CERN-TH/2000-322

Generalised twisted partition functions

V.B. Petkova *
School of Computing and Mathematics
University of Northumbria
NE1 8ST Newcastle upon Tyne, UK
and
J.-B. Zuber ♭
TH Division, CERN, CH-1211 Genève 23

We consider the set of partition functions that result from the insertion of twist operators compatible with conformal invariance in a given 2D Conformal Field Theory (CFT). A consistency equation, which gives a classification of twists, is written and solved in particular cases. This generalises old results on twisted torus boundary conditions, gives a physical interpretation of Ocneanu's algebraic construction, and might offer a new route to the study of properties of CFT.

1. Introduction

The study of possible boundary conditions and of the associated finite size effects is known to be a powerful means of investigation of critical systems. This is particularly true in two dimensions, where conformal invariance gives a very restrictive framework. In this note we discuss a class of twists which may be inserted in a 2D conformal field theory along a non contractible cycle (on a cylinder, say), and which are requested to be compatible with conformal invariance, in a sense to be defined. It is shown that the complete set of partition functions depending of these twists and the associated algebras contain all the information on the system, its physical spectrum and Operator Product Algebra. Our discussion is parallel to the one done recently on boundary conditions on a half plane (or on a strip) [1]. Somehow, the latter explored a chiral subsector of the theory, while the present approach reveals all its structure. At the same time it gives a physical realisation of an algebraic construction proposed by Ocneanu [2] and pursued also by Böckenhauer, Evans and Kawahigashi [3].

* Permanent address: Institute for Nuclear Research and Nuclear Energy, 72 Tzarigradsko Chaussee, 1784 Sofia, Bulgaria. E-mail: valentina.petkova@unn.ac.uk

♭ On leave from: SPhT, CEA Saclay, 91191 Gif-sur-Yvette, France. E-mail: zuber@spht.saclay.cea.fr

2. Twisted boundary conditions

Throughout this paper we make use of the following notations: We consider a rational conformal field theory (RCFT) with a chiral algebra $\mathfrak{A}$, (the Virasoro algebra or one of its extensions), and denote $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$ the finite set of representations of this chiral algebra, $\chi_i(q)$, S_{ij}, ${N_{ij}}^k$ with $i,j,k\in\mathcal{I}$ their characters, modular matrix and fusion rule multiplicities given by the Verlinde formula.

Suppose this RCFT has a spectrum in the plane described by a matrix $Z_{j\bar{j}}$, $j,\bar{j}\in\mathcal{I}$, i.e. a Hilbert space of the form

$$\mathcal{H}_P=\oplus_{j,\bar{j}\in\mathcal{I}}\, Z_{j\bar{j}}\,\mathcal{V}_j\otimes\overline{\mathcal{V}}_{\bar{j}}\,. \tag{2.1}$$

In order to study this CFT, it is a common practice [4] to consider it on a cylinder of perimeter T with a complex coordinate w defined modulo T and to define a Hamiltonian H by the translation operator along the w imaginary axis. Then, taking a finite portion of this cylinder bounded by the circles $\Im m\, w=\pm L$, one may identify the two boundaries by imposing periodic boundary conditions along this imaginary direction: the CFT is regarded as living on a torus, and the partition function is the trace of the "time" evolution operator $\mathcal{T}$ associated with the Hamiltonian, $\mathcal{T}=e^{-2LH}$. Here, however, we shall allow the possibility of inserting one (or several) operator(s) X inside the trace of this evolution operator. This may be interpreted as introducing one or several defect lines $\mathcal{C}$ (" seams") in the system, along non contractible cycles of the cylinder, before closing it into a torus, thus resulting into a certain class of "twisted" boundary conditions. The new partition functions in the presence of X's are thus $Z_X=\mathrm{tr}_{\,\mathcal{H}_P}X\mathcal{T}$, $Z_{X,X'}=\mathrm{tr}_{\,\mathcal{H}_P}XX'\mathcal{T}$, etc.

The X are not arbitrary: we insist that these operators commute with the energy-momentum tensor $T(w),\bar{T}(\bar{w})$, or equivalently with the Virasoro generators

$$[L_n,X]=[\bar{L}_n,X]=0\ . \tag{2.2}$$

Since the Virasoro operators are the generators of infinitesimal diffeomorphisms, this condition says that each operator X is invariant under a distorsion of the line to which it is attached. The operator X is thus attached to the homotopy class of the contour $\mathcal{C}$. If the chiral algebra is larger than Vir, there is a similar set of commutation relations with the generators of $\mathfrak{A}$, whose physical interpretation is however less obvious.

3. Characterising the twists

What is the most general form of operators from $\mathcal{H}_P$ to $\mathcal{H}_P$ commuting with all L_n and $\bar{L}_n$? Following a route that proved useful in a different context [1], we may first restrict ourselves to operators intertwining a pair of components of (2.1), i.e. mapping some $\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}}$ into $\mathcal{V}_{j'} \otimes \overline{\mathcal{V}}_{\bar{j}'}$: irreducibility of the representations $\mathcal{V}_j$ tells us such an X is non trivial only for $j = j'$, $\bar{j} = \bar{j}'$. If the multiplicity $Z_{j\bar{j}}$ is 1, it follows that X must be proportional to the projector $P^j \otimes P^{\bar{j}}$ in $\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}}$. If however $Z_{j\bar{j}} > 1$, X is a linear combination of operators intertwining the different copies of $\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}}$

$$P^{(j,\bar{j};\alpha,\alpha')} \;:\; (\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}})^{(\alpha')} \to (\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}})^{(\alpha)} \quad \alpha, \alpha' = 1, \cdots, Z_{j\bar{j}} \;, \tag{3.1}$$

and acting as $P^j \otimes P^{\bar{j}}$ in each. The notation encompasses the case $Z_{j\bar{j}} = 1$. If $|j, \mathbf{n}\rangle \otimes |\bar{j}, \bar{\mathbf{n}}\rangle$ denotes an orthonormal basis of $\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}}$ labelled by multi-indices $\mathbf{n}, \bar{\mathbf{n}}$, we may write

$$P^{(j,\bar{j};\alpha,\alpha')} = \sum_{\mathbf{n},\bar{\mathbf{n}}} \big(|j,\mathbf{n}\rangle \otimes |\bar{j},\bar{\mathbf{n}}\rangle\big)^{(\alpha)} \big(\langle j,\mathbf{n}| \otimes \langle \bar{j},\bar{\mathbf{n}}|\big)^{(\alpha')} \quad \alpha, \alpha' = 1, \cdots Z_{j\bar{j}} \;. \tag{3.2}$$

There are thus $\sum_{j,\bar{j}} |Z_{j\bar{j}}|^2$ linearly independent solutions of equations (2.2). If these equations are extended to the generators of the full chiral algebra $\mathfrak{A}$, there may be more general solutions $P_U^{(j,\bar{j};\alpha,\alpha')}$ with, say, $|\bar{j}, \mathbf{n}\rangle$ replaced by $U\,|\bar{j}, \mathbf{n}\rangle$, where U is a unitary operator implementing some automorphism of $\mathfrak{A}$, a freedom reminiscent to the "gluing automorphism" in the boundary CFT, which has the effect of changing $(j, \bar{j})$ to some $(j, \omega(\bar{j}))$.

The P's satisfy

$$P^{(j_1,\bar{j}_1;\alpha_1,\alpha'_1)} P^{(j_2,\bar{j}_2;\alpha_2,\alpha'_2)} = \delta_{j_1 j_2} \delta_{\bar{j}_1 \bar{j}_2} \delta_{\alpha'_1 \alpha_2} \, P^{(j_1,\bar{j}_1;\alpha_1,\alpha'_2)} \;. \tag{3.3}$$

Note also that they play here the rôle of the Ishibashi states in the problem of boundary conditions in the half plane. We then write the most general linear combination of these basic operators as

$$X_x = \sum_{j\bar{j},\alpha,\alpha'} \frac{\Psi_x^{(j,\bar{j};\alpha,\alpha')}}{\sqrt{S_{1j} S_{1\bar{j}}}} \, P^{(j,\bar{j};\alpha,\alpha')} \;, \tag{3.4}$$

with x a label taking $n = \sum_{j,\bar{j}} (Z_{j\bar{j}})^2$ values and Ψ an a priori arbitrary complex $n \times n$ matrix. The denominator $\sqrt{S_{1j} S_{1\bar{j}}}$ is introduced for later convenience. We shall denote by $\widetilde{\mathcal{V}}$ the set of labels x and use the label $x = 1$ for the identity operator

$$X_1 := \mathrm{Id} = \sum_{j\bar{j},\alpha} P^{(j,\bar{j};\alpha,\alpha)} \;, \tag{3.5}$$

for which

$$\Psi_1^{(j,\bar{j};\alpha,\alpha')} = \sqrt{S_{1j}S_{1\bar{j}}}\,\delta_{\alpha\alpha'} =: \Psi_1^{(j,\bar{j})}\,\delta_{\alpha\alpha'}\ . \tag{3.6}$$

Using (3.3) and the hermitian conjugation properties of the projectors

$$(P^{(j,\bar{j};\alpha,\alpha')})^\dagger = P^{(j,\bar{j};\alpha',\alpha)} \tag{3.7}$$

we may compose two such X as

$$X_x^\dagger\,X_y = \sum_{j,\bar{j},\alpha,\alpha',\alpha''} \frac{\Psi_x^{(j,\bar{j};\alpha,\alpha')\,*}\ \Psi_y^{(j,\bar{j};\alpha'',\alpha')}}{S_{1j}S_{1\bar{j}}}\,P^{(j,\bar{j};\alpha,\alpha'')}\ . \tag{3.8}$$

For our purposes, insertion of one or two such X will be sufficient.

4. The consistency equation

As usual, it is convenient to map the cylinder into the complex plane with coordinate ζ by $\zeta = \exp -2i\pi\frac{w}{T}$. The toroidal domain is mapped into an annulus with identified boundaries along the circles $|\zeta| = |\tilde{q}|^{\pm 1/2}$ (here $\tilde{\tau} = 2iL/T$ and $\tilde{q} = \exp 2i\pi\tilde{\tau}$). One then reexpresses the partition function in terms of Virasoro generators acting in that plane. This is a well known calculation [4], which is hardly affected by the insertion of operators X and we find

$$Z_X = \mathrm{tr}_{\,\mathcal{H}_P}(X\,\tilde{q}^{L_0-c/24}\,\tilde{q}^{\bar{L}_0-c/24})\ , \tag{4.1}$$

and an analogous formula for the insertion of two X. With the help of

$$\mathrm{tr}_{\,\mathcal{H}_P}(P^{(j,\bar{j};\alpha,\alpha')}\tilde{q}^{L_0-c/24}\,\tilde{q}^{\bar{L}_0-c/24}) = \chi_j(\tilde{q})\,\chi_{\bar{j}}(\tilde{q})\ \delta_{\alpha\alpha'}\ , \tag{4.2}$$

and of (3.7) we write the corresponding twisted partition function as

$$Z_{x|y} := Z_{X_x^\dagger\,X_y} = \sum_{\substack{j,\bar{j}\in\mathcal{I}\\ \alpha,\alpha'=1,\cdots,Z_{j\bar{j}}}} \frac{\Psi_x^{(j,\bar{j};\alpha,\alpha')\,*}\Psi_y^{(j,\bar{j};\alpha,\alpha')}}{S_{1j}S_{1\bar{j}}}\ \chi_j(\tilde{q})\,\chi_{\bar{j}}(\tilde{q})\,. \tag{4.3}$$

In particular, for $x = y = 1$, we find

$$Z_{1|1} = \sum_{j,\bar{j},\alpha} \chi_j(\tilde{q})\,\chi_{\bar{j}}(\tilde{q}) = \sum_{j,\bar{j}\in\mathcal{I}} Z_{j\bar{j}}\,\chi_j(\tilde{q})\,\chi_{\bar{j}}(\tilde{q})\ , \tag{4.4}$$

which is, as it should, the modular invariant partition function describing the system with no twist. The above discussion may be generalised to the situation where the underlying chiral algebra of the CFT is a current algebra of generators J and level k and where the energy-momentum of the system includes a term coupled to the Cartan generators: $T'(w) = T(w) - \frac{2i\pi}{T}\sum_p \nu_p J^p(w) - \frac{k}{2}\sum_p \left(\frac{2\pi\nu_p}{T}\right)^2$, and a similar expression for $\bar{T}$. This modification has been shown in [1] to lead to partition functions involving *unspecialised* characters $\chi(\tilde{q},\nu\tilde{\tau})$. Repeating this calculation in the present situation (and real ν) and choosing properly modified projectors P_U, changing $(j,\bar{j})$ to $(j,\bar{j}^*)$ one recovers the analogues of (4.3) and (4.4) with the second character replaced by $\chi_{\bar{j}}(\tilde{q},\nu\tilde{\tau})^*$.

Because of the identification of its two ends, the cylinder considered above may be mapped into another plane, with coordinate $z = \exp(\pi\frac{w}{L})$. The image of the fundamental domain in w is an annulus in that plane with boundaries along the circles $|z| = 1$ and $|z| = |q|^{-1}$ identified, with now $q = \exp 2i\pi\tau$, $\tau = -1/\tilde{\tau} = iT/2L$. Moreover the fact that (2.2) is satisfied implies that the energy momentum $T(w), \bar{T}(\bar{w})$ is well defined on the cylinder and consistent with this identification, and that $T(z), \bar{T}(\bar{z})$ is thus globally defined in the whole plane. On the cylinder, one may also use the Hamiltonian corresponding to the $\Re e\, w$-translation operator. Then the partition function $Z_{x|y}$ is obtained as the trace of the corresponding evolution operator in a Hilbert space

$$\mathcal{H}_{x|y} = \oplus_{i,\bar{i}\in\mathcal{I}}\, \widetilde{V}_{i\bar{i}^*;\,x}{}^{y}\, \mathcal{V}_i \otimes \overline{\mathcal{V}}_{\bar{i}}\ , \tag{4.5}$$

where the non negative integer multiplicities $\widetilde{V}_{i\bar{i};\,x}{}^{y}$ depend on the twists x and y. In the trivial case $x = y = 1$, they must reduce to

$$\widetilde{V}_{i\bar{i}^*;\,1}{}^{1} = Z_{i\bar{i}}\ . \tag{4.6}$$

We can thus complete the calculation as in the absence of the X operator(s) and get

$$\begin{aligned} Z_{x|y} &= \mathrm{tr}_{\,\mathcal{H}_{x|y}}\, q^{L_0-c/24}\, q^{\bar{L}_0-c/24} \\ &= \sum_{i,\bar{i}\in\mathcal{I}} \widetilde{V}_{i\bar{i};\,x}{}^{y}\ \chi_i(q)\,\chi_{\bar{i}}(q)\ . \end{aligned} \tag{4.7}$$

For real q,ν the unspecialised analog of the second character can be rewritten as $\chi_{\bar{i}^*}(q,\nu)^*$ and taking into account (4.6), the partition function $Z_{1|1}(\tau)$ reduces, up to a ν-dependent factor, to the modular invariant.

Identifying the two expressions (4.3) and (4.7) and using the modular transformations of (unspecialised) characters in the form $\chi_j(\tilde{q},\nu\tilde{\tau}) = e^{i\pi k\frac{\nu^2}{\tau}}\, S_{j^*i}\,\chi_i(q,\nu)$ we get

$$\widetilde{V}_{i\bar{i};\,x}{}^{y} = \sum_{j,\bar{j},\alpha,\alpha'} \frac{S_{ij}S_{\bar{i}\bar{j}}}{S_{1j}S_{1\bar{j}}}\ \Psi_x^{(j,\bar{j};\alpha,\alpha')}\ \Psi_y^{(j,\bar{j};\alpha,\alpha')\,*}\,, \qquad i,\bar{i}\in\mathcal{I}\,. \tag{4.8}$$

In the last step we have used the reality of the l.h.s. The similarity of this condition with Cardy's equation in the case of open boundaries [5] is not a coincidence. We shall in fact exploit equation (4.8) in a way parallel to Cardy's [1].

To proceed, we make the additional assumption that the $\Psi_x^{(j,\bar{j};\alpha,\alpha')}$ form a unitary (i.e. orthonormal and complete) change of basis from the $P^{(j,\bar{j},\alpha,\alpha')}$ to the X_x operators. The integer numbers $\widetilde{V}_{i\bar{i};x}{}^y$ will be regarded either as the entries of $|\mathcal{I}| \times |\mathcal{I}|$ matrices $\widetilde{V}_x{}^y$, $x, y \in \widetilde{\mathcal{V}}$, or as those of $|\widetilde{\mathcal{V}}| \times |\widetilde{\mathcal{V}}|$ matrices $\widetilde{V}_{i\bar{i}}$, $i, \bar{i} \in \mathcal{I}$.

Following a standard argument, equation (4.8) may be regarded as the spectral decomposition of the matrices $\widetilde{V}_{i\bar{i}}$ into their orthogonal eigenvectors Ψ and eigenvalues $S_{ij}S_{\bar{i}\bar{j}}/S_{1j}S_{1\bar{j}}$. As the latter form a representation of the tensor product of two copies of Verlinde fusion algebra, the same holds true for the $\widetilde{V}$ matrices:

$$\widetilde{V}_{i_1 j_1}\widetilde{V}_{i_2 j_2} = \sum_{i_3, j_3} N_{i_1 i_2}{}^{i_3} N_{j_1 j_2}{}^{j_3}\ \widetilde{V}_{i_3 j_3}\ . \tag{4.9}$$

Combining (4.6) with (4.9), we have in particular

$$\sum_{i_3 j_3} N_{i_1 i_2}{}^{i_3} N_{j_1 j_2}{}^{j_3}\ Z_{i_3 j_3} = \sum_x \widetilde{V}_{i_1 j_1^*;\,1}{}^x \widetilde{V}_{i_2 j_2^*;\,x}{}^1\ , \tag{4.10}$$

which is the way the matrices $\widetilde{V}_{ij;\,1}{}^x = \widetilde{V}_{i^* j^*;\,x}{}^1$ appeared originally in the work of Ocneanu. As will be explained below, all $\widetilde{V}_x{}^y$ may be reconstructed from the simpler Ocneanu matrices $\widetilde{V}_1{}^x$.

5. Solutions of (4.9)

For the so-called diagonal theories, for which the bulk spectrum is given by $Z_{j\bar{j}} = \delta_{j\bar{j}}$, we know a class of solutions of (4.9). In that case, it is natural to identify the set $\widetilde{\mathcal{V}}$ of twist labels with the set $\mathcal{I}$ of representations, since their cardinality agrees, and to take

$$\widetilde{V}_{ij} = N_i N_j \tag{5.1}$$

understood as a matrix product, in particular $\widetilde{V}_{ij;\,1}{}^k = N_{ij}{}^k$. The corresponding $\Psi_x^{(j,j)}$ are just the modular matrix elements S_{xj}. As a second case, consider a non-diagonal theory with a matrix $Z_{ij} = \delta_{i\zeta(j)}$, where ζ is the conjugation of representations or some other automorphism of the fusion rules (like the Z_2 automorphism in the $D_{2\ell+1}$ cases of $\widehat{sl}(2)$ theories). Then $\widetilde{V}_{ij} = N_i N_{\zeta(j)}$.

The simplest non trivial cases are provided by the $\widehat{sl}(2)$ theories. The latter are known to be classified by ADE Dynkin diagrams. The diagonal A and the $D_{2\ell+1}$ cases have been just discussed. In the other cases with a block-diagonal modular invariant partition function ($D_{2\ell}$, E_6, E_8), we find that the matrices $\widetilde{V}_{ij;\,1}{}^x$ may be expressed simply as bilinear combinations of the matrices $n_{ia}{}^b$ which give the multiplicities of representations when the RCFT lives on a finite width strip or in the upper half-plane [1]: there, the indices $a, b, \cdots$ belong to the set $\mathcal{V}$ of vertices of the Dynkin diagram, and are in one-to-one correspondence with the possible boundary conditions. In general we find that the labels x may be taken of the form (a, b, γ), $a, b \in \mathcal{V}$, γ an extra label, and

$$\widetilde{V}_{ij^*;\,1}{}^{(a,b,\gamma)} = \sum_{c\in T_\gamma} n_{ic}{}^a n_{jc}{}^b \tag{5.2}$$

with c running over a certain subset T_γ of vertices. For the $D_{2\ell}$ case, we take $b = 1$, $\gamma = 0, 1$ and T_γ is the set of vertices of $\mathbb{Z}_2$ grading equal to γ. For the conformal embedding cases E_6, resp. E_8, $b = 1, 2$, resp $b = 1, 2, 3, 8$, the label γ is dropped, and the range of summation of c is the subset $T \subset \mathcal{V}$, identified with the set of representations of the extended fusion algebra, i.e., $T = \{1, 5, 6\}$, resp. $\{1, 7\}$. (For $D_{2\ell}$, $T = T_0$.) Here we are making use of the same labelling of vertices as in [1]. Finally, the case E_7 requires a separate treatment, as all but one of the matrices $\widetilde{V}_1{}^x$ may be represented by a formula similar to (5.2), in terms of the n matrices of the "parent theory" D_{10}, see [6] for more details. Eq. (5.2) provides closed expressions for the matrices $\widetilde{V}_{ij;1}{}^x$ already known from the work of Ocneanu. It expresses a relation between twisted (torus) and boundary (cylinder) partition functions, generalising a well known formula for the modular invariant. It is illustrated on Fig. 1. See also [7] where the partition functions $Z_{x|y}$ for a diagonal theory, cf. (5.1), appear computing boundary partition functions for tensor product theories.

Let us illustrate (5.2) with the simplest example D_4. There $\widetilde{\mathcal{V}}$ has 8 elements, but one finds only 5 independent matrices

$$\widetilde{V}_1{}^1 = \begin{pmatrix} 1&0&0&0&1\\ 0&0&0&0&0\\ 0&0&2&0&0\\ 0&0&0&0&0\\ 1&0&0&0&1 \end{pmatrix} \quad \widetilde{V}_1{}^2 = \begin{pmatrix} 0&0&0&0&0\\ 1&0&2&0&1\\ 0&0&0&0&0\\ 1&0&2&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \widetilde{V}_1{}^3 = \widetilde{V}_1{}^4 = \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&0&0\\ 1&0&1&0&1\\ 0&0&0&0&0\\ 0&0&1&0&0 \end{pmatrix}$$

$$\widetilde{V}_1{}^5 = \widetilde{V}_1{}^7 = \widetilde{V}_1{}^8 = \begin{pmatrix} 0&0&0&0&0\\ 0&1&0&1&0\\ 0&0&0&0&0\\ 0&1&0&1&0\\ 0&0&0&0&0 \end{pmatrix} \qquad \widetilde{V}_1{}^6 = \begin{pmatrix} 0&1&0&1&0\\ 0&0&0&0&0\\ 0&2&0&2&0\\ 0&0&0&0&0\\ 0&1&0&1&0 \end{pmatrix}. \tag{5.3}$$

Here the labels $1,2,3,4$ and $5,6,7,8$ refer respectively to $\gamma=0$ and $\gamma=1$ in (5.2).

To any of these $\widehat{sl}(2)$ cases Ocneanu has associated a graph $\widetilde{G}$ with a set of vertices given by $\widetilde{\mathcal{V}}$. These graphs are generated by the pair of adjacency matrices, $\widetilde{V}_{2,1}$ and $\widetilde{V}_{1,2}$. For example in the $D_{2\ell}$ cases in the basis used above, $\widetilde{V}_{2,1}$ is block-diagonal, with n_2 appearing twice in the diagonal, while in $\widetilde{V}_{1,2}$ these two blocks appear off diagonally. Such graphs have been also constructed in some higher $n\geq 2$ $\widehat{sl}(n)$ cases, [2,3].

The minimal $c<1$ theories are intimately connected to the $\widehat{sl}(2)$ ones, as is well known. For the theory of central charge $c=1-6(g-h)^2/gh$ classified by the pair (A_{h-1},G), with h odd, and g the Coxeter number of G, the set $\mathcal{I}$ of Virasoro representations (r,s) is restricted by $1\leq r\leq h-1$, $1\leq s\leq g-1$ and we choose r odd. Then the twist labels are of the form (r,x), x a twist label of the corresponding $\widehat{sl}(2)$ theory labelled by G and

$$\widetilde{V}_{(rs)(r's');\,1}{}^{(r'',x)}=N_{rr'}{}^{r''}\ \widetilde{V}^{(G)}_{ss';\,1}{}^{x}\ ,\qquad r,r',r''\ \text{odd}\ ,\tag{5.4}$$

in terms of the fusion matrices N_r of $\widehat{sl}(2)_{h-2}$ and of the $\widetilde{V}^{(G)}$ matrices of the G case of $\widehat{sl}(2)_{g-2}$ discussed above.

6. Examples

Some of the twisted partition functions of minimal models have been already encountered, and have a simple realisation in the corresponding lattice models, in terms of defect lines or of twisted boundary conditions imposed on the lattice degrees of freedom. In particular, when the underlying lattice Hamiltonian has some symmetry under a discrete group, one may use any element of this group to twist the boundary conditions along a line $\mathcal{C}$, and the invariance of the Hamiltonian guarantees the independence with respect to deformations of $\mathcal{C}$ [8]: this is the lattice equivalent of the property (2.2) above.

Consider for example the critical Ising model: this is a diagonal minimal model described by three representations of the Virasoro algebra. Here we depart from our previous conventions and denote the representations by their conformal weight, 0, $\frac{1}{2}$ and $\frac{1}{16}$. By the previous discussion, we know that there are three possible twists, whose matrix $\widetilde{V}_1{}^x$ is given by the ordinary fusion matrix N_x. The one labelled by 0 corresponds to no twist at all, and the two others lead to a partition function $Z_{0|x}$ which reads

$$Z_{0|\frac{1}{2}}=|\chi_0(\tilde q)|^2+|\chi_{\frac{1}{2}}(\tilde q)|^2-|\chi_{\frac{1}{16}}(\tilde q)|^2=\chi_0(q)\chi_{\frac{1}{2}}(q)^*+\text{c.c.}+|\chi_{\frac{1}{16}}(q)|^2\tag{6.1a}$$

$$Z_{0|\frac{1}{16}}=|\chi_0(\tilde q)|^2-|\chi_{\frac{1}{2}(\tilde q)}|^2=\big(\chi_0(q)+\chi_{\frac{1}{2}}(q)\big)\chi_{\frac{1}{16}}(q)^*+\text{c.c.}\ .\tag{6.1b}$$

Other partition functions are then obtained by fusion in the sense that $Z_{y|z} = \sum_x N_{yx}{}^z Z_{0|x}$. In this case, only $Z_{\frac{1}{16}|\frac{1}{16}} = Z_{0|0} + Z_{0|\frac{1}{2}}$ is distinct from the previous ones.

The physical interpretation of the $\tilde{q}$ form of (6.1a) is clear: the three primary operators of the theory are weighted by their $\mathbb{Z}_2$ charge. This is the well known partition function of the Ising model on which periodic boundary conditions are imposed on the spin in one direction and antiperiodic ones in the other [5,8]. In contrast, (6.1b) doesn't seem to have been discussed before. In general, eq. (5.4) in the diagonal case $G = A$ reproduces for $r'' = 1$, $x = s'' = g - 1$, g – even, the $\mathbb{Z}_2$ twisted partition functions due to antiperiodic boundary conditions in [5,8], see also [9]. The physical meaning and implementation in the lattice model of the others is less clear and would require some further investigation. See, however, reference [10], where new Boltzmann weights that preserve Yang-Baxter integrability and commutation of the transfer matrices are inserted recovering the diagonal series with $\widetilde{V}_{ij;1}{}^x = N_{ij}{}^x$.

A similar discussion of the 3-state Potts model, classified as (A_4, D_4), follows easily from the formulae (5.3), (5.4) above. The resulting ten independent partition functions have been listed in Table 1: $Z_{1|1}$ is the standard modular invariant, $Z_{1|3}$ the one studied in [11,5,8] and denoted "C" in [5]: it corresponds to the assignment to each operator of the spectrum of its $\mathbb{Z}_3$ charge, ω or $\bar{\omega}$ for the Potts spin and parafermion, 1 for the others; $Z_{1|5}$ is what was denoted "T" in [5].

Table 1: Twisted partition functions of the 3-state Potts model

The labels (r, s) of characters label as usual the Virasoro representations ($h = 5\,,\ g = 6$); ζ is the golden ratio $(1 + \sqrt{5})/2$.

$$
\begin{aligned}
Z_{1|1} &= |\chi_{1,1}(q) + \chi_{1,5}(q)|^2 + |\chi_{3,5}(q) + \chi_{3,1}(q)|^2 + 2|\chi_{1,3}(q)|^2 + 2|\chi_{3,3}(q)|^2 \\
Z_{1|2} &= (\chi_{1,2}(q) + \chi_{1,4}(q))(\chi_{1,1}(q) + \chi_{1,5}(q) + 2\chi_{1,3}(q))^* + (\chi_{3,2}(q) + \chi_{3,4}(q))(\chi_{3,1}(q) + \chi_{3,5}(q) + 2\chi_{3,3}(q))^* \\
&= \sqrt{3}\big((\chi_{1,1}(\tilde{q}) + \chi_{1,5}(\tilde{q}))(\chi_{1,1}(\tilde{q}) - \chi_{1,5}(\tilde{q}))^* + (\chi_{3,1}(\tilde{q}) + \chi_{3,5}(\tilde{q}))(\chi_{3,1}(\tilde{q}) - \chi_{3,5}(\tilde{q}))^*\big) \\
Z_{1|3} &= Z_{1|4} = ((\chi_{1,1}(q) + \chi_{1,5}(q))\chi_{1,3}(q)^* + (\chi_{3,1}(q) + \chi_{3,5}(q))\chi_{3,3}(q)^* + \text{c.c.}) + |\chi_{1,3}(q)|^2 + |\chi_{3,3}(q)|^2 \\
&= |\chi_{1,1}(\tilde{q}) + \chi_{1,5}(\tilde{q})|^2 + |\chi_{3,1}(\tilde{q}) + \chi_{3,5}(\tilde{q})|^2 - |\chi_{1,3}(\tilde{q})|^2 - |\chi_{3,3}(\tilde{q})|^2 \\
Z_{1|5} &= Z_{1|7} = Z_{1|8} = |\chi_{1,2}(q) + \chi_{1,4}(q)|^2 + |\chi_{3,2}(q) + \chi_{3,4}(q)|^2 \\
&= |\chi_{1,1}(\tilde{q}) - \chi_{1,5}(\tilde{q})|^2 + |\chi_{3,1}(\tilde{q}) - \chi_{3,5}(\tilde{q})|^2 \\
Z_{1|6} &= Z^*_{1|2} \\
Z_{1|9} &= ((\chi_{1,1}(q) + \chi_{1,5}(q))(\chi_{3,1}(q) + \chi_{3,5}(q))^* + \text{c.c.}) + |\chi_{3,1}(q) + \chi_{3,5}(q)|^2 + 2(\chi_{3,3}(q)\chi_{1,3}(q)^* + \text{c.c.}) + 2|\chi_{3,3}(q)|^2 \\
&= \zeta(|\chi_{1,1}(\tilde{q}) + \chi_{1,5}(\tilde{q})|^2 + 2|\chi_{1,3}(\tilde{q})|^2) - \zeta^{-1}(|\chi_{3,1}(\tilde{q}) + \chi_{3,5}(\tilde{q})|^2 + 2|\chi_{3,3}(\tilde{q})|^2) \\
Z_{1|10} &= (\chi_{3,2}(q) + \chi_{3,4}(q))(\chi_{1,1}(q) + \chi_{1,5}(q) + \chi_{3,1}(q) + \chi_{3,5}(q) + 2\chi_{1,3}(q) + 2\chi_{3,3}(q))^* + (\chi_{1,2}(q) + \chi_{1,4}(q))(\chi_{3,1}(q) \\
&\quad + \chi_{3,5}(q) + 2\chi_{3,3}(q))^* = \sqrt{3}\zeta(\chi_{1,1}(\tilde{q}) - \chi_{1,5}(\tilde{q}))(\chi_{1,1}(\tilde{q}) + \chi_{1,5}(\tilde{q}))^* - \sqrt{3}\zeta^{-1}(\chi_{3,1}(\tilde{q}) - \chi_{3,5}(\tilde{q}))(\chi_{3,1}(\tilde{q}) + \chi_{3,5}(\tilde{q}))^* \\
Z_{1|11} &= Z_{1|12} = (\chi_{3,3}(q)(\chi_{1,1}(q) + \chi_{1,5}(q) + \chi_{3,1}(q) + \chi_{3,5}(q) + \chi_{1,3}(q))^* + \chi_{1,3}(q)(\chi_{3,1}(q) + \chi_{3,5}(q))^* + \text{c.c}) + |\chi_{3,3}(q)|^2 \\
&= \zeta(|\chi_{1,1}(\tilde{q}) + \chi_{1,5}(\tilde{q})|^2 - 2|\chi_{1,3}(\tilde{q})|^2) - \zeta^{-1}(|\chi_{3,1}(\tilde{q}) + \chi_{3,5}(\tilde{q})|^2 - 2|\chi_{3,3}(\tilde{q})|^2) \\
Z_{1|13} &= Z_{1|15} = Z_{1|16} = ((\chi_{3,2}(q) + \chi_{3,4}(q))(\chi_{1,2}(q) + \chi_{1,4}(q))^* + \text{c.c.}) + |\chi_{3,2}(q) + \chi_{3,4}(q)|^2 \\
&= \zeta|\chi_{1,1}(\tilde{q}) - \chi_{1,5}(\tilde{q})|^2 - \zeta^{-1}|\chi_{3,1}(\tilde{q}) - \chi_{3,5}(\tilde{q})|^2 \\
Z_{1|14} &= Z^*_{1|10}
\end{aligned}
$$

$$Z_{1|x} = \sum_c{}'$$

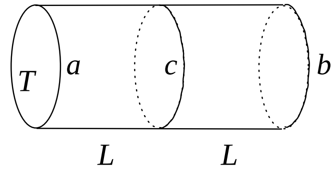

Fig 1. Partition function with the twist $x = (a, b, \gamma)$ as in (5.2).

7. The $\widetilde{N}$ algebra

In the diagonal case, formula (5.1) implies that $\widetilde{V}_y{}^z$ are linear combinations of $\widetilde{V}_1{}^x$, i.e., $\widetilde{V}_y{}^z = \sum_x N_{yx}{}^z \widetilde{V}_1{}^x$. This formula generalises to other cases, the Verlinde matrix being replaced by a new nonnegative integer valued matrix $\widetilde{N}_{xy}{}^z$. In terms of the partition functions we have

$$Z_{y|z} = \sum_x \widetilde{N}_{yx}{}^z \; Z_{1|x} \;, \tag{7.1}$$

where

$$\widetilde{N}_{yx}{}^z = \sum_{j,\bar{j};\alpha} \sum_{\beta,\gamma} \Psi_y^{(j,\bar{j};\,\alpha,\beta)} \frac{\Psi_x^{(j,\bar{j};\,\beta,\gamma)}}{\Psi_1^{(j,\bar{j})}} \Psi_z^{(j,\bar{j};\,\alpha,\gamma)\,*} \,. \tag{7.2}$$

The matrices $\widetilde{N}_x := \{\widetilde{N}_{yx}{}^z\}$ form an associative algebra $\widetilde{N}_x \widetilde{N}_y = \sum_z \widetilde{N}_{xy}{}^z \widetilde{N}_z$ ("fusion algebra of defect lines"). It is noncommutative whenever the corresponding modular invariant matrix $Z_{j\bar{j}}$ has entries larger than 1, like e.g. in the $\widehat{sl}(2)$ $D_{2\ell}$ cases. In the commutative cases, (7.2) reduces to the spectral representation of $\widetilde{N}$. It is easy to check that in all $\widehat{sl}(2)$ cases, $\widetilde{N}_{yx}{}^z$ in (7.2) are indeed non negative integers. This holds true in general and finds a natural explanation in the framework of the subfactor theory [2,3].

The representations of this fusion algebra are labelled by $(j, \bar{j})$, such that $Z_{j\bar{j}} \neq 0$, and appear with multiplicity $Z_{j\bar{j}}$, i.e. they are in one-to-one correspondence with the physical spectrum $(j, \bar{j};\, \alpha)$ of the bulk theory. It turns out that a subset of the structure constants of the associated (commutative) algebra 'dual' to the $\widetilde{N}$ - algebra relates to the squared moduli of the OPE coefficients of the physical (local) fields, [12], [6]. Thus all the information about the bulk theory is encoded in the eigenvector matrices Ψ of the Ocneanu graphs $\widetilde{G}$. We recall that the graphs $G\,, \widetilde{G}$ and the various multiplicities – the sets of integers $N_{ij}{}^k\,, n_{ja}{}^b\,, \widetilde{V}_{ij;x}{}^y\,, \widetilde{N}_{xy}{}^z\,, \tilde{n}_{ax}{}^b$ – are related to the existence of a quantum symmetry of the CFT, the Ocneanu "double triangle algebra", [12,6], studied in the more mathematical language of subfactors in [2,3]. The last of the above multiplicities, $(\tilde{n})_{ax}{}^b$, furnishes a representation of the $\widetilde{N}$ algebra,

$$\tilde{n}_{ax}{}^{b} = \sum_{j,\,\alpha,\beta} \psi_a^{(j\,,\alpha)} \, \frac{\Psi_x^{(j,j;\,\alpha,\beta)}}{\Psi_1^{(j,j)}} \, \psi_b^{(j\,,\beta)\,*} \,, \tag{7.3}$$

where $\psi_a^{(j\,,\alpha)}$ is the eigenvector matrix which diagonalises the cylinder multiplicities n_i.

Note added: Answering a question of Patrick Dorey – it is easy to repeat the calculation on a cylinder in the presence of twist operators and with boundary states $|a\rangle$ and $\langle b|$ at each end. In the simplest case of one such insertion $X_x^\dagger$, one gets a partition function linear in the characters with multiplicities given by $(n_i\tilde{n}_x)_{a^*}{}^{b^*}$, with the notations of [1].

Acknowledgements

Our warmest thanks to Adrian Ocneanu for generous explanations of his recent work, which instigated the present work. We thank Will Orrick, Paul Pearce and Andreas Recknagel for explaining the results of their work to us prior to publication and Luis Alvarez-Gaumé, John Cardy and César Gomez for discussions. V.B.P. acknowledges partial support of the Bulgarian National Research Foundation (Contract Φ-643).

References

[1] R.E. Behrend, P.A. Pearce, V.B. Petkova and J.-B. Zuber, Phys. Lett. **B 444** (1998) 163-166, `hep-th/9809097`; Nucl. Phys. **B 579** [FS] (2000) 707-773, `hep-th/9908036`.

[2] A. Ocneanu, *Paths on Coxeter Diagrams: From Platonic Solids and Singularities to Minimal Models and Subfactors* in *Lectures on Operator Theory,* Fields Institute, Waterloo, Ontario, April 26–30, 1995, (Notes taken by S. Goto) Fields Institute Monographies, AMS 1999, Rajarama Bhat et al, eds.; *Quantum symmetries for $SU(3)$ CFT Models,* Lectures at Bariloche Summer School, Argentina, Jan 2000, to appear in AMS Contemporary Mathematics, R. Coquereaux, A. Garcia and R. Trinchero, eds.

[3] J. Böckenhauer and D. E. Evans, Comm. Math. Phys. **205** (1999) 183-228 `hep-th/9812110`; J. Böckenhauer , D. E. Evans and Y. Kawahigashi, Comm. Math. Phys. **208** (1999) 429-487 `math.OA/9904109`; Comm. Math. Phys. **210** (2000) 733-784 `math.OA/9907149`.

[4] J.Cardy, Nucl. Phys. **B 270** [FS16] (1986) 186-204.

[5] J.Cardy, Nucl. Phys. **B 275** [FS17] (1986) 200-218.

[6] V.B. Petkova and J.-B. Zuber, *The many faces of Ocneanu cells*, in preparation.

[7] A. Recknagel, in preparation.

[8] J.-B. Zuber, Phys. Lett. **B 176** (1986) 127-128.

[9] S. Liénart, P. Ruelle and O. Verhoeven, *On discrete symmetries in $su(2)$ and $su(3)$ affine theories and related graphs*, `hep-th/0007095`.

[10] W. Orrick and P. Pearce, to be published.

[11] G. von Gehlen and V. Rittenberg, J. Phys. **A19** (1986) L625-L629.

[12] V.B. Petkova and J.-B. Zuber, PRHEP-tmr2000/038 (Proceedings of the TMR network conference *Nonperturbative Quantum Effects 2000*), `hep-th/0009219`.