# Strings from Logic 

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#### Abstract

What are strings made of? The possibility is discussed that strings are purely mathematical objects, made of logical axioms. More precisely, proofs in simple logical calculi are represented by graphs that can be interpreted as the Feynman diagrams of certain large $-N$ field theories. Each vertex represents an axiom. Strings arise, because these large $-N$ theories are dual to string theories. These "logical quantum field theories" map theorems into the space of functions of two parameters: $N$ and the coupling constant. Undecidable theorems might be related to nonperturbative field theory effects.


Based on a talk given at CERN (Nov 7, 2000)

[^0]
## 1. Introduction



Apples are made of atoms, atoms are made of elementary particles, and elementary particles are presumably excitation modes of superstrings. It is natural to ask what superstrings are made of. More generally, it is natural to wonder whether this chain of questions for more and more fundamental constituents of the things we observe can ever end, or whether it goes on ad infinitum.

Here I would like to present some observations that can perhaps be regarded as a realization of an ancient and vague idea how the chain of questions might end. According to this ancient idea, the copuscles of the four elements (water, air, earth and fire) represent objects of pure mathematics (such as regular polyhedra) [1]. The chain of questions ends, because it is meaningless to further ask for the origin of such mathematical objects - they "are simply there". The present proposal does not involve polyhedra. But it involves string world-sheets that are made of axioms of mathematical logic in a sense that will be made precise.

These "logical" string world-sheets are what one stumbles upon in the course of pursuing a different project that at first seems unrelated to string theory. Its starting point is the standard formalization of branches of mathematics such as number theory: one introduces variables that represent propositions $p \in\{$ true, false $\}$, variables that represent numbers $n \in\{1,2,3, \ldots\}$, and symbols such as $\wedge, \vee, \neg, \exists,=,+, \ldots$ out of which new propositions are formed. Starting from a basic set of axioms (propositions that are defined to be true), theorems are constructed ("proven") by manipulating the symbols according to given "rules of inference". By applying the axioms in all possible ways, one obtains the set of all theorems that can be proven within this formal system.

The project mentioned is to look for what in the following will be called "logical quantum field theories": field theories, whose Feynman diagrams graphically represent the proofs in a given formal system, with each vertex representing a single axiom. We'll find that at least for some very simple subsystems of the propositional and predicate calculus such quantum field theories exist. They are well-known zero-dimensional $S U(N)$ field theories whose coupling constant will be called $\beta$. Their correlation functions correspond to theorems and are

1. zero, if the theorem is false;
2. $e^{-\beta L} p(N)$ for $\beta \rightarrow \infty$ if the theorem is true,
where $L$ is the "length" of the shortest proof and $p(N)$ is a power series. These theories thus provide a map from theorems to functions of $\beta$ and $N$. One original motivation for looking for such field theories was to relate undecidable theorems that are true but can not be proven within a given formal system to features of field theory that are nonperturbative in $\beta$ (and perhaps in $N$ ). This will only be mentioned here and will be worked out elsewhere.

My emphasis here is on a different aspect of these logical quantum field theories: they are dual to string theories; the string world-sheets are built out of the axioms of the formal system. I will also indicate how more complicated formal systems might involve superstrings and membranes. This will then raise the question whether these logical string world-sheets can be identified with the physical ones, thereby motivating the picture drawn above.

In section 2, the basic idea is explained at a primitive toy model of a formal system. Section 3 begins to generalize this model to the standard propositional calculus. After it is explained what the propositional calculus has to do with strings, open issues are discussed in section 4. It is suggested how membranes and superstrings may arise, and unprovable theorems are also commented on. Section 5 then tries to take seriously the proposal that the physical strings could be such logical strings.

The ideas that I will mention here are part of more general views that have developed over the past 20 years. They were particularly stimulated by discussions with Jürgen Schmidhuber, whose study of the ensemble of computable bit-sequences (interpreted as desribing universes) is inspired by similar thoughts [2]. As I noticed recently, Tegmark [3] also advocates identifying the physical world with a purely mathematical one, and suggests that logical calculi play some role (see also earlier books and articles cited in [3] that point in this direction). Our observations may be regarded as a concrete (and stringy) realization of such general suggestions.

## 2. Toy Model

First, I would like to explain the basic ideas at the example of a primitive toy model. Theoreticians who are familiar with the "old matrix model" will find that I essentially reinterpret it in terms of logic. The reason for doing this will become clear in section 3: there, the toy model will serve as a prototype for part of the standard propositional calculus. In order to get to the point quickly, the style of this section is a bit sloppy; definitions of symbols and phrases are omitted if their use is intuitively clear.

Our toy formal system consists of the following elements:

1. Variables $a, b, \ldots$ which we assume to be already defined (denoting, e.g., numbers).
2. A predicate symbol " $=$ ".
3. Two axioms: If ( $a=b$ and $b=c$ ), then $a=c$

$$
\text { If } a=c \text {, then there is a value of } b \text { such that }(a=b \text { and } b=c) .
$$

We want to
(i) identify a "logical quantum field theory" whose Feynman rules represent the axioms, whose Feynman diagrams represent proofs, and whose correlators represent theorems.
(ii) illustrate in what sense the axioms are the constituents of string world-sheets.

To this end, let us graphically represent the axioms in terms of double line diagrams as shown in the figure (left and center). Each single line represents a variable and each pair of lines represents a proposition such as $a=b$. The diagrams are read from top to bottom. In this sense, the two axioms are just "time-reversed" interpretations of the same graph.


Figure 2: Two axioms and a proof.

By gluing axioms together, "theorems" can be proven. The right graph shows an example of a proof that uses the first axiom twice:

$$
\begin{align*}
\text { Axiom: } & \text { If }(a=b \text { and } b=c), \text { then } a=c  \tag{2.1}\\
+ \text { Axiom: } & \text { If }(a=c \text { and } c=d), \text { then } a=d  \tag{2.2}\\
\rightarrow \text { Theorem: } & \text { If }(a=b \text { and } b=c \text { and } c=d), \text { then } a=d \tag{2.3}
\end{align*}
$$

Note that there are different theorems that are represented by the same graph. E.g., the right graph can also be read "crossed", such that it represents a proof of the theorem

$$
\text { If }(b=a \text { and } a=d) \text {, then there is a } c \text { such that }(b=c \text { and } c=d) \text {. }
$$

We now imagine a "proof machine" that systematically derives all true statements by performing all possible successive applications of the axioms. This machine will stupidly produce many redundant proofs, such as

$$
\begin{align*}
& \text { Axiom: } \text { If } a=b, \text { then there is a } c \text { such that }(a=c \text { and } b=c)  \tag{2.4}\\
& \text { + Axiom: } \text { If }(a=c \text { and } b=c), \text { then } a=b  \tag{2.5}\\
& \rightarrow \text { Theorem: }  \tag{2.6}\\
& \text { If } a=b, \text { then } a=b .
\end{align*}
$$

This proof corresponds to a loop diagram (see figure 3, left) with an auxiliary variable $c$ inside the loop. By gluing the axioms together in all possible ways one obtains the set of all theorems and proofs, represented in terms of double line graphs. One of them is shown in figure 3 (center).


Figure 3: More proofs.

All theorems that can be proven by our machine are of course completely trivial. This should not bother us, because in this note we are not interested in any nontrivial content of individual theorems. Instead, we are going to consider the set of theorems and proofs in simple formal systems as statistical-mechanical ensembles and study properties of the ensembles. This is a rough analog of the situation in thermodynamics, where one does not care about the shape of the individual gas molecules. One only cares about statistical properties of the ensemble of gas molecules such as temperature, pressure, etc. In the case at hand, the ensemble of proofs has two equivalent representations, which we now discuss.

First, our double-line diagrams are well-known to be the dual graphs of random triangulations of a two-dimensional surface with boundary [4] - i.e., of a Euclidean open string world-sheet (figure 3, right). The vertices of the double-line graphs correspond to the centers of the triangles, and the external double lines (which make up the theorem to be proven) correspond to the boundary of the triangulation. The diagrams drawn so far are "planar" in the sense that the dual surface has the topology of a disk. It is possible to interpret higher-genus diagrams in terms of proofs in which some of the external or internal variables are identified. Note that theorems in which an external variable appears an odd number of times cannot be proven within this formal system.

Second, our double line graphs are of course also well-known to be the Feynman diagrams of a zero-dimensional field theory of a real symmetric $N \times N$ matrix $M$ with Lagrangean [5]

$$
\begin{equation*}
\mathcal{L}=N \operatorname{tr}\left(M^{2}+e^{-\beta} M^{3}\right) \tag{2.7}
\end{equation*}
$$

Here, the "coupling constant" $\beta$ is a free real parameter and the number $N$ of $\mathrm{SO}(N)$ "colors" is a free integer parameter. This is the "old matrix model", which is the "logical quantum field theory" for our toy formal system. The value of a Feynman diagram with $L$ vertices, representing a proof with $L$ axiomatic steps, is well-known to be $e^{-\beta L} N^{2-2 g}$, where $g$ is the genus of the corresponding surface. So the correlation function (or Wilson loop) corresponding to a theorem $T$ has an expansion of the form

$$
\begin{equation*}
f_{T}(\beta, N) \sim \sum_{p(T)} e^{-\beta L(p)} N^{2-2 g(p)} \tag{2.8}
\end{equation*}
$$

where $p(T)$ runs over all inequivalent proofs of $T$. In the "classical limit" $\beta \rightarrow \infty$, the shortest proof dominates as advertised in the introduction.

Alternatively, $\beta$ can be fine-tuned to a certain critical value $\beta_{0}$ (as is also well-known), where the continuum limit is reached and large proofs just begin to dominate in (2.8). In this
limit, the ensemble of the random triangulations that are dual to our proofs is in the same universality class as the ensemble of continuum string world-sheets [6, 7]. (2.8) becomes a correlation function in a non-critical Euclidean string theory, whose string world-sheets are literally "made of axioms". In section 5, it will be discussed whether these logical strings can be identified with the physical strings.

To summarize: in the case of the toy model, the logical quantum field theory is a zerodimensional $\mathrm{SO}(N)$ theory with coupling constant $\beta$. It defines a map from theorems into the space of functions $f(\beta, N)$ of two parameters. These functions are the corresponding correlation functions. The situation is, at least superficially, analogous to the situation in topological quantum field theories, where knots and links are mapped onto functions of two parameters $k$ and $N[8]$ (which motivates the name "logical quantum field theories").

## 3. Propositional calculus

In this section we begin to generalize the discussion to the standard propositional calculus, which is a basis for most of the interesting higher calculi, such as number theory. The predicate calculus is commented on in the next section.

### 3.1. Setup ${ }^{1}$

The propositional calculus contains only propositions and no numbers or other variables. Propositions will be denoted by $a, b, c, \ldots \in\{$ true, false $\}$. New propositions such as

$$
\neg a, \quad a \vee b
$$

can be built from old ones via $\neg($ "not"),$\wedge($ "and" $)$, $\supset($ "implies") and $\vee(" o r ")$, which we take to mean "either $a$ or $b$ or both". We denote the negation of $a$ by $\bar{a} \equiv \neg a$. Brackets $($,$) and quantifiers \exists, \forall$ will be used in the standard way. $\wedge$ and $\supset$ may be eliminated as independent symbols by expressing them in terms of $\vee$ and $\neg$ :

$$
a \wedge b \equiv \neg(\bar{a} \vee \bar{b}) \quad, \quad a \supset b \equiv(\bar{a} \vee b)
$$

In the propositional calculus, statements can always be decided via "truth tables": a statement is a theorem if it is true for all values (true or false) of all the variables it consists of, e.g. $\bar{a} \vee a$. In more complicated calculi, where variables can, e.g., be numbers, this is

[^1]generally not possible, since it would take an infinite amount of time to check infinitely many possible values of the variables. In this case one needs to prove theorems using axioms and rules of inference. This approach can of course be used even for the propositional calculus.

For the propositional calculus, there are many equivalent formal systems, i.e., sets of axioms and rules of inference that are complete (all true statements of the propositional calculus can be derived) and sound (no false statements can be derived). The system that will be used here is a variation of that of Russell and Whitehead [9]. The latter contains 4 axioms ${ }^{2}$. The first three are:

$$
\begin{array}{ll}
\text { Axiom 1: } & (a \vee a) \quad \supset a \\
\text { Axiom 2: } & a \vee(a \vee b) \\
\text { Axiom 3: } & (a \vee b) \supset(b \vee a) . \tag{3.3}
\end{array}
$$

The fourth and most interesting axiom is

$$
(b \supset a) \supset[(\bar{c} \vee b) \supset(\bar{c} \vee a)] .
$$

To make the relation with strings more manifest (see below), it is useful to rewrite it as

$$
\begin{equation*}
\text { Axiom 4: } \quad\{(a \vee \bar{b}) \wedge(b \vee \bar{c})\} \supset(a \vee \bar{c}) \tag{3.4}
\end{equation*}
$$

In rewriting this axiom, axiom 3 and associativity of $\vee$ must be used. Associativity can be proven with the original 4 axioms, but we have not succeeded in proving it with the new 4th axiom instead of the old one. To be safe, we thus add associativity as a new axiom:

$$
\begin{equation*}
\text { Axiom 5: } \quad\{(a \vee b) \vee c\} \quad \supset\{a \vee(b \vee c)\} . \tag{3.5}
\end{equation*}
$$

In addition to the axioms, there are two rules of inference:

1. the modus ponens: if $A$ and $A \supset B$ are theorems, then $B$ is a theorem.
2. the rule of substitution: if $P(a)$ is a theorem, then $P(f)$ is a theorem, where $a$ is replaced by any composite (but not necessarily true) proposition $f\left(a_{1}, a_{2}, \ldots\right)$.

Similarly as in section 2, the goal is to construct "logical quantum field theories" whose Feynman diagrams represent the proofs that can be constructed from axioms 1-5 using these two rules. We will proceed as follows. In subsection 3.2, proofs will be discussed

[^2]that can be built from only the fourth axiom using the modus ponens. In subsection 3.3, axioms 1-3 will be incorporated. Section 4 contains open issues: subsection 4.1 speculates how to include the rule of substitution and axiom 5, which are crucial for proving interesting theorems. Subsection 4.2 adds elements of the lower predicate calculus. Subsection 4.3 suggests a relation between undecidable theorems and nonperturbative field theory.

### 3.2. Axiom 4

To see how strings arise in the propositional calculus, we first consider only the subset of theorems that can be proven using axiom 4 (3.4) alone (and its "time reversal" - see below). Let us graphically represent this axiom in terms of double lines as shown in figure 4 (left).


Figure 4: Axiom 4.
The difference to the diagram for the toy model in figure 2 is that each line now represents a proposition $a, b, \ldots \in\{$ true, false $\}$ rather than a general variable such as a number. Also, the lines carry arrows. Let us define "time" as the direction in which the diagram is read. Arrows that run backwards in time are understood to represent the negated propositions. If time runs from top to bottom, axiom 4 is recovered. The diagram still represents a true statement, though, if time runs in any other direction: reading the graph from the lower left to the upper right yields the same axiom with $a, b$ and $c$ permuted; reading the graph from bottom to top yields another true statement, provided that we insert an " $\exists$ ":

$$
(a \vee \bar{c}) \supset \exists b:(a \vee \bar{b}) \wedge(b \vee \bar{c}) .
$$

This is drawn in figure 4 (center). (The introduction of the symbol $\exists$, which is usually reserved for the predicate calculus, is convenient but not really necessary here: in the propositional calculus, $\exists b: f(b)$ can be written out as $f(1) \vee f(0)$.)

The modus ponens implies that, as in the toy model, new theorems such as

$$
\{(a \vee \bar{b}) \wedge(b \vee \bar{c}) \wedge(c \vee \bar{d})\} \quad \supset \quad(a \vee \bar{d})
$$

can now be proven by gluing together axioms as shown in figure 4 (right). If there are closed lines (similarly as in figure 3, left), the corresponding variable $c$ in the loop is assumed to come with an $\exists$. Again, by combining graphs representing axiom 4 in all possible ways, one can derive all theorems that can be proven with this axiom alone. This yields a small and rather trivial subset of all theorems of the propositional calculus. As in the toy model, the corresponding diagrams are the dual graphs of random triangulations of a two-dimensional surface with boundary (as in figure 3). The only difference is that each line now has two possible orientations.

The logical quantum field theory whose Feynman diagrams are the proofs built from axiom 4 is again an "old matrix model" with some coupling constant $\beta$, but the two possible orientations translate into having complex instead of real symmetric matrices. The color indices $a, b, c \in\{1,2, \ldots, N\}$ that label the single lines are now $S U(N)$ indices. Correlation functions again correspond to theorems and are functions of $\beta$ and $N$ of the form (2.8).

### 3.3. The first three axioms

So far, only the fourth axiom has been graphically represented. How about the first three axioms of Russell and Whitehead? As for axiom 3 (3.3), it is already implicit in our formalism, since the double lines can be twisted (figure 5, left). For our random surfaces, this possibility of twisting implies that they are in general not orientable. As for axioms 1 and 2 , those can be written as

$$
\begin{array}{ll}
\text { Axiom 1: } & (a \vee a) \supset(a \vee 0) \wedge(1 \vee a) \\
\text { Axiom 2: } & (a \vee 0) \wedge(1 \vee b) \supset(a \vee b) \tag{3.7}
\end{array}
$$

Here we use $a \wedge 1=a, a \vee 0=a, a \vee 1=1, a \wedge 0=0$. We regard these four statements as auxiliary axioms, defining " 0 " and " 1 ". In the form (3.6),(3.7), axioms 1 and 2 become analogous to axiom 4. This is also drawn in figure 5 (second and third graph), where the dashed line represents " 0 " or its negation " 1 ".

So it seems that it is not difficult to include the first three axioms in our proof diagrams: this involves dashing some of the internal or external lines, so that they denote 0's or 1's. Note that with the help of 0's and 1's in external lines we can now also prove theorems in
which a given variable occurs an odd number of times. Note also a curious aspect of axiom 2: it violates "time reversal" in the sense that it cannot be read from bottom to top: there is no theorem that says $(\bar{a} \vee \bar{b}) \supset(\bar{a} \vee 1) \wedge(0 \vee \bar{b})$, except when $a=b$ as in the first axiom. It remains to be understood what this means for the string world-sheets. Note finally that the modus ponens, if applied to single lines rather than double lines, is also contained in our diagrams (figure 5, right), since it can be written as

$$
(0 \vee A) \wedge(\bar{A} \vee B) \supset(0 \vee B) .
$$



Figure 5: Axioms 3, 1, 2, and the modus ponens.

## 4. Open issues and conjectures

In this section, three open problems and their possible solutions are discussed:

1. It remains to graphically represent axiom 5 (3.5) and the rule of substitution. As I will explain, this might require extending the strings to membranes.
2. The discussion should be generalized to the lower predicate calculus. I will suggest that this involves higher-dimensional string embedding spaces and may involve superstrings.
3. How do undecidable theorems show up in our formalism? I will speculate that they show up as nonperturbative effects in the logical quantum field theories.

### 4.1. Axiom 5 and the rule of substitution

So far, our proof diagrams do not account for the the possibility that a proposition variable that is part of a theorem may be substituted by a composite proposition. Without including
this "rule of substitution", we can only prove a tiny and rather trivial subset of theorems of the propositional calculus. In terms of the double-line diagrams of Fig. 4, substitution means that the propositions $p$ and $q$ in a double line that represents $p \vee q$ may be composed of more elementary propositions. In other words, a closer look at any of the single lines may reveal that the single line is itself a pair of lines, or a pair of pairs of lines, and so on.

A natural way to represent this graphically is the following. We first use the fact that every well-formed (but not necessarily true) proposition can be drawn as an electric cirquit. The example $p \vee q$ with $p=(a \vee b)$ and $q=\{b \vee \neg(c \vee d)\}$ is drawn in figure 6 , left. In the short-hand notation of figure 6 (center), the cirquit becomes a branched tree: the negation is represented by a dashed line and the 3 -vertex represents either $V$ or a branching in the inputs $a, b$, depending on whether it branches upwards or downwards.


Figure 6: Substitution.

Let us now introduce a third dimension in the planar Feynman diagrams of Fig. 4, perpendicular to the plane on which the planar diagram is drawn (figure 6, right). We extend the double lines of the planar diagram into the third dimension. This is done such that they become branched sheets, a cross section through which is the branched tree just constructed (dashed lines are represented in the figure by shaded areas). In this representation, the sum over proofs becomes a sum over three-dimensional branched structures. But these structures can not only branch in the direction of the third dimension. They can also branch in the directions parallel to the original plane, in the sense that the trees can be manipulated during the proof process, e.g., by axiom 5:

$$
\left(p_{1} \vee p_{2}\right) \vee p_{3} \quad \rightarrow \quad p_{1} \vee\left(p_{2} \vee p_{3}\right)
$$

The open problem is now to describe the statistical-mechanical properties of the ensemble of such branched sheets. This looks like a tough problem, but there is an immediate conjecture: that the ensemble of such three-dimensional graphs is effectively represented by dynamical membranes; and that the associated "logical quantum field theory" is related to M-theory. ${ }^{3}$ Of course, even if this can be made precise, one is left with the usual problems of "random membranes": the sum over three-dimensional topologies is not understood ${ }^{4}$, and there is no renormalizable theory of dynamical three-dimensional gravity, so we cannot expect anything like a second-order phase transition. We must leave the subject for the future; perhaps looking for relations with matrix theory [11] may help.

### 4.2. Lower predicate calculus

In the propositional calculus, a proposition can be built out of other propositions, such as $a \vee b$. At the next level, the lower predicate calculus, propositions can also be built from other elements, such as numbers or group elements $m, n$. One then needs a predicate such as " $=$ " that makes a proposition $m=n$ out of two numbers or group elements. The variables $m, n$ themselves are defined through their own axioms such as the Peano axioms in the case of numbers; in addition there are axioms that define how to make a new number out of two numbers (e.g. via,$+ \cdot$ ) or a new group element out of two group elements (via *).


Figure 7: Axioms and proofs with predicates.

The open problem to be discussed is how to represent all these new axioms in terms of Feynman diagrams of some field theory. Here I will only comment on axioms involving the

[^3]predicate. The latter is generally required to obey the transitivity axiom
$$
(m=n \wedge n=k) \supset m=k .
$$

This axiom is graphically represented in figure 7 (left). It is precisely the axiom that has been discussed in the toy model of section 2 . So the proofs that can be constructed with the transitivity axiom alone are again the Feynman diagrams of a zero-dimensional $\mathrm{SO}(N)$ theory. The dual graphs are triangulated string world-sheets with boundary, as for the propositional calculus.

Strings on group spaces
Next, I would like to argue that, after including the other axioms, the space of values of the variables $m, n, \ldots$ becomes an embedding space for the boundary of the string world-sheet. To this end, let us assume that $m, n, k, \ldots$ are elements of a $d$-dimensional Lie group $G$ and denote group multiplication by "*": $m * k \in G$. Let us graphically represent one of the standard axioms,

$$
(m=n) \quad \supset \quad(m * k=n * k)
$$

as shown in figure 7 (second graph), where the dashed line denotes multiplication with $k$. The third graph in figure 7 then represents a proof of $(m=l) \wedge(l=n) \supset(m * k=n * k)$. Figure 7 (right) shows a more general proof in which group elements $k_{i}$ are associated with each boundary edge of the dual triangulation. This defines a map from the boundary to the $d$-dimensional group space. Likewise, the bulk of the triangulated string world-sheet can be mapped onto group space. One may hope that a suitable continuum limit exists, where the boundary becomes a continuous curve that is embedded in the group space. The boundary would then correspond to a correlation function of the corresponding logical quantum field theory that resembles a Wilson loop. Like the loop, this theory would live in $d$ dimensions.

## D-branes

If the boundary lives in $d$ embedding dimensions, then the bulk of the world-sheet actually lives in $d+1$ dimensions. This is a well-known feature of string representations of large- $N$ field theories (such as our logical quantum field theories): there is a new embedding dimension that arises from the world-sheet conformal factor [12]. The $d$-dimensional hypersurface on which the world-sheet ends is a so-called "D-brane" of the $d+1$-dimensional string theory. The bulk of the world-sheet represents proofs, and the boundary represents theorems; so the ensemble of both proofs and theorems would be represented by the ensemble of closed
strings that are allowed to end on a D -brane of codimension 1.
Of course, once $d>1$, one really needs superstrings in order to avoid the tachyon problem of the bosonic string. Do superstrings arise in logic?

## Superstrings?

In the above, proofs built from the transitivity axiom and proofs built from axiom 4 (3.4) have both been represented by double line graphs that are dual to triangulated surfaces. The main difference is that the individual lines represent numbers or group elements $g \in G$ in one case, and propositions $p \in\{$ true, false $\}$ in the other case. Next we should combine both types of proofs. This requires including the rule of substitution and must therefore be left for the future. One hope is that numbers (or group elements) and propositions become superpartners, so that the logical quantum field theory lives in a superspace - or equivalently, is a supersymmetric large- N theory that lives on a D -brane of some dual superstring theory.

### 4.3. Nonperturbative Field Theory and Undecidable Theorems

Let me finally mention another original motivation for defining "logical quantum field theories". At least before adding the rule of substitution, they provide a map from theorems $T$ into the space of functions $f_{T}(\beta, N)$ of a real parameter $\beta$ and an integer parameter $N$ - the corresponding correlation functions. These functions are

1. zero, if the theorem is false.
2. $f_{T}(\beta, N)=\sum_{p(T)} e^{-\beta L(p)} f_{p}(N)$ if the theorem is true and can be proven,
where $L$ is the length of a proof $p$ of a theorem $T$, and $f_{p}$ is a power of $N$. What about theorems that are true but cannot be proven within a given formal system? Such theorems are well-known to exist in number theory, and can already exist in truncations of the propositional calculus. If there is no proof of a theorem, this means that there is no Feynman diagram that contributes to the corresponding correlation function. So the correlation function must be zero in a perturbation expansion in $\beta$. But this leaves open the possibility that the correlation function is
3. nonperturbatively nonzero, if the theorem is true but cannot be proven.

In other words, the logical quantum field theory might know more about theorems than the formal system: it might know about theorems that are undecidable within this formal sys-
tem. This is a version of the standard fact that a field theory knows more about correlation functions than its Feynman diagrams: it knows about nonperturbative effects (although nonperturbative effects do show up in the Feynman diagram expansion in terms of singularities of the Borel transform).

If such a relation between nonperturbative field theory and undecidable theorems exists, it would raise a fascinating question: are there dualities in logic that interchange provable and unprovable theorems and that correspond to the strong-weak coupling dualities known from string theory? Indeed, theorems that cannot be proven in one formalization of a given branch of mathematics may be provable in another formalization, and vice versa. These two formalizations would then correspond to different perturbation expansions of the same logical quantum field theory. In switching between them, the functions $f_{T}(\beta, N)$ that we have associated with theorems would undergo a duality transformation. This is planned to be investigated in the future.

## 5. Possible interpretation

Let me conclude by first stating from a slightly different perspective what we have done. We have considered the sets of proofs in formal systems as statistical-mechanical ensembles, and we have discussed their statistical-mechanical properties. That is, we have considered partition functions of the type

$$
\begin{equation*}
Z(\beta, \ldots) \sim \sum_{p} e^{-\beta L(p)+\ldots}, \tag{5.1}
\end{equation*}
$$

where $L$ denotes the length of a proof $p$, measured in axiomatic steps, and the dots represent possible other parameters in the action (in addition to $\beta$ ). One could now interpret $\beta^{-1}$ as a temperature, and define thermodynamic quantities for the ensemble of proofs, such as the specific heat

$$
c(\beta) \sim \partial_{\beta}^{2} \log Z(\beta)
$$

One could then study the specific heat as a function of the temperature $\beta^{-1}$, and ask questions such as whether there is a second order phase transition at some critical temperature.

At least for simple subsystems of the standard propositional and predicate calculus, we have mapped the proofs onto triangulated random surfaces. These surfaces are made of logical axioms in the sense that each triangle represents one of the axioms from which the proof is constructed. If the temperature $\beta^{-1}$ is fine-tuned "by hand" to a critical value, then
indeed a second-order phase transition can be studied where the triangulated surfaces can be interpreted as continuum string world-sheets. (5.1) then has an interpretation as a string partition funtion. For more general cases, it has been speculated (in subsection 4.2) that the ensemble of theorems and their proofs becomes the ensemble of string world-sheets whose boundaries live on a D-brane of codimension 1 .

The question then arises whether these "logical string world-sheets" can be identified with the string world--sheets that the real world is assumed to be made of (after switching to Minkowskian signature). This question will be discussed next.

The "Mathscape"
Let me first apologize for the philosophical and therefore vague character of the following remarks. Even if they are vague, I want to mention them as a main motivation for trying to make a connection between string theory and logic.

Identifying the logical with the physical strings seems to require that we take a "platonic" attitude towards mathematics, regarding it as an abstract reality that exists independently of mathematicians, in the sense that mathematicians can discover it by means of logical reasoning but they cannot change it. E.g., we certainly cannot change the fact that 335149 is a prime number, but we can find out that it is true.

The theorem "335149 is a prime number" is only one out of an infinite number of facts that can be discovered: given the axioms of number theory, what are the theorems that can be proven with them? Moreover, how long is the shortest proof of a given theorem? How many distinct proofs are there with given length $L$ ? More generally, what are the properties of the ensemble of theorems and their proofs? Let us call this ensemble of theorems and proofs in general formal systems the "mathscape". ${ }^{5}$ Let us take the viewpoint that it makes no sense to further ask where the mathscape comes from - like the list of prime numbers, it is "simply there" - but that it makes sense to study its properties.

Regarded as a statistical-mechanical system, what does this mathscape "look like"? To describe it, it first of all seems sufficient to consider the set of theorems and proofs in number theory: following [14], number theory is "sufficiently powerful" in the sense that theorems and proofs in all other formal systems can be mapped onto theorems and proofs in number theory.

We have considered the beginnings of number theory, namely the propositional calculus

[^4]and simple extensions of it. In these cases we have argued that the mathscape resembles the real world: it contains strings and (perhaps) membranes. Moreover, these strings are automatically first-quantized: first quantization of the logical strings simply reflects the standard relation between field theory and statistical mechanics, applied to the ensemble of proofs. If the string representation of general proofs in number theory could be derived rigorously, this would support the hypothesis that the "real world" is the mathscape, or at least some basic part of it.

## Two objections

The idea that the physical strings are logical strings of the type discussed here may raise eyebrows for many reasons. Let me try to reply to two of the immediate objections one might have:

1. These logical strings are "unphysical" - they are "abstract" strings that live in an abstract space of logical proofs. The reply is that we - the observers - would ourselves be unphysical in the same sense, since we are also made of strings and therefore axioms, and therefore we would live in exactly the same abstract space. And for abstract observers, abstract strings may be as "real" as physical strings are for physical observers. Similar remarks hold for the computer-generated observers in computer-generated universes discussed in [2], and are also made in [3].
2. Who fine-tunes $\beta$ in partition functions of the type (5.1) to the critical value? If $\beta$ were not fine-tuned, we would not see continuum strings in the real world, but, if anything, discretized strings. The reply to this is that this fine-tuning problem is nothing but the tachyon problem of bosonic string theory: at least in the simple models discussed here, $\beta$ can be regarded as the world-sheet cosmological constant, i.e., as a tachyon zero mode. So it remains to get rid of the tachyon, e.g. by supersymmetry or in whatever way the hypothetical QCD string gets rid of it.

## Renormalization group flows

Once $\beta$ in (5.1) is near its critical value, the statistical-mechanical properties of the "mathscape" can be studied using renormalization group methods. The simple formal systems represented in sections 2 and 3 contain only two parameters, $N$ and $\beta$. In the case of more general predicate calculi, where one has higher-dimensional string embedding spaces (see subsection 4.2), the action will contain many parameters, corresponding to all the string
fields. These paramenters will flow as one considers the mathscape at larger and larger scales in the sense of larger and larger proofs. Renormalization group trajectories correspond to $\phi$-dependent solutions of string theory, where $\phi$ denotes the additional embedding dimension that arises from the conformal factor of the world-sheet [15].

As in [16] (which was already motivated in part by these ideas, but where $\phi$ was "time"), one would flow from some UV fixed point to some stable IR fixed point. The IR fixed point of the mathscape would correspond to a stable string vacuum. What is the UV fixed point? It would describe the "bare" mathscape. It would be interesting to find solutions where the UV fixed point corresponds to a topological string theory.
$\underline{\text { Relation to previous papers }}$
To conclude, let me relate the remarks in this section to two previous papers $[2,3]$.
In [2], Jürgen Schmidhuber studies the ensemble of bit sequences that can be computed by computer programs. This includes the ensembles of proofs and theorems in formal systems, since those can be encoded as bit sequences that can be computed. Our universe is interpreted in terms of such bit sequences. One of the aspects of [2] that has no analog here is that bit sequences are weighted by their Kolmogorov complexity, rather than by the length of the computation (whose analog, the length of proofs, has led us to string theory here).

Tegmark [3] considers a step-wise generalization of properties of the observable world, from varying the parameters of the Standard Model to making space-time discrete. It is concluded that the most general universe should be related to the most general logical calculus in a sense that does not seem to be specified. The present note specifies such a relation between logic and particle physics by suggesting an explicit map from proofs in formal systems to first-quantized string world-sheets.

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[^1]:    ${ }^{1}$ All the background that we need here is contained in the brief review in the Encyclopedia Britannica.

[^2]:    ${ }^{2}$ The original system contained 5 axioms, but the fifth one was later derived from the other 4 .

[^3]:    ${ }^{3} \mathrm{~A}$ perhaps related hint at membranes is the observation that general propositions can be written as $\left(a_{1} \vee a_{2} \vee a_{3}\right) \wedge\left(b_{1} \vee b_{2} \vee b_{3}\right) \wedge \ldots$ and can be represented by triple-line (rather than double-line) graphs. Those may be dual to discretized membranes.
    ${ }^{4}$ Although a subclass of topologies, Seifert manifolds, can be described in a field theory context [10].

[^4]:    ${ }^{5}$ Modifying R. Rucker's term "mindscape" [13].

