Heavy → Light semileptonic decays of pseudoscalar mesons from lattice QCD

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Abstract

We have computed the form factors for $B \to \pi$ and $D \to K(\pi)$ semileptonic decays on the lattice by using full non-perturbative $\mathcal{O}(a)$ improvement, in the quenched approximation. Our results are expressed in terms of few parameters which describe the q^2-dependence and normalization of the form factors. After combining our results for $B \to \pi \ell \nu_\ell$ decay with the measured branching ratio, we obtain $|V_{ub}| = (4.1 \pm 1.1) \cdot 10^{-3}$.

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1 Introduction

In order to extract the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{ub}|$ from the experimentally measured semileptonic branching ratio, $B(B^0 \to \pi^- \ell^+ \nu)$ [1], it is essential to have a good theoretical control over the corresponding decay amplitude. This is difficult to solve from first principles because of non-perturbative hadronic effects.

The relevant matrix element for a generic heavy(H)→light(P) decay of pseudoscalar mesons is parameterized as

$$\langle P(p)|V_\mu|H(p_H)\rangle = \left(p_H + p - q m_H^2 - m_P^2 q^2\right)_\mu F_+(q^2) + m_H^2 - m_P^2 q^2 q_\mu F_0(q^2) , \qquad (1)$$

where $V_\mu = \bar{Q}\gamma_\mu q$ is the vector current involving the heavy (Q) and light (q) quark fields, $F_{0/+}(q^2)$ are the form factors which carry the dynamical information on non-perturbative QCD, and $q = (p_H - p)$ is the momentum transferred from the parent to the daughter meson.

In the case of massless leptons (which is an excellent approximation for $\ell = e,\ \mu$), the differential decay rate reads

$$\frac{d\Gamma}{dq^2}(H^0 \to P^- \ell^+ \nu_\ell) = \frac{G_F^2 |V_{qQ}|^2}{192\pi^3 m_H^3} \lambda^{3/2}(q^2) |F_+(q^2)|^2 , \qquad (2)$$

with $\lambda(q^2) = (q^2 + m_H^2 - m_P^2)^2 - 4m_H^2 m_P^2$, being the usual triangle function. To obtain the total width, one should integrate (2) over the entire (large) physical region, $0 \leq q^2 \leq (m_H - m_P)^2$, which requires the precise knowledge of the normalization (say $F_+(0)$) and the q^2-dependence of the form factor.

There have been many studies devoted to the computation of these form factors. Various quark models have been employed (see *e.g.* ref. [2]), which in many aspects helped phenomenological understanding of the heavy→light transitions. More quantitative predictions, however, can be only obtained with approaches which are based on first QCD principles: QCD sum rules [3] [1] and lattice QCD simulations [5].

In this paper we present new results on the form factors $F_{+/0}(q^2)$, as obtained from our computation of the matrix element (1) on the lattice. In order to reduce lattice artifacts, we work on a large lattice ($24^3 \times 64$) with a small spacing ($a \simeq 0.07$ fm, or $a^{-1} = 2.7(1)$ GeV), using the non-perturbatively $\mathcal{O}(a)$ improved Wilson action and operators [6], *i.e.* all leading lattice discretization effects proportional to the lattice spacing are absent. Due to the limitations on computational resources, however, one is obliged to make several approximations, of which the most important ones are the use of the quenched approximation and the inability to resolve hadrons heavier than 3 GeV. The latter implies that the final answer for $B \to \pi$ transition has to be reached through an extrapolation to m_B, which introduces large systematic uncertainties (which are of order 15%). An alternative is to discretize some effective theory, such as NRQCD, and to compute (1) from the opposite side, that is by extrapolating from infinitely heavy decaying mesons and eventually including higher order $1/m_H$ corrections [7]. We also mention that alternative strategies to treat $B \to \pi \ell \nu$ decay on the lattice have been developed in refs. [8, 9].

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[1] The most recent results for $B \to \pi$ transition, obtained by using the light cone QCD sum rules (LCSR), can be found in ref. [4].

The form factors are conveniently expressed in terms of three parameters (c_H, α_H, β_H), which define the normalization and the q^2 dependence and which follow simple scaling laws as functions of the heavy quark mass. In terms of these parameters we have:

$$F_+(q^2) \;=\; c_H \; (1 - \alpha_H)(1 - \tilde{q}^2) \; (1 - \alpha_H \tilde{q}^2) \,,$$

$$F_0(q^2) \;=\; c_H \; (1 - \alpha_H) 1 - \tilde{q}^2 \beta_H \,, \tag{3}$$

where $\tilde{q}^2 = q^2/m_{H^*}^2$, and m_{H^*} is the lightest meson that couples to the vector current in eq. (1). This form is convenient to be used in Monte Carlo simulations for the experimental analyses of semileptonic decays. The values of the parameters are given below, in eq. (4). The parameterization (3), which we refer to as BK, has been introduced in ref. [10], and it encodes most of the known constraints on the form factors. This includes:

i) the kinematical constraint, $F_+(0) = F_0(0)$;

ii) the heavy quark (meson) scaling laws predicted by the heavy quark effective theory (HQET), $F_+ \sim \sqrt{m_H}$ and $F_0 \sim 1/\sqrt{m_H}$, which are applicable in the zero recoil region $(\vec{q} \to 0,\ i.e.\ q^2 \to q_{\max}^2)$ [11];

iii) the heavy quark scaling law predicted by the large energy effective theory (LEET) and explicitly realized in the LCSR framework [2], $F_{+/0} \sim 1/m_H^{3/2}$, applicable in the large recoil region, $q^2 \to 0$ [13, 14];

iv) the knowledge of the position of the first pole in the crossed channel $(q^2 = m_{H^*}^2)$, determining the q^2-dependence of $F_+(q^2)$ in the small recoil region;

v) the symmetry relation between the two form factors $F_0 = (2E_P/m_H)F_+$, which is valid when the energy released to the light meson is large [13] [3].

In the absence of the last constraint v), one additional parameter would appear in eq. (3) [10]. However, we verified that the relation, $F_0 = (2E_P/m_H)F_+$, holds when applied to our data (at large recoils), within the statistical accuracy. Its validity has also been verified in the LCSR approach [16].

In the infinite quark mass limit, the quantities $(c_H\sqrt{m_H}, (1 - \alpha_H)m_H, (\beta_H - 1)m_H)$ should scale as a constant. As usual, the $1/m_H$ and $1/m_H^2$ corrections can be estimated from the fit with our lattice data. This allows us to extrapolate to the B meson mass. This strategy is employed in this paper and we call it *Method I*. Another way to handle the extrapolations has been recently proposed by the UKQCD collaboration [17]. In this case the assumptions are different, namely the extrapolations are performed first in the light meson at fixed q^2, and then in the heavy one at a fixed value of $v \cdot p$ where v is

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[2]This scaling law has been pointed out long ago by Chernyak and Zhitnitsky in ref. [12].

[3]Very recently, it has been shown in ref. [15] that this relation also holds to a high accuracy when radiative corrections are included.

the four-velocity of the heavy meson. The q^2 dependence of the extrapolated form factors is described by the parameterization (3). We also employed this strategy, which we call *Method II*. Both methods lead to fully consistent results. They are also in a good agreement with those of UKQCD [17, 18].

In the form (3), and by using the two different methods, we obtain:

$$
\begin{array}{cc}
\underline{Method\ I\ :} & \underline{Method\ II\ :} \\[1em]
F(0) = 0.26(5)(4) & F(0) = 0.28(6)(5) \\[1em]
c_B = 0.42(13)(4) & c_B = 0.51(8)(1) \\[1em]
\alpha_B = 0.40(15)(9) & \alpha_B = 0.45(17)^{+.06}_{-.13} \\[1em]
\beta_B = 1.22(14)^{+.12}_{-.03} & \beta_B = 1.20(13)^{+.15}_{-.00}
\end{array}
\tag{4}
$$

where the first errors are statistical and the second are systematic. The result for $F(0)$ is displayed for the reader's convenience, although it is clear that $F(0) = c_B(1 - \alpha_B)$. We note that the results of the two methods are also in good agreement with the LCSR results of ref. [16].

Knowing the normalization and the q^2-dependence of $F_+(q^2)$, we can integrate eq. (2) to get the decay width. We quote:

$$
1|V_{ub}|^2\ \Gamma(\bar{B}^0 \to \pi^+ \ell\bar{\nu}) = (7.0 \pm 2.9)\ \text{ps}^{-1}
\tag{5}
$$

obtained by combining the results of the two methods and adding the statistical and systematic errors in the quadrature. From this result, by using the measured branching ratio $B(\bar{B}^0 \to \pi^+ \ell\bar{\nu}) = (1.8 \pm 0.6) \cdot 10^{-4}$ [1] and the average B^0 meson lifetime $\tau_{B^0} = 1.548(32)$ ps [19], we get

$$
|V_{ub}| = (4.1 \pm 1.1) \cdot 10^{-3}\ .
\tag{6}
$$

In addition, we also study $\bar{D}^0 \to \pi^+ \ell\bar{\nu}$ and $\bar{D}^0 \to K^+ \ell\bar{\nu}$ decays and compare the results with the experimental data.

The remainder of this paper is organized as follows: in sec. 2 we give the information on the lattice setup and describe the extraction of the raw form factor data; in sec. 3 we explain the details used in extrapolation procedures to get the physical results which are presented in sec. 4 for $B \to \pi\ell\nu$ and in sec. 5 for $D \to K(\pi)\ell\nu$; we briefly conclude in sec. 6.

2   Lattice Setup and Computation of the Matrix Element

In this section we give the essential information about the lattice setup used in our simulation. We also sketch the strategy to compute the matrix element (1) improved at $\mathcal{O}(a)$.

c_{SW}	[20]	1.614
$\kappa_{heavy} \equiv \kappa_Q$		0.1250; 0.1220; 0.1190; 0.1150
$\kappa_{light} \equiv \kappa_q$		0.1344; 0.1349; 0.1352
$a^{-1}(m_{K^*})$		2.7(1) GeV
κ_{cr}		0.13585(2)
$Z_V^{(0)}$	[21, 22]	0.79
c_V	[22]	-0.09
b_V	[21, 22]	1.40

Table 1: Parameters used in this work: the Wilson hopping parameters corresponding to the heavy (κ_Q) and light (κ_q) quark masses; c_{SW} provides the non-perturbative $\mathcal{O}(a)$ improvement of the Wilson action; c_V, b_V, $Z_V^{(0)}$ ensure that the vector current is free of $\mathcal{O}(a)$ effects (see eq. (8)). For each parameter we quote the reference where the quantity has been computed non-perturbatively. The critical hopping parameter (κ_{cr}) and the inverse lattice spacing (a^{-1}) are fixed as explained in ref. [23].

The results presented in this paper are obtained from a simulation on a lattice of size $24^3 \times 64$, using a sample of 200 independent gauge field configurations generated, at $\beta = 6.2$, in the quenched approximation. The values of the Wilson hopping parameters corresponding to the heavy and light quarks are listed in tab. 1. In the same table, we give the values of the other parameters necessary for the $\mathcal{O}(a)$ improvement which we implement in this work (note that all the parameters are determined non-perturbatively). The statistical errors are estimated by using the standard jackknife procedure, by decimating each time 5 configurations from the whole sample. In our previous papers [23, 24], we discussed the mass spectrum and decay constants using the $\mathcal{O}(a)$ improvement as obtained on a subset of 100 configurations [4]. The updated values of our heavy-light pseudoscalar (H_d) and vector (H_d^*) meson masses in physical units, obtained after extrapolating to the average $up - down$ quark mass (see ref. [24] for details) are:

$$m_{H_d}[\text{GeV}] \;\; = \;\; \{\; 1.74(7),\; 2.01(8),\; 2.25(9),\; 2.56(10) \;\}$$

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[4]The values of the heavy-light decay constants were updated in refs. [25, 26].

$$m_{H_d^*}[\text{GeV}] \;\;=\;\; \{\; 1.86(7),\; 2.10(8),\; 2.33(9),\; 2.63(11) \;\}\;. \tag{7}$$

As in our previous publications, we denote the meson masses in lattice units by capital letters, whereas the small case letters are used for the same masses in physical units, *e.g.* $M_H = m_H a$.

To compute the matrix element (1) on the lattice with improved Wilson fermions, the appropriate definition of the vector current reads [21]

$$\hat{V}_\mu = Z_V^{(0)} \left(1 + \frac{b_V}{2}(am_Q + am_q) \right) \left[\bar{Q}\gamma_\mu q \; + \; ac_V \cdot i\partial_\nu \bar{Q}\sigma_{\mu\nu} q \right] \tag{8}$$

with $\sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]$, and where the renormalization constants $Z_V^{(0)}$ and b_V, as well as the subtraction constant c_V, are known quantities (in tab. 1 we also quote the reference in which a given quantity has been computed non-perturbatively). However, since we deal with the heavy-light operator, one can suspect that higher order terms, $\propto (am)^n$ $(n \geq 2)$, may be important. Their cancellation at tree level can be achieved by multiplying Z_V by the KLM factor [27] which we modify to include $\mathcal{O}(a)$ improvement [24], *i.e.*

$$Z_V^{\text{KLM}} = Z_V^{(0)} \sqrt{1 + am_q}\sqrt{1 + am_Q} 1 + a\bar{m}\left(1 + b_V a\bar{m}\right)\;, \tag{9}$$

where $am_{q\{Q\}} = (1/\kappa_{q\{Q\}} - 1/\kappa_{cr})/2$ and $\bar{m} = (m_q + m_Q)/2$.

To access the matrix element (1) on the lattice, one computes the following three- and two-point correlation functions

$$\mathcal{C}_\mu^{(3)}(t_x, t_y; \vec{q}, \vec{p}_H) \;\; = \;\; \int d\vec{x}d\vec{y}\; e^{i(\vec{q}\cdot\vec{y} - \vec{p}_H\cdot\vec{x})}\langle P(0)\hat{V}_\mu(y)H^\dagger(x)\rangle\,,$$

$$\mathcal{C}_{JJ}^{(2)}(t) \;\; = \;\; \int d\vec{x}\; e^{i\vec{p}\cdot\vec{x}}\langle J(\vec{x}, t)J^\dagger(0)\rangle\,, \tag{10}$$

where $H = \bar{q}i\gamma_5 Q$ and $P = \bar{q}i\gamma_5 q$ are the source operators of the heavy-light and light-light pseudoscalar mesons, respectively, and J denotes any of the two (H or P). On the temporal axis in $\mathcal{C}_\mu^{(3)}$, we fixed $H(x)$ to $t_x = 27$, and the improved renormalized vector current $\hat{V}_\mu(y)$ defined in eq. (8) is free $(0 \leq t_y \leq 63)$. $\vec{p}_H$ is the three-momentum given to the decaying meson, and $\vec{q}$ denotes the momentum transferred to the daughter meson $(\vec{q} = \vec{p}_H - \vec{p})$. In this work we consider the following 13 kinematical configurations:

$$\underline{\vec{p}_H = (0,0,0)} \quad \vec{q} = \{\, (0,0,0)\,;(1,0,0)\,;(1,1,0)\,\} \tag{11}$$

$$\underline{\vec{p}_H = (1,0,0)} \quad \vec{q} = \{\, (0,0,0)\,;(1,0,0)\,;(0,1,0)\,;(2,0,0)\,;(1,1,0)\,;(1,1,1)\,;(2,1,0)\,\}$$

$$\underline{\vec{p}_H = (1,1,0)} \quad \vec{q} = \{\, (1,0,0)\,;(1,1,0)\,;(2,1,0)\,\}$$

where each component is given in units of elementary momentum on the lattice $(2\pi/La)$. As usual, to get a better statistical quality of the signal, we averaged over the equivalent
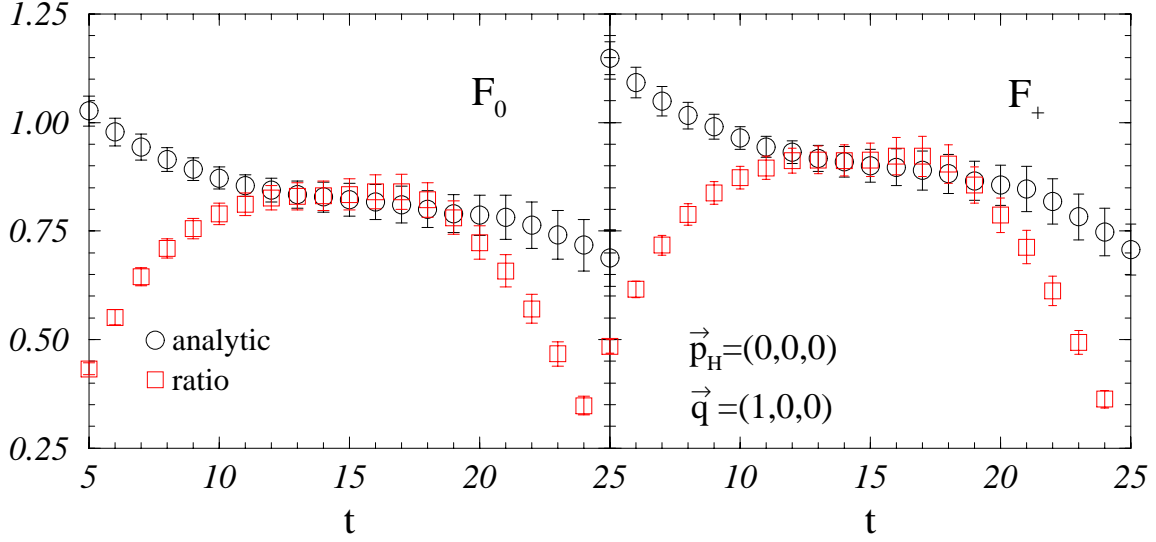
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Figure 1: Plateaus of the two form factors for the specific combination of the heavy and light quark masses (corresponding to $\kappa_Q = 0.1220$ and $\kappa_q = 0.1349$). The form factors are extracted from the fit in $t \in [12, 15]$.

configurations, *i.e.* the ones that can be obtained by applying the cubic lattice symmetry, as well as the parity and charge conjugation transformations.

From the asymptotic behavior of the three-point correlation function

$$\mathcal{C}_\mu^{(3)}(t_x, t_y; \vec{q}, \vec{p}_H) \xrightarrow{t_x \gg t_y \gg 0} \sqrt{\mathcal{Z}_P} 2E_P \ e^{-E_P t_y} \cdot \langle P(\vec{p}_H - \vec{q})|\hat{V}_\mu(0)|H(\vec{p}_H)\rangle \cdot \sqrt{\mathcal{Z}_H} 2E_H \ e^{-E_H(t_x - t_y)} , \quad (12)$$

it is obvious that the removal of the exponential factors can be achieved by considering the ratio

$$R_\mu(t_y) = \mathcal{C}_\mu^{(3)}(t_x, t_y; \vec{q}, \vec{p}_H) \quad \mathcal{C}_{PP}^{(2)}(t_y, \vec{p}_H - \vec{q}) \ \mathcal{C}_{HH}^{(2)}(t_x - t_y, \vec{p}_H) \quad \cdot \sqrt{\mathcal{Z}_H} \sqrt{\mathcal{Z}_P} , \quad (13)$$

where $\sqrt{\mathcal{Z}_H} = |\langle 0|\bar{q}i\gamma_5 Q|H\rangle|$ and $\sqrt{\mathcal{Z}_P} = |\langle 0|\bar{q}i\gamma_5 q|P\rangle|$, are obtained from the fit with asymptotic behavior of the two-point correlation functions.

$$\mathcal{C}_{JJ}^{(2)}(t) \xrightarrow{t \gg 0} \mathcal{Z}_J 2E_J e^{-E_J t} . \quad (14)$$

When the operators in the ratio (13) are sufficiently separated, one observes the stable signal (plateau), which is the desired matrix element:

$$R_\mu(t_y) \xrightarrow{t_x \gg t_y \gg 0} \langle P(p)|\hat{V}_\mu|H(p_H)\rangle . \quad (15)$$

The plateaus are typically found for $12 \le t_y \le 15$. Fitting $R_\mu(t_y)$ to a constant in that interval, we extract the matrix element (1). The other possibility is to express $R_\mu(t_y)$ in terms of form factors for each time-slice, $t \equiv t_y$, and then to fit each form factor to a constant in the common plateau interval. As an important cross check of this procedure, we employ the following two methods [28]:
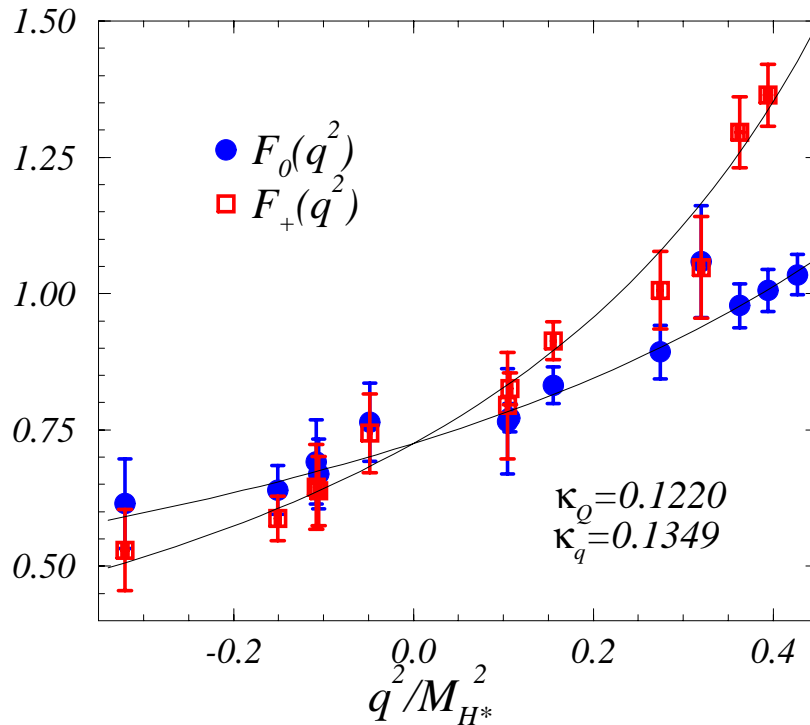
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Figure 2: The form factors $F_{0/+}(q^2)$ for $\kappa_q = 0.1349$ and $\kappa_Q = 0.1220$, plotted as functions of q^2/M_V^2, where M_V is the mass of the lightest vector meson exchanged in the t-channel. The lines show the fit to the parameterization (3).

– **Ratio method** consists in forming the ratio (13) of the correlation functions computed on the lattice, and using $\mathcal{Z}_P$ and $\mathcal{Z}_H$ obtained from the fit with the corresponding two-point correlation functions in the same time-interval as the one at which the plateau has been attained.

– **Analytic method** refers to the procedure in which we replace the two-point correlation functions in eq. (13) by their asymptotic expressions (14) and $\mathcal{Z}_P$ and $\mathcal{Z}_H$ are extracted from the separate study of the two-point correlation functions. In this case we also take into account the symmetry of $\mathcal{C}_{JJ}^{(2)}(t)$ with respect to the time inversion, $t \to (64 - t)$. In this work we use the latticized free boson dispersion relation:

$$\sinh^2 (E_J(\vec{p})2) = \sinh^2 (M_J 2) + \sin^2 (\vec{p}2) \ , \qquad (16)$$

which appears to be appropriate in describing our data [23, 24].

The differences in the results obtained by using the ratio and analytic methods are always smaller than the statistical errors. In fig. 1, we illustrate the form factors $F_{0/+}(t)$ as functions of time $t \equiv t_y$, obtained from the ratio (13) by using both the ratio and the analytic methods. From now on, our central results will always be those obtained by using the ratio method. The difference between that result and the one we get by applying the analytic method, will be added to the systematic uncertainties.

3   Getting to the physical results: Extrapolations

Once we computed the form factors for all 13 momentum combinations (11), we need to address the question of the mass extrapolations. Firstly, since our light meson masses are in the range $0.54 \text{ GeV} \lesssim m_P \lesssim 0.84 \text{ GeV}$, a small extrapolation to the physical kaon and a longer one to the pion mass are needed. Secondly, the extrapolation in the heavy quark mass is needed, if we are to reach the physical results relevant for $B \to \pi$ decay. These extrapolations are closely related to the q^2 fit of the form factors because both extrapolations, in light and heavy mesons, obviously enlarge the physical region accessible from the semileptonic decay, $q^2 \in [0, (m_H - m_P)^2]$. Before discussing the extrapolations, we list in tab. 2 the BK-parameters (3) resulting from the fit to our data for each combination of the heavy and light quarks. One such a fit is shown in fig. 2.

$\kappa_Q - \kappa_q$	c_H	α_H	β_H	$F^{H \to P}(0)$
$1250 - 1344$	$1.23(12)$	$0.30(7)$	$1.38(11)$	$0.87(2)$
$1250 - 1349$	$1.09(13)$	$0.29(9)$	$1.37(12)$	$0.77(3)$
$1250 - 1352$	$0.99(13)$	$0.29(11)$	$1.33(13)$	$0.70(4)$
$1220 - 1344$	$1.13(12)$	$0.28(9)$	$1.42(13)$	$0.82(2)$
$1220 - 1349$	$0.99(12)$	$0.26(10)$	$1.41(13)$	$0.73(3)$
$1220 - 1352$	$0.88(11)$	$0.26(12)$	$1.35(14)$	$0.65(4)$
$1190 - 1344$	$1.08(12)$	$0.28(9)$	$1.43(13)$	$0.78(3)$
$1190 - 1349$	$0.93(11)$	$0.26(11)$	$1.42(14)$	$0.69(3)$
$1190 - 1352$	$0.83(10)$	$0.27(11)$	$1.33(12)$	$0.61(4)$
$1150 - 1344$	$1.02(11)$	$0.28(10)$	$1.43(14)$	$0.73(3)$
$1150 - 1349$	$0.88(10)$	$0.26(12)$	$1.42(14)$	$0.65(4)$
$1150 - 1352$	$0.77(9)$	$0.25(12)$	$1.35(13)$	$0.58(4)$

Table 2: The results of the fit of our form factor data to the parameterization given in eq. (3), for all combinations of $\kappa_Q - \kappa_q$.

3.1   Method I: Extrapolation of the parameters

The parameters $\varphi_i = (c_H, \alpha_H, \beta_H)$, for each pair of the heavy and light hopping parameters (see tab. 2) give us information on the normalization and the q^2-dependences of both form factors. For fixed heavy quark, one can apply the lattice plane method [29] to extrapolate to the case when the final meson is either a kaon or a pion. This essentially means a linear
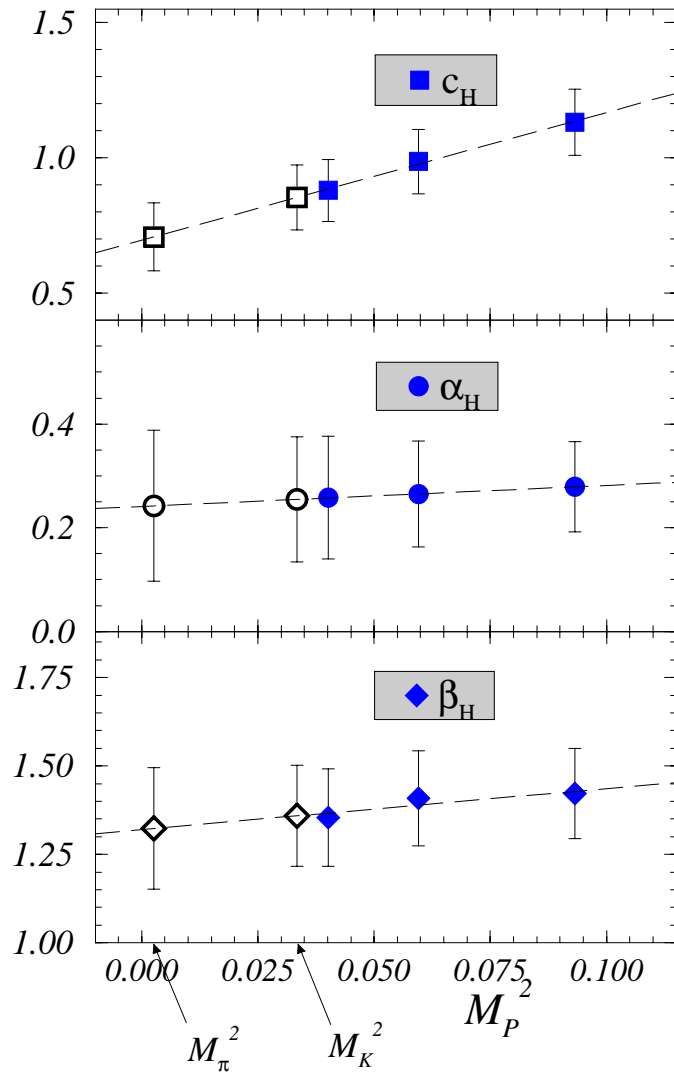
Figure 3: Extrapolation of each of the parameters (c_H, α_H, β_H) to the final kaon/pion. The plot is made for the heavy quark corresponding to $\kappa_Q = 0.1220$.

	Final meson $-$ kaon				Final meson $-$ pion			
κ_Q	c_H	α_H	β_H	$F(0)$	c_H	α_H	β_H	$F(0)$
0.1250	0.96(14)	0.29(11)	1.33(13)	0.69(4)	0.82(16)	0.29(14)	1.30(17)	0.59(6)
0.1220	0.85(12)	0.25(12)	1.36(14)	0.64(4)	0.71(12)	0.24(15)	1.32(17)	0.55(6)
0.1190	0.80(11)	0.26(12)	1.34(13)	0.60(4)	0.66(11)	0.25(14)	1.29(15)	0.51(6)
0.1150	0.74(10)	0.24(13)	1.35(13)	0.57(4)	0.60(10)	0.23(15)	1.31(15)	0.48(6)

Table 3: Values of the BK-parameters after extrapolating to the final π and K meson. The extrapolation of the form factor at $q^2 = 0$ provides a cross check of the extrapolation.

fit in the light quark mass ($M_P^2 \propto m_q$) for each parameter:

$$\varphi_i = A_i + B_i\,M_P^2\,, \tag{17}$$

and then an extrapolation to M_π^2, determined in the (M_P^2, M_V) plane by fixing the ratio M_V/M_P to the physical point m_ρ/m_π. Similarly, the value of M_K^2 is determined from the physical m_{K^*}/m_K ratio. The result of that extrapolation is illustrated in fig. 3, and the extrapolated parameters are listed in tab. 3. The form factor at $q^2 = 0$ is just the combination $c_H(1 - \alpha_H)$. We note that whether we extrapolate the parameters c_H and α_H to the final pion (kaon) separately and then combine them to $F(0)$, or we combine the parameters to obtain $F(0)$ and then extrapolate to the pion (kaon), the result remains the same. The latter number is also given in tab. 3.

After the extrapolation in the light mass, one needs to fit the parameters in the inverse heavy meson mass and to interpolate to the D or extrapolate to the B meson mass. The quantities which scale in the infinitely heavy quark limit (ϕ), are fitted as

$$\phi = b_0 + b_1 M_H + b_2 M_H^2\,, \quad \text{where} \quad \phi \in \left\{ c_H\sqrt{M_H},\ (1 - \alpha_H)M_H,\ (\beta_H - 1)M_H \right\}\,, \tag{18}$$

where we neglect the logarithmic corrections. In fig. 4 we show the $1/M_H$ behavior of each parameter for the case in which the final light meson is the pion. Again, as a cross check, one can also fit $F(0)M_H^{3/2}$ to the form (18). The results for $F(0)$ extrapolated to M_B in that way is fully consistent with $F(0)$ obtained by combining separately the extrapolated values of c_B and α_B to $F(0) = c_B(1 - \alpha_B)$. As can be seen from fig. 4, however, a systematic difference in the extrapolated values of the parameters is obtained depending on whether we perform a linear ($b_2 = 0$) or a quadratic (b_2 free) fit in the inverse heavy meson mass. Indeed, besides the quenched approximation, this difference represents the main systematic uncertainty present in our calculation. To make a comparison with other approaches easier, we give the values of the $1/m_H$ corrections (in physical units) that are obtained from the linear and quadratic fit with our data:

$$F(0) = (3.1 \pm 0.5)\ \text{GeV}^{3/2} m_H^{3/2} \times [\,1\ -\ (0.98 \pm 0.09)\ \text{GeV} m_H\,]\,,$$
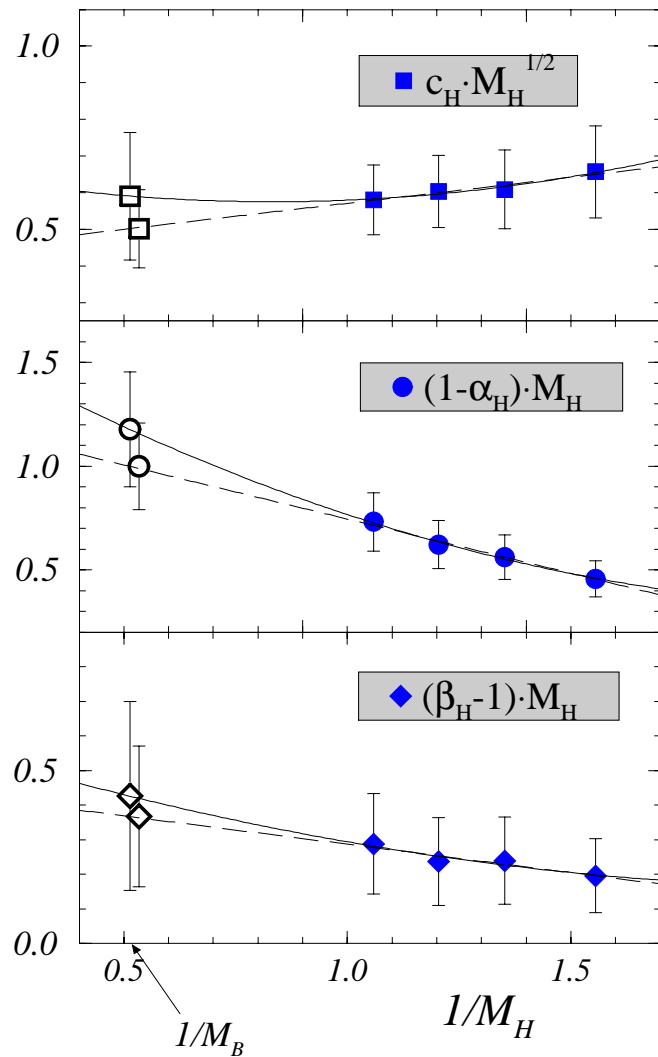
Figure 4: Extrapolation in the heavy meson mass. The curves represent the results of the quadratic (full) and linear (dashed) extrapolations.

$$F(0) = (4.7 \pm 1.1) \text{ GeV}^{3/2} m_H^{3/2} \times \left[\ 1\ -\ (2.0 \pm 0.3) \text{ GeV} m_H\ +\ (1.2 \pm 0.1 \text{ GeV})^2 m_H^2\ \right] \tag{19}$$

We observe, in passing, that the $1/m_H$ corrections are large and comparable to the ones which appear with the heavy-light decay constants [24].

At this point we should decide which result, from either the linear or quadratic fit, to quote as the central one. As a criterion for that, we test whether the result of the extrapolation satisfies the soft pion theorem, *i.e.* the Callan-Treiman relation applied to the case of $B \to \pi \ell \nu$ [30]: [5]

$$F_0(m_B^2)\ =\ f_B\ f_\pi\ , \tag{20}$$

where f_B and f_π are the corresponding decay constants. The values of the ratio f_B/f_π will be given in the next section. It turns out that after the quadratic extrapolation in $1/M_H$ (18), we find a better consistency of $F_0(m_B^2)$ with f_B/f_π. Thus, the result of the quadratic $1/M_H$-extrapolation will be taken as our central value, and the difference between the quadratically and linearly extrapolated values will be accounted for in the estimate of the systematic error. All the numerical results of the extrapolations will be given in next section, where we discuss our physical results.

3.2   Method II: UKQCD

The second method that we use in this paper is the one proposed by the UKQCD collaboration in ref. [17]. As in the previous subsection, the starting point are the results given in tab. 2. For a given value of the heavy quark mass, one extrapolates the form factors to the final pion (kaon) by using (17). In order to separate the intrinsic light quark mass dependence of the form factors from the one that arises from the variation in q^2, the extrapolation in the light quark mass should be made at fixed q^2 [17]. In other words, we consider the extrapolation to the pion for a constant $v \cdot p$, which is defined as [6]

$$v \cdot p = M_{H_d}^2 + M_\pi^2 - q^2 2 M_{H_d}\ , \tag{21}$$

where v is the four-velocity of the heavy meson and p is the momentum of the light meson. We note that the lattice results for the form factors are in the region $0.2 \le v \cdot p \le 0.5$. In this window we could fix 7 equidistant points $v \cdot p = 0.20, 0.25, \ldots 0.50$, where the distance between the points is taken to be the same as in ref. [17]. We used the fit to the parameterization (3) [7] to interpolate to the chosen values of $v \cdot p$. For each fixed $v \cdot p$, we then extrapolate to the final pion as described in the previous subsection. A typical situation is displayed in fig. 5, where the heavy quark corresponds to $\kappa_Q = 0.1220$, *i.e.*

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[5]An extensive high statistic study of this relation on the lattice, in the NRQCD framework, has been made in ref. [7].

[6]Note that the light meson mass in eq. (21) is the physical pion mass.

[7]Other interpolation forms are also applicable, such as the pole formula (with the pole mass left as a free parameter), which is what has been done in ref. [17]. We checked that the final results on the form factors do not depend on the form used in this interpolation.
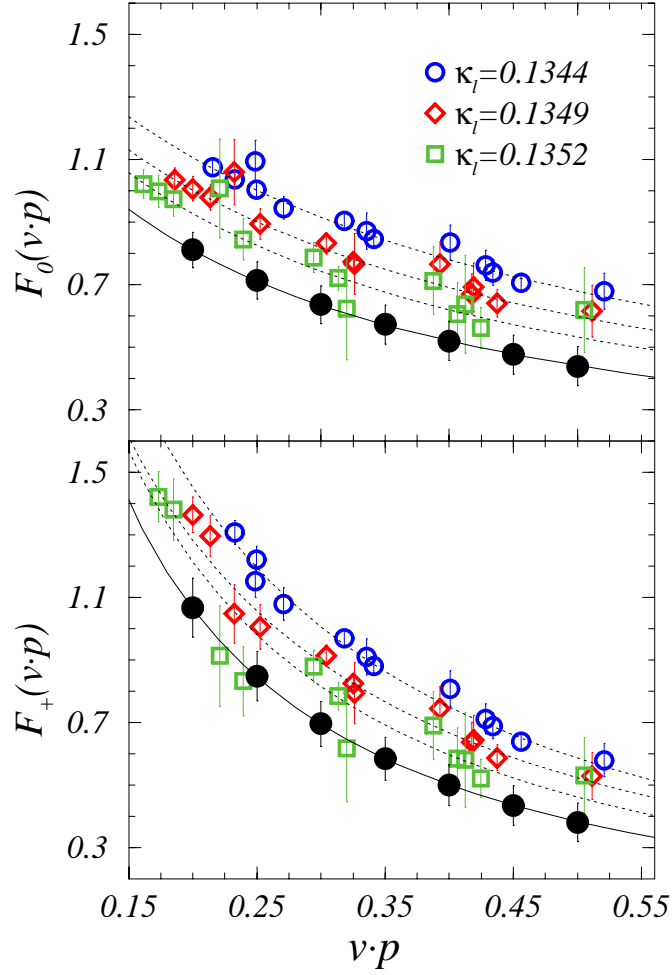
Figure 5: Lattice data for the form factors $F_{0/+}(v \cdot p)$ with the heavy quark fixed to $\kappa_Q = 0.1220$, and for three different values of light quark masses. Dotted (full) curves represent the fit to the parameterization (3). Filled symbols denote the form factors extrapolated to the final pion at a given value of $v \cdot p$ which has been fixed according to eq. (21).
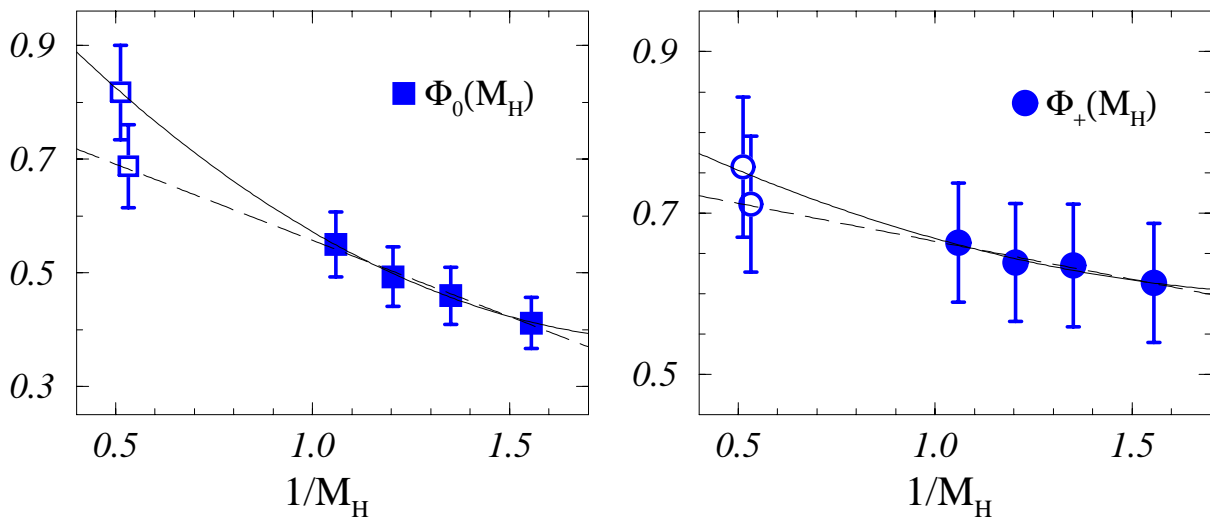
14

Figure 6: Extrapolation of the form factors (22) at $v \cdot p = 0.35$ in the inverse heavy meson mass to the B_d meson. Dashed and full curves correspond to the linear and quadratic extrapolation, respectively.

$M_{H_d} = 0.739(2)$ in lattice units. The result of the extrapolation is marked by the filled circles in fig. 5.

We now have to extrapolate in the heavy quark mass for which one uses the heavy quark scaling laws. For each fixed value of $v \cdot p$, we extrapolate

$$\Phi(M_H) = a_0 + a_1 M_H + a_2 M_H , \qquad (22)$$

where

$$\Phi(M_H) = (\alpha_s(M_B)\alpha_s(M_H))^{-\widetilde{\gamma}_0/2\beta_0} \times \left\{ F_0(v \cdot p)\sqrt{M_H} , F_+(v \cdot p)\sqrt{M_H} \right\} \qquad (23)$$

The term multiplying both form factors gives the logarithmic dependence on the heavy meson mass predicted by the HQET in which the vector current has the anomalous dimension whose leading order coefficient is $\widetilde{\gamma}_0 = -4$ [31] [8]. As usual, the extrapolation in $1/M_H$ (22) is performed either linearly ($a_2 = 0$), or quadratically (a_2 free). Obviously, the HQET scaling law [11] is valid in the region close to the zero-recoil ($v \cdot p \to 0$), and the extrapolation from the points which are rather far from zero-recoil is an assumption. The difference between the results obtained after the linear and quadratic extrapolations will be included in the systematic uncertainty. In fig. 6, we show this extrapolation for the case of $v \cdot p = 0.35$. As in the previous subsection we quote the result of the quadratic extrapolation as our central value, and the difference with the linearly extrapolated value is included in the estimate of the systematic error. The resulting form factors for the $B \to \pi$ semileptonic decay are given in tab. 4. The last step is the fit of the data from tab. 4 to the form (3), which is what we discuss next.

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[8]It should be noted, however, that the effect of the logarithmic corrections in eq. (23) on the extrapolation to m_B is practically negligible.

q^2 [GeV^2]	$F_0^{B\to\pi}(q^2)$	$F_+^{B\to\pi}(q^2)$
13.6	$0.46(7)^{+.05}_{-.08}$	$0.70(9)^{+.10}_{-.03}$
15.0	$0.49(7)^{+.06}_{-.08}$	$0.79(10)^{+.10}_{-.04}$
16.4	$0.54(6)^{+.05}_{-.09}$	$0.90(10)^{+.10}_{-.04}$
17.9	$0.59(6)^{+.04}_{-.10}$	$1.05(11)^{+.10}_{-.06}$
19.3	$0.64(6)^{+.04}_{-.10}$	$1.25(13)^{+.09}_{-.08}$
20.7	$0.71(6)^{+.03}_{-.10}$	$1.53(17)^{+.08}_{-.11}$
22.1	$0.80(6)^{+.01}_{-.12}$	$1.96(23)^{+.06}_{-.18}$

Table 4: The values of the semileptonic $B \to \pi$ form factors at several values of q^2. First errors are the statistical and the second are systematic whose estimate is explained in the text.

4 Physical results

4.1 $B \to \pi\ell\nu_\ell$

The main physical results as obtained by employing the *Method I* and *Method II*, are listed in tab. 5. In the same table we also give the results obtained by using the LCSR which are fitted to the parameterization (3) in ref. [16]. First, we see that the final results of the two methods agree very well with each other. We also notice a very good agreement with the results reported in ref. [17] where the *Method II* has been used. This is shown in fig. 7. Moreover, we see that there is a pleasant agreement with the LCSR results. We verified that the Callan-Treiman relation (20) is well satisfied when the quadratic extrapolation in the heavy meson mass is performed. We recall that on the same set of configurations, we have $f_B/f_\pi = 1.29(10)$ and $1.46(16)$, as obtained from the linear, and quadratic extrapolation of the ratio f_H/f_π in $1/M_H$, respectively. The latter value has to be compared with the value of $F_0(m_B^2)$ given in tab. 5.

We were also able to examine the effect of the SU(3) breaking at $q^2 = 0$, by replacing the final pion by a kaon. This ratio (see tab. 5) turns out to be large compared to one and, although with large errors, quite compatible with the findings of the LCSR.

We now discuss the sources of systematic errors which are combined in the quadrature.

- Extraction of the matrix element: The central results are obtained by using the *ratio method* (see sec. 2). The whole analysis has also been performed with the matrix element extracted by using the *analytic method*. The difference in the results is always smaller than the statistical error, and we include it in the systematic uncertainty.

- Discretization errors: The main source of discretization errors is expected to be of $\mathcal{O}((am_H)^2)$, where m_H is the heavy meson mass. Our central results are obtained by correcting the $\mathcal{O}(a)$ improved renormalization constant by the KLM factor as given in eq. (9). We repeated the analysis without the KLM factor and found only a small discrepancy with the central results of the parameter c_B (of about 5 %)

Quantity	Method I	Method II	LCSR [16]
c_B	0.42(13)(4)	0.51(8)(1)	0.41(12)
α_B	0.40(15)(9)	$0.45(17)^{+.06}_{-.13}$	$0.32^{+.21}_{-.07}$
β_B	$1.22(14)^{+.12}_{-.03}$	$1.20(13)^{+.15}_{-.00}$	—
$F^{B\to\pi}(0)$	0.26(5)(4)	0.28(6)(5)	0.28(5)
$F_0^{B\to\pi}(m_B^2)$	$1.3(6)^{+.0}_{-.4}$	$1.5(5)^{+.0}_{-.4}$	—
$F^{B\to K}(0)/F^{B\to\pi}(0)$	$1.21(9)^{+.00}_{-.09}$	$1.19(11)^{+.03}_{-.11}$	$1.28^{+.18}_{-.10}$
$|V_{ub}|^{-2}\cdot\Gamma(\bar{B}^0\to\pi^+\ell\bar\nu)$ [ps^{-1}]	$6.3\pm2.4\pm1.6$	$8.5\pm3.8\pm2.2$	7.3 ± 2.5
$|V_{ub}|/10^{-3}$	$4.3\pm0.9\pm0.6$	$4.0\pm1.0\pm0.5$	$4.0\pm0.7\pm0.7$

Table 5: Main results of this paper, as obtained by using the two methods explained in the text. For easier comparison, we also list the results obtained by using the LCSR [16].
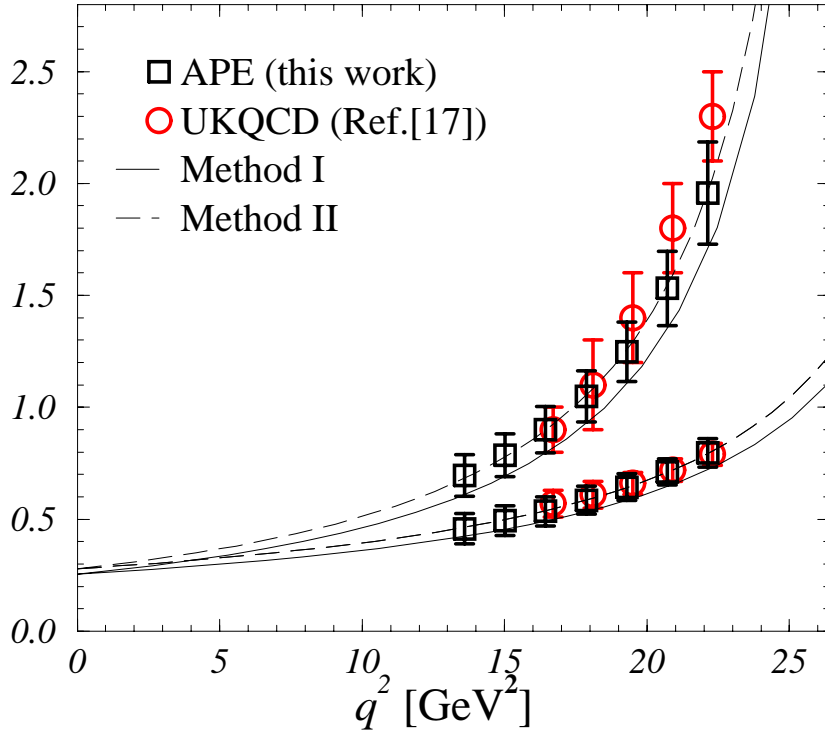

Figure 7: The form factors relevant for $B\to\pi\ell\bar\nu$ decay obtained by using *Method II*, are directly compared to the result of the UKQCD group (only statistical errors are shown). We also draw the curves describing the q^2 dependence with the parameters given in tab. 5.
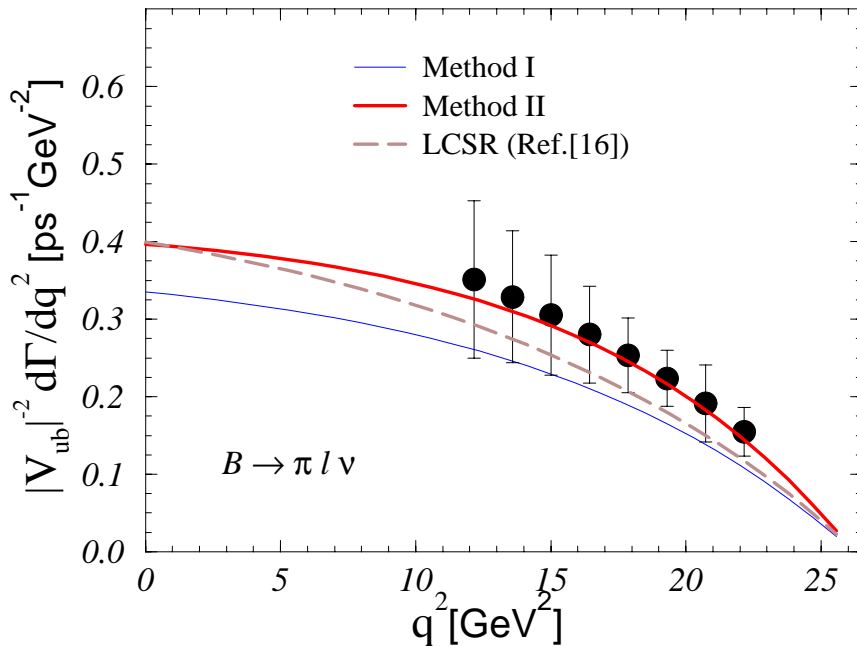
Figure 8: Differential decay rate for $B \to \pi \ell \bar{\nu}$ decay. We show the curves as obtained by both methods. We also plot the result of ref. [16], obtained using the LCSR.

which is included in the systematics. The other parameters remain unchanged. We emphasize that this comparison can only give us idea about the size of discretization errors. Besides, we also used the value of $a^{-1}(f_K) = 2.8(1)$ GeV and repeated the whole analysis which leads to the results only slightly different (less than 5%) from our central values. This difference is, however, added in the systematic uncertainty.

– **Heavy meson extrapolation**: As already mentioned, we quote as central values the results obtained by using the quadratic extrapolation in $1/M_H$. The difference between this and the results of the linear extrapolation is added to the systematics. Beside quenching, this is the main source of systematic uncertainty present in our analysis (it is of order 15%).

The small differences of BK-parameters given in tab. 5 give a more pronounced effect in the q^2 spectrum of the differential decay rate (2). This is shown in fig. 8. After integrating eq. (2) we get the decay width (divided by $|V_{ub}|^2$), which is also given in tab. 5. As mentioned in the introduction, after comparing this result to the measured branching ratio $B(\bar{B}^0 \to \pi^+ \ell \bar{\nu})$ [1], we are able to make an estimate of $|V_{ub}|$ [9]. We also checked that if the pole/dipole form is used to describe the q^2-dependence of the form factors that we obtained by using *Method II*, the result is fully consistent with the one we quote in tab. 5, namely we obtain $|V_{ub}|^{-2} \cdot \Gamma(\bar{B}^0 \to \pi^+ \ell \bar{\nu}) = 7.8 \pm 3.2 \pm 2.1$ ps^{-1}.

To make a unique estimate, we combined in quadrature the statistical and systematic errors for $|V_{ub}|^{-2} \Gamma(\bar{B}^0 \to \pi^+ \ell \bar{\nu})$, and made a weighted average of the results of two

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[9]The last number, however, should be taken cautiously because the experimental result has been obtained by relying on various different theoretic estimates.
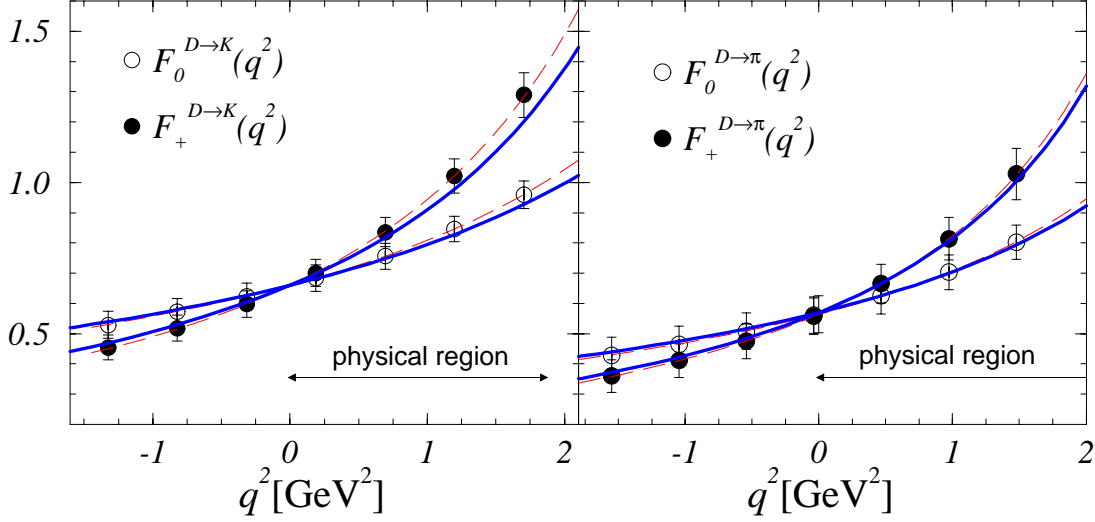
Figure 9: Form factors for $D \to K$ and $D \to \pi$ semileptonic decays. The full line correspond to the results of *Method I*, and the dashed one to the results of *Method II*. The physical region $0 \leq q^2 \leq (m_D - m_{K/\pi})^2$ is indicated in both cases.

methods to get, $|V_{ub}|^{-2} \, \Gamma(\bar{B}^0 \to \pi^+ \ell \bar{\nu}) = (7.0 \pm 2.9)$ ps. Now, as before, if we compare this result to the measured branching ratio, we obtain $|V_{ub}| = (4.1 \pm 1.1) \cdot 10^{-3}$, which is consistent with the world averaged value from inclusive decays, presented at this year's ICHEP conference [32]: $|V_{ub}| = (4.1 \pm 0.6 \pm 0.2) \cdot 10^{-3}$.

5 $\quad D \to \pi \ell \nu_\ell$ and $D \to K \ell \nu_\ell$

In this section we briefly discuss the results relevant for the semileptonic decays of D meson. As in the previous section, we apply both *Method I* and *Method II*. In this case, however, the most important source of systematic uncertainty is negligible, namely in order to reach the physical value of the heavy meson mass, only a smooth interpolation of the data from tab. 3, by using eq. (18), is required. Notice that for these decays, a bulk of final lattice points is out of the physical region (this is illustrated in fig. 9). The values of the form factors whose q^2's are within the physical region for the semileptonic decays are listed in tab. 6.

Our final results are given in tab. 7, where we again compare to the numbers obtained from the LCSR study of ref. [16].

As in the previous section, we plot the q^2 dependence of the partial decay width for

q^2 [GeV^2]	-0.04	0.47	0.97	1.48
$F_0^{D\to\pi}(q^2)$	$0.56(6)^{+.02}_{-.00}$	$0.62(6)^{+.02}_{-.00}$	$0.70(6)^{+.01}_{-.00}$	$0.80(6)^{+.01}_{-.00}$
$F_+^{D\to\pi}(q^2)$	$0.56(6)^{+.02}_{-.00}$	$0.67(6)^{+.01}_{-.00}$	$0.81(7)^{+.02}_{-.00}$	$1.03(9)^{+.01}_{-.00}$

q^2 [GeV^2]	0.19	0.69	1.20	1.70
$F_0^{D\to K}(q^2)$	$0.68(4)(0)$	$0.76(4)(0)$	$0.85(4)(0)$	$0.96(4)(0)$
$F_+^{D\to K}(q^2)$	$0.70(5)(0)$	$0.84(5)(0)$	$1.02(6)(0)$	$1.29(7)(0)$

Table 6: The semileptonic $D \to \pi$ and $D \to K$ form factors at several q^2's obtained by using *Method II*. Only the values at q^2 belonging to the physical region for the semileptonic decays are listed, with the exception of $q^2 = -0.04$ GeV^2 which is very close to $q^2 = 0$. The systematic errors are estimated in the way explained in the text.

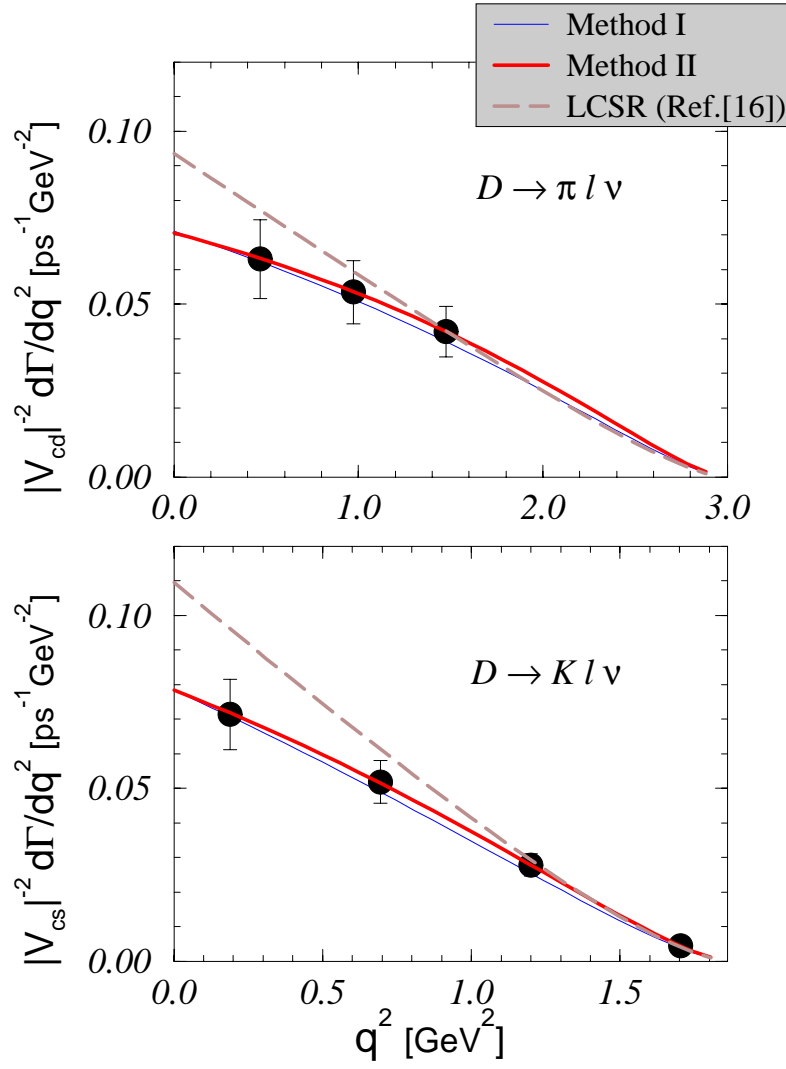| *Quantity*	*Method I*	*Method II*	*LCSR* [16]
c_{DK}	$0.90(12)$	$1.17(23)$	—
α_{DK}	$0.27(11)^{+.00}_{-.01}$	$0.43(12)^{+.00}_{-.02}$	$-0.07^{+.15}_{-.07}$
β_{DK}	$1.34(14)^{+.05}_{-.00}$	$1.22(13)^{+.05}_{-.00}$	—
$c_{D\pi}$	$0.76(13)^{+.01}_{-.06}$	$0.88(18)^{+.00}_{-.06}$	—
$\alpha_{D\pi}$	$0.27(14)^{+.00}_{-.06}$	$0.36(16)^{+.00}_{-.07}$	$0.01^{+.11}_{-.07}$
$\beta_{D\pi}$	$1.31(17)^{+.13}_{-.00}$	$1.23(17)^{+.15}_{-.00}$	—
$F^{D\to K}(0)$	$0.66(4)^{+.01}_{-.00}$	$0.66(4)^{+.01}_{-.00}$	$0.78(11)$
$F^{D\to\pi}(0)$	$0.57(6)^{+.01}_{-.00}$	$0.57(6)^{+.02}_{-.00}$	$0.65(11)$
$F_0^{D\to\pi}(m_D^2)$	$1.6(3)^{+.0}_{-.2}$	$1.9(5)^{+.0}_{-.3}$	—
$F^{D\to K}(0)/F^{D\to\pi}(0)$	$1.16(5)^{+.00}_{-.03}$	$1.22(6)$	$\simeq 1.2$

Table 7: The same as in tab. 5.

Figure 10: The same as in fig. (8) but for D meson semileptonic decays.

	$\Gamma(D^0 \to K^- \ell\nu)$ [ps^{-1}]	$\Gamma(D^0 \to \pi^- \ell\nu)$ [ps^{-1}]
Experiment [19]	$(8.5 \pm 0.5) \cdot 10^{-2}$	$(8.6 \pm 1.8) \cdot 10^{-3}$
This work (*Method I*)	$(6.9 \pm 1.1) \cdot 10^{-2}$	$(5.6 \pm 1.4) \cdot 10^{-3}$
This work (*Method II*)	$(7.3 \pm 1.1) \cdot 10^{-2}$	$(5.9 \pm 1.5) \cdot 10^{-3}$
APE 95 [36]	$(9.1 \pm 2.0) \cdot 10^{-2}$	$(8 \pm 2) \cdot 10^{-3}$
UKQCD 95 [37]	$(7.0 \pm 1.7) \cdot 10^{-2}$	$(5.2 \pm 1.8) \cdot 10^{-3}$
LANL 95 [38]	$(9.4 \pm 1.5) \cdot 10^{-2}$	$(7.4 \pm 1.1) \cdot 10^{-3}$
LCSR [16]	$(9.2 \pm 3.5) \cdot 10^{-2}$	$(6.3 \pm 2.4) \cdot 10^{-3}$

Table 8: Comparison of the full decay widths obtained in this work, to the experimental value, to the several recent lattice results and to the prediction of ref. [16]. In the computation we used $|V_{cs}| = 0.9745(8)$ and $|V_{cd}| = 0.220(3)$ [19].

both decays ($D \to K$ and $D \to \pi$). The agreement with the results of the LCSR [10] is less good than it was for $B \to \pi\ell\nu$. Our central numbers for the form factors at $q^2 = 0$ are smaller. This region (of small q^2's) dominates when integrating over the phase space. On the other hand the discrepancy in the q^2-dependence of the form factors is less important since the kinematically accessible region from these decays is small. Our results for the parameter α, larger than those obtained by LCSR, suggest that the contribution of the higher excited states to the form factor $F_+(q^2)$ is not absent. LCSR, instead, predict that the q^2-dependence of $F_+^{D \to \pi(K)}(q^2)$ is very close to a pure pole behavior (with the pole mass being $m_{D^*_{(s)}}$). We also mention that c_D gives access to the value of the strong coupling of D and π to the first pole D^* (known as $g_{D^*D\pi}$), namely $g_{D^*D\pi} = 2c_D m_{D^*}/f_{D^*}$. With $f_{D^*} = 258(14)(6)$ MeV [26], this amounts to $g_{D^*D\pi} = 12 \pm 2$ and $g_{D^*D\pi} = 14 \pm 5$, by using the *Method I* and *Method II*, respectively. The most recent LCSR prediction is $g_{D^*D\pi} = 10.5 \pm 3$ [33] [11]. A detailed discussion on this coupling with a complete list of references can be found in refs. [35].

Finally, in tab. 8 we also give our results for the total decay rates, which are in a fair agreement with the experimental values [19]. As in the previous subsection, we can also compare our result for $|V_{cs}|^{-2}\Gamma(D^0 \to \pi^- \ell\nu)$ to the experimental measurement $\Gamma^{(\text{exp.})}(D^0 \to K^- \ell\bar{\nu}) = (8.5 \pm 0.5) \cdot 10^{-2} \text{ps}^{-1}$. We obtain,

$$|V_{cs}| = 1.07 \pm 0.09 , \qquad (24)$$

which is to be compared to $|V_{cs}| = 1.04 \pm 0.16$ [19], extracted from the $D \to K$ semileptonic decay where the vector meson dominance for the form factor $F_+(q^2)$ has been assumed.

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[10]The quoted LCSR results for $D \to K\ell\nu$ [16] are obtained by using $m_s(\mu = m_c) = 150$ MeV.

[11]The coupling $g_{D^*D\pi}$ is one of the main parameters in the approach based on the phenomenological lagrangians [34].

Both central values are larger than 1, although consistent with $|V_{cs}| = 0.9745(8)$ [19], the value obtained by using the unitarity of the CKM matrix. The more precise experimental result for $B(D^0 \to K^- \ell \nu)$ expected from CLEO and E791 [32] will hopefully help in clarifying the situation.

6 Conclusion

In this paper we have computed the form factors relevant for the semileptonic decays of the heavy-to-light pseudoscalar mesons on the lattice. We used the fully $\mathcal{O}(a)$ improved Wilson action and vector current. Since the actual computation is performed on a lattice with an inverse lattice spacing smaller than 3 GeV, the physical results for the most interesting decay in this class, namely $B \to \pi \ell \nu$, had to be reached through an extrapolation in the inverse heavy meson mass, for which HQET and LEET provide us helpful scaling laws. To describe the interplay of the mass dependence and the q^2-dependence of the form factors, we used two different methods which are described in the text. We summarize our findings as follows:

– There is a good agreement of the results obtained by using two different methods to reach the physically interesting results of our study;

– We note a very good agreement with the results of ref. [17] in which the same (improved) lattice action has been used;

– Our results agree with the ones of ref. [16] where the LCSR were employed. This is especially the case for $B \to \pi \ell \nu$ decay. We also note that, although with somewhat large errors, we see SU(3) breaking effects for the form factors at $q^2 = 0$, consistent with the findings of the LCSR studies;

– We also verified that our form factor data are consistent with the soft pion theorem (Callan-Treiman relation (20)).

The main results of our study are given in the introduction and in tables 5–8. A remaining uncertainty for the $B \to \pi$ decay arises from our inability to work with the B meson directly on the lattice. This uncertainty can be further reduced by repeating the calculation on a larger and finer lattice. Besides, our lattice simulation has been performed in the quenched approximation. It is obviously desirable to attempt the unquenched study of the semileptonic decays.

Acknowledgements

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