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THE ALBANESE MAP OF A 3-FOLD OF GENERAL TYPE WHOSE CANONICAL MAP IS COMPOSED WITH A PENCIL

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Introduction

Let X be a complex projective manifold of general type. A natural question is, if X enjoys such and such analytic properties, then how is the topology of X? In this note we will study some topological properities of X when the canonical map Φ_X of X is composed with a pencil.

It is known that, if X is a surface of general type whose canonical map is composed with a pencil, then the irregularity $q(X) \leq 2$ [X1]. (We refer the reader to [Be1] in which the canonical map of surface of general type was first systematically studied.) However, when $\dim X \geq 3$, simple examples show that $q(X)$ can be arbitrarily large whenever Φ_X is composed with a pencil.

We recall that, the Albanese dimension $a(X)$ of a manifold X, is defined to be the dimension of the image of the Albanese map $alb: X \to \mathrm{Alb}(X)$, and X is said *of maximal Albanese dimension* or *of Albanese general type* if $a(X) = \dim X$. We note that $a(X)$ is a topological invariant of X (cf. [Ca, Proposition 1.4]). Our main result is the following.

Theorem 1. *Let X be a smooth projective 3-fold of general type over the complex number field. Assume that the canonical linear system is composed with a pencil, and the irregularity $q(X) \geq 6$. Then X is not of maximal Albanese dimension.*

The condition that $q(X) \geq 6$ is technical (see Remark below). Examples of 3-folds X with i) $\dim(\mathrm{Im}\,\Phi_X) = 2$ and $a(X) = 3$, and ii) $\dim(\mathrm{Im}\,\Phi_X) = 1$ and $a(X) = 2$ are easily constructed (see Example 1). Theorem 1 will be proved in section 1. In section 2 we will consider the behaviour of the canonical map under smooth deformations.

Theorem 2. *Let X be a smooth projective 3-fold of general type whose canonical map is composed with a pencil. Let $f: X \to C$ be the fiber space associated to the canonical map. If $g(C) \geq 2$, then the canonical map of any (global) smooth deformation of X is composed with a pencil.*

We use standard notations as in [BPV] or [Mo].

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1. Proof of Theorem 1

Let X be a smooth complex projective 3-fold with $p_g(X) \geq 2$. The canonical map

$$\Phi_X: X - - \to \mathbb{P}^{p_g(X)-1}$$

is defined by the moving part of the canonical linear system $|K_X|$. Assume that Φ_X is composed with a pencil. Let $\pi: X' \to X$ be a resolution of the base locus of the moving part of $|K_X|$.

We have the following commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & C \\ \pi\downarrow & & \downarrow \\ X & \overset{\Phi_X}{\dashrightarrow} & \mathbb{P}^{p_g(X)-1}, \end{array}$$

where $f\colon X' \to C$ is the Stein factorization of $\Phi_X \circ \pi$. We call $f\colon X' \to C$ the *canonical fibration* associated with the canonical map. Clearly, such a fibration is defined up to birational equivalence. But this does not matter, since we only consider birational invariants (e.g., the Albanese map, $\mathrm{a}(X)$, $q(X)$, etc.). Using X' instead of X, for convenience, such a canonical fibration is denoted by $f\colon X \to C$.

Proposition 1. *(cf.* [CC]*) Let X be a 3-fold with $p_g(X) \geq 2$. Assume that the canonical map Φ_X is composed with a pencil. Let $f\colon X \to C$ be the fiber space associated with Φ_X, and F a general fiber of f. If $g(C) \geq 2$, then we have that either $p_g(F) = 1$ or $p_g(F) = p_g(X) = g(C) = 2$.*

Proof. Since $p_g(X) \neq 0$, we have $p_g(F) > 0$. Now we assume that $p_g(F) \geq 2$. Let M and E be the moving part and the fixed part of $|K_X|$ respectively. Since Φ_X factors through f, $\mathcal{O}_X(M) = f^*\mathcal{L}'$ for some $\mathcal{L}' \in \operatorname{Pic} C$. Thus $h^0(\mathcal{L}') = p_g(X)$ and $\omega_X = f^*\mathcal{L}' \otimes \mathcal{O}_X(E)$. There is an injective map of sheaves $\mathcal{L}' \hookrightarrow f_*\omega_X$. Let $\mathcal{L}$ be the saturated subsheaf of $\mathcal{L}'$ in $f_*\omega_X$. Then $\mathcal{L}$ is invertible and the quotient $\mathcal{Q} := f_*\omega_X/\mathcal{L}$ is torsion free, hence locally free of rank $p_g(F) - 1$. We have an exact sequence of sheaves

$$0 \to \mathcal{L} \to f_*\omega_X \to \mathcal{Q} \to 0.$$

Consider the natural map $H^0(\omega_C \otimes \mathcal{L}^{-1}) \times H^0(\mathcal{L}) \to H^0(\omega_C)$, note that $h^0(\mathcal{L}) \geq h^0(\mathcal{L}') \geq 2$, by [Ha, Lemma IV.5.5]. We have

$$\begin{aligned} g(C) - 1 \geq& h^0(\omega_C \otimes \mathcal{L}^{-1}) \\ = h^1(\mathcal{L}) \geq& h^0(\mathcal{Q}) \quad \text{(note that } h^0(f_*\omega_X) = h^0(\mathcal{L})) \\ \geq& (p_g(F) - 1)(g(C) - 1). \end{aligned}$$

The last inequality follows by the Riemann-Roch Theorem and the semipositivity of $f_*\omega_{X/C} = f_*\omega_X \otimes \omega_C^{-1}$ (cf. [Fu]). This implies $p_g(F) = 2$ and $h^1(\mathcal{L}) = g(C) - 1$. Now by Clifford's theorem, we have $\deg \mathcal{L} = 2$ and $\mathcal{L} \simeq \omega_C$. So $p_g(X) = g(C) = 2$. □

Lemma 1. *Let X be a 3-fold of general type whose canonical map is composed with a pencil, and $f\colon X \to C$ the fiber space associated with Φ_X. If $q(X) \geq 5$ and $g(C) \geq 2$, then $alb\colon X \to \mathrm{Alb}(X)$ is not of Albanese general type.*

Proof. Let F be a general fiber of f. We have $q(X) - g(C) \leq q(F)$. Since F is (a surface) of general type, $q(F) \leq p_g(F)$. If $g(C) \geq 2$, by Proposition 1, we have that either $q(X) - g(C) \leq 1$

or $q(X) \leq g(C) + q(F) \leq 2 + p_g(F) = 4$. Thus $f^*H^0(\Omega^1_C)$ is of codimension ≤ 1 in $H^0(\Omega^1_X)$ by the assumption. So for any $\alpha, \beta, \gamma \in H^0(\Omega^1_X)$, we have $\alpha \wedge \beta \wedge \gamma = 0$. Note that X is of Albanese general type if and only if $\mathrm{Image}(\overset{3}{\wedge} H^0(\Omega^1_X) \to H^0(\omega_X)) \neq 0$. (cf. [Ca]) $\square$

Notation. *Let X be a complex projective manifold. For any $0 \neq \alpha \in H^0(\Omega^n_X)$ ($1 \leq n \leq \dim X$), we denote by $\mathrm{Z}(\alpha)$ the zero-locus of the holomorphic n-form α. When $n = \dim X$, we have $\mathrm{Z}(\alpha) = \mathrm{div}(\alpha)$.*

Lemma 2. *Let $f: X \to C$ be as in Lemma 1. Assume that $g(C) \leq 1$, and that there exist linearly independent 1-forms $\alpha, \beta \in H^0(\Omega^1_X)$, such that $\alpha \wedge \beta = 0$. Then $alb: X \to \mathrm{Alb}(X)$ is not of Albanese general type.*

Proof. By a well-known theorem of Castelnuovo-de Franchis (see e.g. [B-P-V]), there exists a surjective morphism $h: X \to B$ to a curve B (of genus ≥ 2) such that $\alpha, \beta \in h^*H^0(\Omega^1_B)$. Let H be a general fiber of h. Then H is not contained in the fixed part of $|K_X|$. Since $g(C) \leq 1$ by the assumption, f and h are different fibrations of X. So H is not contained in the moving part of $|K_X|$. Since $\alpha, \beta \in h^*H^0(\Omega^1_B)$, we have that $H \subset Z(t\alpha + t'\beta)$ ($t\alpha + t'\beta \neq 0$) for a suitable choice $t, t' \in \mathbb{C}$. If $\mathrm{a}(X) = 3$, then there exist two 1-forms $\gamma_1, \gamma_2 \in H^0(\Omega^1_X)$ such that $\omega := (t\alpha + t'\beta) \wedge \gamma_1 \wedge \gamma_2 \neq 0$. Then $H \subset \mathrm{div}(\omega) \in |K_X|$. This is a contradiction. $\square$

Proof of Theorem 1.

Notations as in Lemma 1. By Lemma 1, we can assume that $g(C) \leq 1$. We consider the Albanese map $alb: X \to \mathrm{alb}(X) \subset \mathrm{Alb}(X)$. Suppose that X is of Albanese general type.

If $\kappa(\mathrm{alb}(X)) = 3$, then $\Phi_{\mathrm{alb}(X)}$ and hence Φ_X is generically finite by a theorem of Griffiths and Harris (cf. [Mo, Theorem 3.9]). This contradicts the assumption that Φ_X is composed with a pencil.

Since $\mathrm{alb}(X)$ is a proper subvariety of $\mathrm{Alb}(X)$ generating $\mathrm{Alb}(X)$, one has that $\kappa(\mathrm{alb}(X)) > 0$. By a theorem of Ueno [Ue, Theorem 10.9], there exist an abelian subvariety A of $\mathrm{Alb}(X)$ and a projective variety S' which is a subvariety of an abelian variety such that $\mathrm{alb}(X)$ is a fiber bundle over S' (denoted by u') whose fiber is A, and $\kappa(S') = \dim(S') = \kappa(\mathrm{alb}(X))$.

Let

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{alb}} & \mathrm{alb}(X) \\ {\scriptstyle u}\downarrow & & {\scriptstyle u'}\downarrow \\ S & \longrightarrow & S' \end{array}$$

be the Stein factorization of $u' \circ alb$. Since the invariants we consider are birational, by a standard argument, we can assume that S is smooth.

If $\kappa(\mathrm{alb}(X)) = 1$, then S is a curve of genus ≥ 2. This is impossible by Lemma 2.

In what follows, we treat the remaining case when $\kappa(\mathrm{alb}(X)) = 2$. In this case, S is a surface of general type, and $q(S) = q(X) - 1$. (Note that $q(\mathrm{alb}(X)) - q(S') \leq q(A) = 1$, and if $q(S) = q(X)$ then X is not of Albanese general type.)

We distinguish two cases according to whether f factors through u or not.

Case 1. f does not factor through u.

Let $s \in S$ be a general point such that

i) $u^{-1}s$ is not contained in the fixed part of $|K_X|$, and

ii) $f|_{u^{-1}s}: u^{-1}s \to C$ is surjective.

Since $q(S) > 3$, there exists $0 \neq \alpha \in H^0(\Omega^1_S)$ such that $s \in Z(\alpha)$. We have that $u^{-1}s \subset Z(u^*\alpha)$. Let $0 \neq \beta \in H^0(\Omega^1_X) \setminus u^*H^0(\Omega^1_S)$, and $\gamma \in H^0(\Omega^1_S)$ such that $u^*\alpha$, β, $u^*\gamma$ are linearly independent. By Lemma 2, we can assume that $\alpha \wedge \gamma \neq 0$. Since β is closed, it is easy to verify that $\omega := \beta \wedge u^*\alpha \wedge u^*\gamma \neq 0$. Now $u^{-1}s \subset \mathrm{div}(\omega) \in |K_X|$. This implies that there exists an irreducible component Γ of $\mathrm{div}(\omega)$ such that $u^{-1}s \subset \Gamma$. Clearly Γ is not a fiber of f since $f|_{u^{-1}s}: u^{-1}s \to C$ is surjective by the choice of s. So Γ is an irreducible component of the fixed part of $|K_X|$. This is a contradiction.

Case 2. f factors through u.

Let $h: S \to C$ be the rational map such that $f = h \circ u$. By a suitable choice of the birational model of S and hence of X, we can assume that h is a morphism.

We prove the case when $g(C) = 1$. The proof of the case when $g(C) = 0$ is similar and left to the reader.

Let $s_1, s_2 \in S$ be general points, such that

i) $h(s_1) \neq h(s_2)$ and $F_i = f^*(h(s_i))$ $(i = 1, 2)$ are general fibers of f;

ii) $u^{-1}s_i$ ($i = 1$ and 2) is not contained in the fixed part of $|K_X|$.

Since $q(S) = q(X) - 1 \geq 5$, we have that there exists $0 \neq \alpha \in H^0(\Omega^1_S)$ such that $s_i \in Z(\alpha)$ for $i = 1$ and 2. If $F_i \subset Z(u^*\alpha)$ for $i = 1$ and 2, then $f^*t \cdot u^*\alpha$ is a holomorphic 1-form on X, where t is a meromorphic function having simple poles exactly at $h(s_1)$ and $h(s_2)$. Clearly $f^*t \cdot u^*\alpha$ and $u^*\alpha$ are linearly independent and $(f^*t \cdot u^*\alpha) \wedge u^*\alpha = 0$. By Lemma 2, we get a contradiction.

Now we can assume that $F_i \not\subset Z(u^*\alpha)$ for some i (say $i = 1$). Let $p \in F_1$ be a general point. Then $p \notin \mathrm{Z}(u^*\alpha)$. Let $0 \neq \beta \in H^0(\Omega^1_X) \setminus u^*H^0(\Omega^1_S)$, and $\gamma_1, \gamma_2 \in H^0(\Omega^1_S)$ such that $\gamma_1, \gamma_2, \alpha$ are linearly independent. By Lemma 2, we can assume that $\alpha \wedge (t_1\gamma_1 + t_2\gamma_2) \neq 0$ for any linear combination $t_1\gamma_1 + t_2\gamma_2 \neq 0$. Then we have that $\omega_i := \beta \wedge u^*\alpha \wedge u^*\gamma_i \neq 0$, and ω_i ($i = 1$ and 2) are $\mathbb{C}$-linearly independent. Thus $\mathrm{div}(\omega_i)$ ($i = 1$ and 2) generate a linear pencil Λ of X. Since p is a general point of F_1 and F_1 is a general fiber of f, we can assume that p is not in the base locus of Λ. So we have that $p \notin \mathrm{div}(\omega_i)$ for some i. Fix such an i. Since $u^{-1}s_1 \subset \mathrm{div}(\omega_i) \in |K_X|$, there exists an irreducible component Γ of $\mathrm{div}(\omega_i)$ such that $u^{-1}s \subset \Gamma$. By the above argument, $\Gamma \neq F_i$. So Γ is a component of the fixed part of $|K_X|$. This contradicts the choice of s_1. □

Corollary. *Let X be as in Theorem 1. Assume that the canonical map Φ_X is composed with a pencil, and $q(X) \geq 6$. Let $f: X \to C$ be the fiber space associated with Φ_X. Assume that $g(C) \geq 2$. Then $\kappa(\mathrm{alb}(X)) = 1$.*

Proof. By Theorem 1 we have $\mathrm{a}(X) \le 2$. The corollary is trivial when $\mathrm{a}(X) = 1$.

Now we assume that $\mathrm{alb}(X)$ is a surface. We claim that f factors through *alb*. Otherwise, there is a dominant rational map $X -- \to C \times \mathrm{alb}(X)$, which implies $q(X) \ge q(\mathrm{alb}(X)) + g(C)$. We get necessarily $g(C) = 0$, contradicting the assumption. Let F be a general fiber of f. Since F is of general type, we have $p_g(F) \ge q(F)$. By Proposition 1, we have that either $p_g(F) = 1$ or $p_g(F) = g(C) = 2$. The latter case does not occur since $q(X) \le g(C) + q(F) \le 2 + p_g(F) = 4$, contradicting the assumption. Now if $\kappa(\mathrm{alb}(X)) = 2$, then the image $\mathrm{alb}_{\mathrm{X}}(F)$ of F under the Albanese map of X is a curve of genus ≥ 2. So $p_g(F) \ge q(F) \ge g(\mathrm{alb}_{\mathrm{X}}(F)) \ge 2$. This is a contradiction. □

Example 1. 1) Let S be a smooth surface of general type with $q(S) = 2$ such that i) $\mathrm{alb}: S \to \mathrm{Alb}(S)$ is surjective and ii) Φ_S is composed with a pencil (cf. [X2, p. 600] for such examples). Let $X = S \times C$, where C is a smooth curve of genus ≥ 2. Then X is of Albanese general type and $\dim(\mathrm{Im}\, \Phi_X) = 2$.

2) Let $X = S \times C$, where S is a smooth surface of general type with $p_g(S) = q(S) = 1$ and C is a smooth curve of genus ≥ 2. Then $\mathrm{a}(X) = 2$ and Φ_X is composed with a pencil.

Remark. 1) The condition $q(X) \ge 6$ in Theorem 1 is technically needed in the proof of Case 2 when $g(C) = 1$. When $g(C) = 0$, a similar argument of Case 2 works only assuming $q(X) \ge 4$. Thus from our proof we see that Theorem 1 holds if either i) $q(X) \ge 4$ and $g(C) = 0$ or $g(C) \ge 3$, or ii) $q(X) \ge 6$ and $g(C) = 1$, or iii) $q(X) \ge 5$ and $g(C) = 2$.

We conjecture that Theorem 1 is true under the assumption $q(X) \ge 4$.

Examples of 3-fold X of general type whose canonical map is composed with a pencil, with $q(X) = 3$ and $a(X) = 3$ are easily constructed. Let (A, Θ) be a principal polarized Abelian n-fold ($n \ge 3$), and $D \in |2\Theta|$ a smooth divisor. Let $X \to A$ be the double cover determined by the double cover data (D, Θ). Then $p_g(X) = 2$ and $\mathrm{alb}(X) \simeq A$.

2) The condition that X is of general type in Theorem 1 is indispensable. For example, let $X = A \times C$, where A is an abelian variety and C is a smooth curve of genus ≥ 2. Then X is of Albanese general type and the map induced by $|K_X|$ is composed with a pencil.

3) Let $n \ge 3$ be an integer, and X be a smooth projective n-fold X of general type and of maximal Albanese dimension. Then it is easy to see that $p_g(X) \ge 1$ and by a theorem of Ueno [Ue], one has $p_g(X) \ge 2$ if $q(X) > n$.

Question. Is there such an n-fold X with $p_g(X) = 1$?

Clearly, a statement similar to Theorem 1 for $(n + 1)$-folds of general type is false if the above question has a positive answer.

2. Deformations of the canonical map

Let $f: X \to Y$ be a morphism of smooth projective manifolds with connected fibers. It is known that the morphism $f: X \to Y$ (of the whole triple) has a semiuniversal deformation.

Its base space is denoted by $\mathrm{Def}(f\colon X \to Y)$. There is a natural map

$$\mathcal{P}\colon \mathrm{Def}(f\colon X \to Y) \to \mathrm{Def}(X),$$

where $\mathrm{Def}(X)$ is the base space of the semiuniversal deformation of X.

Fact 1. *If $h^0(R^1 f_* \mathcal{O}_X \otimes \mathcal{T}_Y) = 0$, where $\mathcal{T}_Y$ is the dual of Ω_Y, then $\mathcal{P}$ is surjective.*

We refer the reader to [Ca, Il, Fl, Ho, Pa, Ra] for the basic results on deformations of morphisms and the proof of Fact 1.

Proposition 2. *Let $f\colon X \to C$ be a fibered 3-fold whose general fiber F is a surface with $p_g(F) = 1$. Then Φ_X is composed with a pencil if $p_g(X) \geq 2$.*

Proof. Since $p_g(F) = 1$ by assumption, $\mathcal{L} := f_* \omega_X$ is an invertible sheaf. We have an exact sequence of sheaves

$$0 \to f^* \mathcal{L} \to \omega_X$$

with $H^0(f^* \mathcal{L}) = H^0(\omega_X)$. This implies that Φ_X factors through f. $\square$

We need the following result. For a proof, see [Be2] or [Si].

Fact 2. *(Siu-Beauville) Let $g \geq 2$ be an integer, and $\pi_1(g)$ be the fundamental group of a Riemann surface of genus g. Let X be a compact Kaehler manifold of dimension ≥ 2. Then X has a pencil of genus $\geq g$ if and only if there exists a surjective homomorphism from $\pi_1(X)$ to $\pi_1(g)$.*

Theorem 2. *Let X be a smooth projective 3-fold of general type whose canonical map is composed with a pencil. Let $f\colon X \to C$ be the fiber space associated with Φ_X. If $g(C) \geq 2$, then the canonical map of any (global) smooth deformation of X is composed with a pencil.*

Proof. Since the geometric genus is a global deformation invariant, the theorem is trivially true when $p_g(X) = 2$. So we can assume that $p_g(X) \geq 3$. By Proposition 1, we have that $p_g(F) = 1$, where F is a general fiber of f.

Let X' be any (global) smooth deformation of X, that is, there is a relatively projective smooth family $\pi\colon \mathcal{X} \to B$ such that $\mathcal{X}_b \simeq X$, $\mathcal{X}_{b'} \simeq X'$ $(b, b' \in B)$ and B is connected. Let $t \in B$ be any given point. By Fact 2, there is a surjective morphism $f_t\colon \mathcal{X}_t \to C_t$ to a curve C_t of genus $\geq g(C)$. Note that $q(X) \leq g(C) + q(F) \leq g(C) + 1$ ($q(F) \leq p_g(F)$ since F is of general type), we have that $g(C_t) = g(C)$ again by Fact 2. Since

$$h^0(R^1 f_{t*} \mathcal{O}_{\mathcal{X}_t} \otimes \mathcal{T}_{C_t}) = h^1(R^1 f_{t*} \omega_{\mathcal{X}_t} \otimes \omega_{C_t}) = 0$$

by [Ko, Theorem 2.1], $\mathrm{Def}(f_t\colon \mathcal{X}_t \to C_t) \to \mathrm{Def}(\mathcal{X}_t)$ is surjective by Fact 1, there is a sufficiently small neighborhood U_t of $t \in B$, such that for any $t' \in U_t$, $\mathcal{X}_{t'}$ admits a holomorphic map $\tilde{f}_{t'}\colon \mathcal{X}_{t'} \to \tilde{C}_{t'}$ which is a smooth deformation of $f_t\colon \mathcal{X}_t \to C_t$. On the other hand, $\mathcal{X}_{t'}$ admits at most one holomorphic map to a curve of genus $g(C)$ up to isomorphism since $q(\mathcal{X}_{t'}) \leq g(C)+1$. So $\tilde{f}_{t'}$ and $f_{t'}$ are the same fibrations up to isomorphism. This implies that the geometric genus of the general fiber of $f_{t'}: \mathcal{X}_{t'} \to C_{t'}$ ($t' \in U_t$) is constant on U_t. Consequently, since B is connected, we have that $p_g(F') = p_g(F) = 1$, where F' is a general fiber of $f_{b'}: \mathcal{X}_{b'} \to C_{b'}$. By Proposition 2, we get the result. □

Remark. Let X be a smooth complex projective manifold of general type with $p_g(X) \geq 3$. Assume that Φ_X is composed with a pencil. In general, the canonical map of the (global) deformation of X is not necessarily composed with a pencil even when $\dim X = 2$.

Question. Is there a maximal family (i.e., moduli space component) of smooth projective surfaces of general type with $p_g \gg 0$ whose canonical maps are composed with a pencil?

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