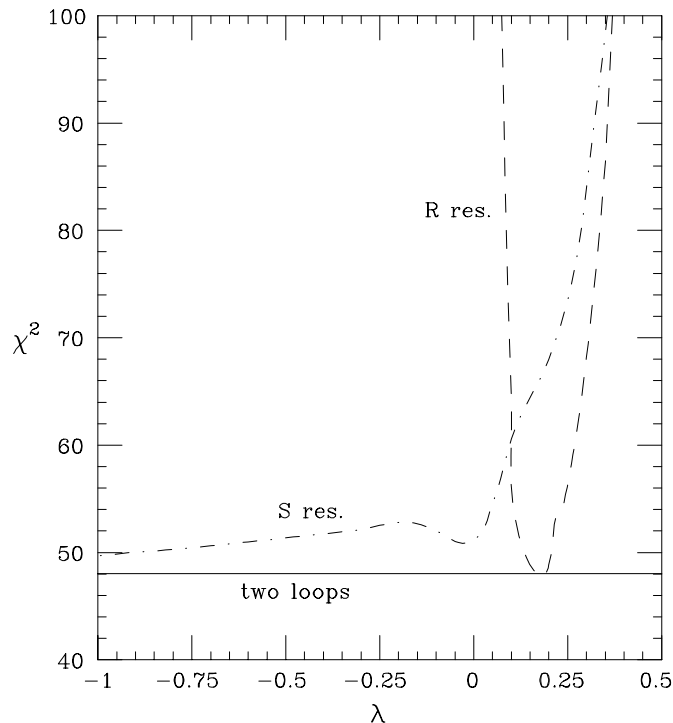


Figure 6: Same as Fig. 1, but with  $\alpha_s(m_z^2)$  now left as a free parameter and fitted for each value of  $\lambda$ .

fit deteriorates. This is sufficient to explain why if  $\alpha_s$  is fixed to the world average central value of 0.119, the resummed fits are in better agreement with the data than the two loop fit (Fig. 1). In the R-resummation we find again a rapid variation, though a result for  $\alpha_s(m_z^2)$  similar to the two loop one is reproduced in the small range of  $\lambda$  where the quality of the fit is good.

In all fits discussed so far, the  $Q^2$  behaviour of  $\lambda$  was assumed to be of the form  $\lambda(Q^2) = \alpha_s(Q^2)c$  where  $c$  is a constant, independent of  $Q^2$ . However, as discussed in sec. 3, lacking a more complete understanding of the effects of running on the value of  $\lambda$  and on its  $Q^2$  dependence, it is necessary to study the sensitivity of our results to this assumption. A possible alternative is to assume that, in the domain of the most relevant data, the actual  $\lambda$  can be replaced by some average value, independent of  $Q^2$ . We consider this as a somewhat drastic assumption; however, it seems appropriate to choose a rather extreme alternative given our ignorance of the true behaviour. The fixed  $\lambda$  case is also interesting for the theoretical reasons discussed in sec. 4: first, in this case there is no need to perform an extra subtraction in order to remove the instability of the effective anomalous dimension eq. (4.18), and also, the spurious poles (4.22) that appear in the effective NLLx BFKL function are absent in this case. We have argued in sec. 4 that the effect of both of these problems should in practice be small even when  $\lambda$  does depend on  $Q^2$ , provided the value of  $\lambda$  is itself reasonably small. We can now verify the correctness of this conclusion by checking that the results of the fits do not change very much if  $\lambda$  is taken to be scale-independent. In fig. 8 and 9 we compare the curves of fig. 6 and 7 with the corresponding ones for  $\lambda$  independent of  $Q^2$ . We see that indeed the changes are



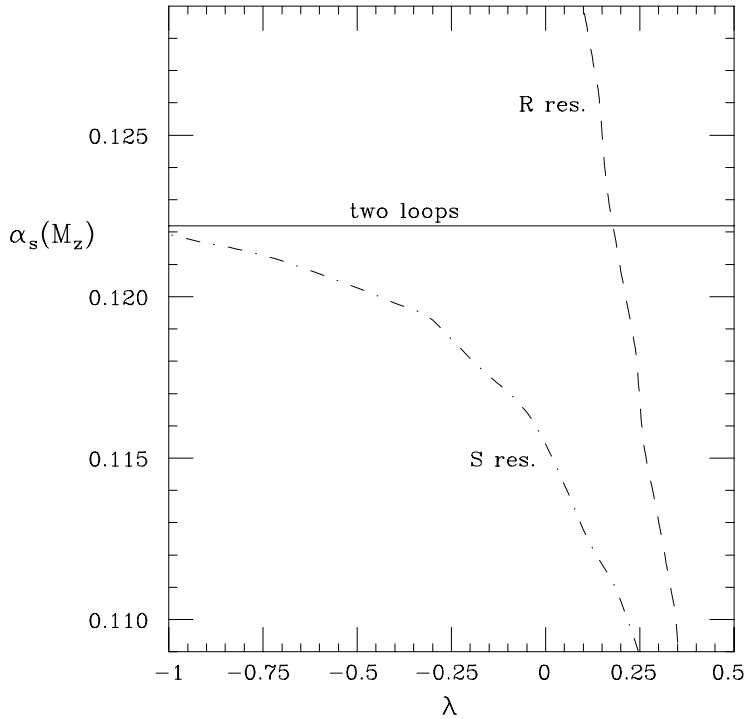


Figure 7: The best-fit values of  $\alpha_s(m_z^2)$  for the fits of Fig. 6.

indeed quite small, particularly in the case of the S-resummation, thereby supporting the reliability of our resummation procedure.

In fig. 8 and 9 we also show an additional horizontal line: this line corresponds to yet another determination of  $\lambda(Q^2)$ , in which we have taken the NLO perturbative result for  $\lambda(Q^2)$ , i.e.

$$\lambda(Q^2) = \alpha_s(Q^2)\chi_0(1/2) + \alpha_s^2(Q^2)\chi_1(1/2). \quad (5.2)$$

We see that with this choice, in the S-resummation, the  $\chi^2$  is essentially the same as the two loop one. This is because the perturbative  $\lambda(Q^2)$  is very small or negative in most of the region of the data and this is what is required to fit the data with the S-resummation. It is however remarkable that the precise form of the perturbative  $\lambda(Q)$  is in such a good agreement with the data. Note however that if the R-resummation is adopted instead, the perturbative  $\lambda(Q)$  is very far from fitting the data because it is much lower than the values of  $\lambda$  where the fit is acceptable. The dependence of the fitted values of  $\alpha_s(m_z^2)$  (Fig. 9) on the choice of these different prescriptions for the  $Q^2$  dependence of  $\lambda$  is similarly small.

In order to assess the impact of higher order corrections it is useful to study the renormalization scale dependence of physical observables, such as  $\alpha_s$  itself. The renormalization scale variation is done by letting

$$\frac{\alpha_s(Q^2)}{2\pi} \rightarrow \frac{\alpha_s(k_R Q^2)}{2\pi} \left[ 1 + \beta_0 \frac{\alpha_s(k_R Q^2)}{4\pi} \ln k_R \right] \quad (5.3)$$

everywhere in our computation and suppressing all subleading terms. In the resummed computations we take  $\lambda$  proportional to  $\alpha_s(k_R Q^2)$  with the constant of proportionality

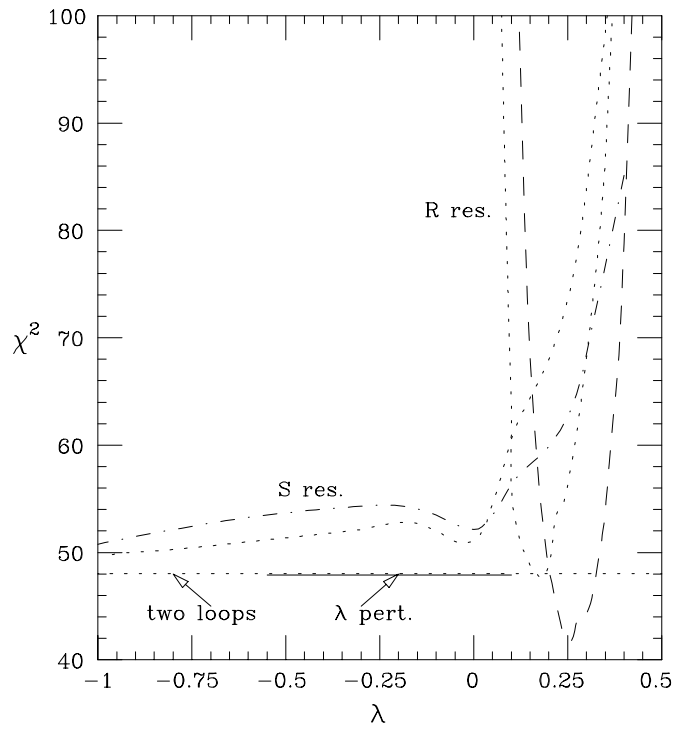


Figure 8: Comparison between the fits of Fig. 6 (dotted lines), the R-resummation (dashed) and S-resummation (dotdashed) fits with  $\lambda$  now taken to be constant, and the S-resummation fit with the perturbative expression eq. (6.2) of  $\lambda$  (solid line).

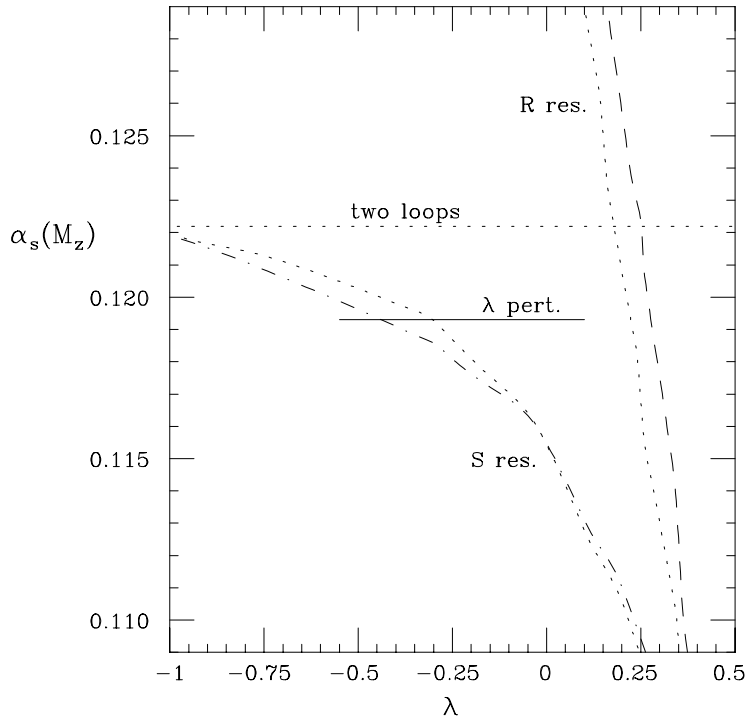


Figure 9: The best-fit values of  $\alpha_s(m_z^2)$  for the fits of Fig. 8.

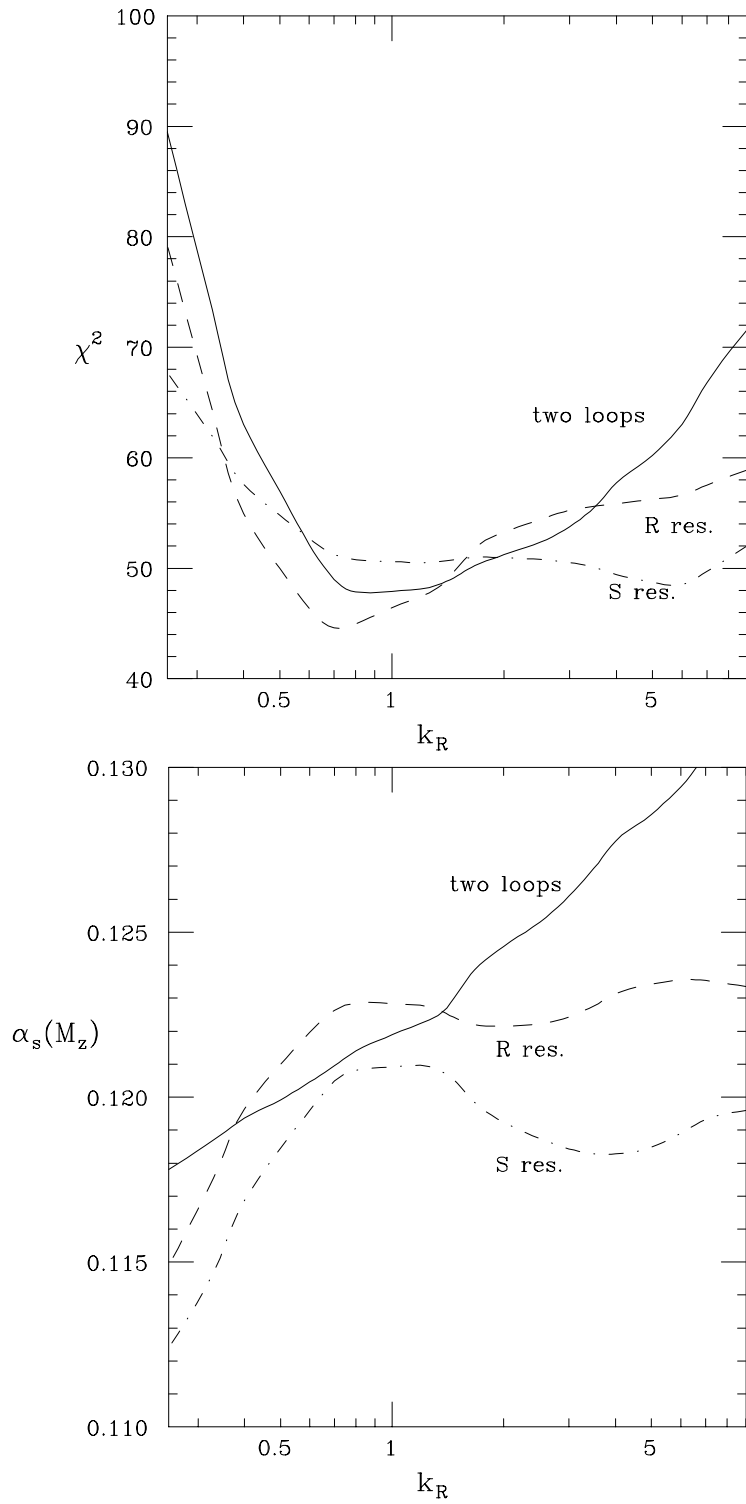


Figure 10: Dependence of the  $\chi^2$  (left) and the best-fit value of  $\alpha_s(m_z^2)$  (right) on the renormalization scale  $k_R$ . The value of  $\lambda$  is fitted for each choice of renormalization scale, along with the starting quark and gluon parameters  $\lambda_q$  and  $\lambda_g$ .

refitted as a function of  $k_R$ . The ensuing  $\chi^2$  and  $\alpha_s(m_z^2)$  are plotted as a function of  $k_R$  in Fig. 10a, b for the two loop, R-resummation and S-resummation fits. In the two loop case, the  $\chi^2$  grows quickly when  $k_R$  is moved away from unity, and  $\alpha_s$  grows monotonically with a particularly rapid rise at large scales. This suggests a large scale uncertainty in the value of  $\alpha_s$  extracted at two loops from HERA data [43,38], due to important higher order terms in the perturbation series. By contrast, both the resummed fits remain good even if the renormalization scale becomes very large, and the corresponding values of  $\alpha_s$  are surprisingly stable. This suggests that the error due to renormalization scale variation in a resummed determination of  $\alpha_s$  at HERA would be significantly less than in a two loop estimate. However, this would have to be offset against the extra uncertainty due to  $\lambda$  discussed above, as is clear from Fig. 10b.

## 6. Summary and Conclusion

Summarizing, we have compared our small  $x$  resummation formalism with structure function data from HERA. We recall that our main starting point is to assume that there is a region of  $x$  and  $Q^2$ , including the HERA kinematic region, where both the leading twist  $Q^2$  evolution and the BFKL small  $x$  evolution equations are simultaneously valid. Using the information from both equations, we construct a double leading expansion for the anomalous dimension. We then reorganise the expansion by factorizing out the small  $x$  behaviour of the splitting function, parametrised in terms of an exponent  $\lambda$ .

A priori, a variety of different resummations are conceivable. Here we discussed two distinct possibilities, the S and R resummations, but intermediate solutions could also be constructed. We find that the data are in good agreement with the resummed structure functions in the S-resummation approach, for a wide range of values of  $\lambda$ , starting from small positive values down to negative ones. Similarly, in the R-resummation there is also good agreement, but here only in a narrow range of positive values of  $\lambda$ . In each case the range of values of  $\lambda$  preferred by the data is such that the anomalous dimension is close to the two loop one. This may be evidence that a correct all-order treatment of the running coupling in the Regge limit would lead to an anomalous dimension with a singularity structure similar to the unresummed two-loop one.

We think that the data provide a very clear indication that the true value of  $\lambda$  is small or negative. It is an open question whether or not  $\lambda$  can be reliably computed in perturbation theory. In this respect it is interesting that in the S-resummation the perturbative evaluation of  $\lambda(\alpha_s)$  at 2 loops gives a particularly good fit to the data with a value of  $\alpha_s(m_z^2)$  in perfect agreement with the world average.

An important consequence of our analysis is that even though the NLO QCD formalism leads to an almost unequalled agreement with the data, still the extracted gluon density and the value of the strong coupling are significantly affected by the resummation, in a way which depends on the precise value of  $\lambda$ .

**Acknowledgements:** We would like to thank S. Catani, M. Ciafaloni, R.K. Ellis and G. Salam for discussions and correspondence. We are especially grateful to F. Zomer for useful discussions and comments on the H1 data. This work was supported in part by EU TMR contract FMRX-CT98-0194 (DG 12 - MIHT).

$n$	$a_n$	$b_n^0$	$b_n^f$
1	1	0	6.191566
2	0	-2.249876	-3.338781
3	0	0.527795	-0.551911
4	0.112797	0.224766	2.271531
5	0	-0.858826	-1.634828
6	0.012658	0.360812	0.058893
7	0.038170	0.016245	1.022436
8	0.001601	-0.417144	-0.980697
9	0.011422	0.245856	0.249555
10	0.017429	-0.064702	0.445905
11	0.002607	-0.206370	-0.590416
12	0.008884	0.157832	0.268752
13	0.009427	-0.084159	0.163843
14	0.003002	-0.097305	-0.341825
15	0.006709	0.094038	0.225093
16	0.005813	-0.077607	0.030933
17	0.003016	-0.041283	-0.185691
18	0.005078	0.050706	0.166627
19	0.004004	-0.062323	-0.024490
20	0.002832	-0.014100	-0.091471
21	0.003904	0.023129	0.113295
22	0.003016	-0.046270	-0.041019
23	0.002564	-0.002371	-0.037742
24	0.003069	0.006793	0.071643
25	0.002425	-0.032648	-0.039719
26	0.002277	0.001495	-0.009422
27	0.002475	-0.002039	0.042104
28	0.002039	-0.022298	-0.031744
29	0.002006	0.001740	0.003773
30	0.002047	-0.006178	0.022695

Table 1: The first thirty coefficients  $a_n$ ,  $b_n^0$ ,  $b_n^f$  from which  $\gamma_s$  and  $\gamma_{ss}$  may be computed.

$n$	$c_n$	$d_n^L$	$d_n^{MS}$
1	0.781460	-0.120225	0.536650
2	0.299133	0.277452	1.263035
3	0.388067	0.106613	0.770611
4	0.252563	0.007350	0.661812
5	0.178383	0.123681	0.786193
6	0.226323	0.050700	0.531599
7	0.154733	0.027507	0.518416
8	0.138915	0.075887	0.552254
9	0.157449	0.034088	0.414474
10	0.116020	0.031678	0.426128
11	0.115329	0.052513	0.425327
12	0.119970	0.027497	0.347879
13	0.095680	0.031055	0.361618
14	0.098463	0.039209	0.347915
15	0.097079	0.024334	0.303998
16	0.082914	0.028822	0.313664
17	0.085594	0.031073	0.296830
18	0.082037	0.022438	0.271865
19	0.073841	0.026192	0.276778
20	0.075504	0.025864	0.260999
21	0.071575	0.021013	0.246641
22	0.066829	0.023660	0.247790
23	0.067477	0.022391	0.234531
24	0.063926	0.019769	0.225970
25	0.061122	0.021414	0.224628
26	0.061018	0.019973	0.214079
27	0.058081	0.018611	0.208590
28	0.056334	0.019498	0.205838
29	0.055760	0.018207	0.197663
30	0.053434	0.017520	0.193751

Table 2: The first thirty coefficients  $c_n$ ,  $d_n^L$ ,  $d_n^{MS}$  with which the quark anomalous dimensions  $\gamma_{ss}^{qq}$ ,  $\gamma_{ss}^{qg}$ , the  $F_L$  coefficient functions  $C_{L,ss}^q$ ,  $C_{L,ss}^g$  and  $F_2$   $\overline{\text{MS}}$  coefficient functions  $C_{2,ss}^q$ ,  $C_{2,ss}^g$  may be computed.

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