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STRUCTURE FUNCTIONS ON THE LATTICE

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We report on a lattice computation of the second moment of the pion matrix element of the twist-2 non-singlet operator corresponding to the average momentum of parton densities. We apply a fully non-perturbatively evaluated running renormalization constant as well as a careful extrapolation of our results to the continuum limit. Thus the only limitation of our final result is the quenched approximation.

1 Introduction

A reliable computation of parton distribution functions from first principles would be very important for future experiments planned e.g. at the LHC. The results of such calculations would offer a unique way to test whether QCD is indeed the correct theory for the strong interactions.

In principle, the lattice regularization offers such a calculational scheme, where the only starting point is the QCD lagrangian, and indeed is able to give results for moments of parton density distributions *in the continuum* and *fully non-perturbatively*, as we will show in this contribution. The work presented here is a summary of a series of publications [1,2,3,4,5] that provide the essential ingredients to reach the above ambitious aim.

The moments of parton density distributions are related to expectation values of certain local operators, which are renormalized multiplicatively by applying appropriate renormalization factors $Z(1/\mu)$ that depend on the energy scale μ. This leads to consider renormalized matrix elements $O_{\rm SF}^{\rm ren}(\mu)$ to be computed in a certain, at this stage not specified, renormalization scheme SF. If the energy scale μ is chosen large enough, it is to be expected that the scale evolution is very well described by perturbation theory, giving rise to the following definition of a *renormalization group invariant matrix element:*

$$O_{\rm INV}^{\rm ren} = O_{\rm SF}^{\rm ren}(\mu) \cdot f^{\rm SF}(\bar{g}^2(\mu)) \qquad (1)$$

with $\bar{g}(\mu)$ the running coupling and

$$f^{\rm SF}(\bar{g}^2) = (\bar{g}^2(\mu))^{-\gamma_0/2b_0} \qquad (2)$$
$$\cdot \exp\left\{ -\int_0^{\bar{g}(\mu)} dg \left[\frac{\gamma(g)}{\beta(g)} - \frac{\gamma_0}{b_0 g} \right] \right\}$$

where $\beta(g)$ and $\gamma(g)$ are the β and anomalous-dimension functions computed to a given order of perturbation theory in the specified scheme, i.e. here the SF scheme. Once we know the value of $O_{\rm INV}^{\rm ren}$ evaluated non-perturbatively, the running matrix element in a preferred scheme can be computed, for example in the $\overline{\rm MS}$ scheme:

$$O_{\overline{\rm MS}}^{\rm ren}(\mu) = O_{\rm INV}^{\rm ren}/f^{\overline{\rm MS}}(\bar{g}^2(\mu)) \qquad (3)$$

with now, of course, the β and γ functions computed in the $\overline{\rm MS}$ scheme.

A non-perturbatively obtained value of the renormalization group invariant matrix element is hence of central importance. Its calculation has to be performed in several steps. The reason is that we have to cover a broad range of energy scales – from the deep perturbative to the non-perturbative region. With the scale-dependent renormalization factor $Z(1/\mu)$ we can write the renormalized matrix element of eq. (1) as

$$O_{\rm INV}^{\rm ren} = \frac{\langle \pi | \mathcal{O}_{\rm NS} | \pi \rangle}{Z^{\rm SF}(1/\mu)} \cdot f^{\rm SF}(\bar{g}^2(\mu)) , \quad (4)$$

with $\langle \pi | \mathcal{O}_{\rm NS} | \pi \rangle$ the expectation value of the (non-singlet) operator under consideration in given states, here the pion states.

So far, all our discussions have been in the continuum. However, if we think of the

lattice regularisation and eventual numerical simulations to obtain non-perturbative results, it would be convenient to compute the renormalized matrix element at only one convenient (i.e. small hadronic) scale μ_0. We therefore rewrite the r.h.s. of eq. (4) as:

$$\frac{\langle \pi | \mathcal{O}_{\mathrm{NS}} | \pi \rangle}{Z^{\mathrm{SF}}(1/\mu_0)} \cdot \underbrace{\frac{Z^{\mathrm{SF}}(1/\mu_0)}{Z^{\mathrm{SF}}(1/\mu)}}_{\equiv \sigma(\mu/\mu_0, \bar{g}(\mu))} \cdot f^{\mathrm{SF}}(\bar{g}^2(\mu)) \quad (5)$$

where we introduce the step scaling function $\sigma(\mu/\mu_0, \bar{g}(\mu))$, which describes the evolution of the renormalization factor from a scale μ_0 to a scale μ. The advantage of concentrating on the step scaling function instead of the renormalization factor itself, is that the step scaling function is well defined in the continuum and hence suitable for eventual continuum extrapolations of lattice results.

We finally write the r.h.s. of eq. (4) as

$$O_{\mathrm{SF}}^{\mathrm{ren}}(\mu_0) \underbrace{\sigma(\mu/\mu_0, \bar{g}(\mu)) \cdot f^{\mathrm{SF}}(\bar{g}^2(\mu))}_{\equiv \mathfrak{S}_{\mathrm{INV}}^{\mathrm{UV}}(\mu_0)} \quad (6)$$

with $O_{\mathrm{SF}}^{\mathrm{ren}}(\mu_0)$ the renormalized matrix element, which is to be computed only once at a scale μ_0 and the (ultraviolet) invariant step scaling function $\mathfrak{S}_{\mathrm{INV}}^{\mathrm{UV}}(\mu_0)$, which still depends on the infrared scale μ_0. The following sections are devoted to a description of how these two basic ingredients can be reliably computed on the lattice using non-perturbative methods, i.e. numerical simulations.

2 The renormalization group invariant step scaling function

Let us start this section by disclosing what is hidden behind the fictitious SF scheme mentioned in the introduction. SF stands for Schrödinger functional and denotes a finite physical volume, $V = L^3 \cdot T$, renormalization scheme where the energy scale μ is identified with the inverse spatial length of the box itself, e.g. $\mu = 1/L$. The peculiarity of the Schrödinger functional set-up is that fixed

boundary conditions in time x_0 are imposed with classical fields at the time boundaries at time $x_0 = 0$ and $x_0 = T$. For a more detailed discussion we refer the reader to [7].

To discuss the renormalization of operators related to moments of parton distribution functions, we first have to provide a renormalization condition. Denoting by $|SF\rangle$ a classical SF state, i.e. a classical quark field at a time boundary with external momentum $\mathbf{p}$, the renormalization condition that we will use reads

$$\langle SF | \mathcal{O}^{\mathrm{ren}} \left(\mu = \frac{1}{L} \right) | SF \rangle = \langle SF | \mathcal{O}^{\mathrm{tree}} | SF \rangle . \quad (7)$$

The relation between the expectation value of the bare operator and the renormalized one is established through a scale-dependent renormalization constant:

$$O^R(\mu) = Z^{-1}(1/\mu) O^{\mathrm{bare}}(1/L) . \quad (8)$$

In perturbation theory, on the 1-loop level, we have $Z(1/\mu) = 1 - \bar{g}^2(\mu) \left[\gamma^{(0)} \ln(\mu) + B_0 \right]$ with $\gamma^{(0)}$ the anomalous dimension and B_0 the constant part. Up to this point, the discussion is given solely in the continuum where the SF renormalization scheme is a perfectly acceptable one. Different schemes such as the $\overline{\mathrm{MS}}$ scheme can be related to the SF scheme as usual in perturbation theory.

If we are interested, however, in a non-perturbative calculation, we have to detour for a short time (which means, however, a substantial computer time) to a finite lattice with non-zero lattice spacing a that allows for numerical simulations. A lattice representation of the twist-2 non-singlet operator, which is the only case we are considering here, is given by

$$\mathcal{O}_{12}(x) = \frac{1}{4} \bar{\psi}(x) \gamma_{\{1} \overset{\leftrightarrow}{D}_{2\}} \frac{\tau^3}{2} \psi(x) , \quad (9)$$

where $\overset{\leftrightarrow}{D}_\mu$ is the covariant derivative and the bracket around indices means symmetrization. The operator is probed by boundary

quark fields ζ and $\bar{\zeta}$, which reside at $x_0 = 0$ and a correlation function is constructed

$$f_{O_{12}}(\frac{x_0}{a}) = \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} \langle e^{i\mathbf{p}(\mathbf{y}-\mathbf{z})} \mathcal{O}_{12}(\mathrm{x}) \bar{\zeta}(\mathrm{y}) \gamma_2 \tau^3 \zeta(\mathrm{z}) \rangle \tag{10}$$

with $\mathbf{p}$ the spatial 3-momentum.

To take into account the effects of the boundary fields, we also consider the boundary operators defined at the time boundaries $x_0 = 0$ and $x_0 = T$:

$$\mathcal{O}_0 = \frac{a^6}{L^3} \sum_{\mathbf{y},\mathbf{z}} \bar{\zeta}(\mathbf{y}) \gamma_5 \frac{\tau^3}{2} \zeta(\mathbf{z}),$$

$$\mathcal{O}_T = \frac{a^6}{L^3} \sum_{\mathbf{y},\mathbf{z}} \bar{\zeta}'(\mathbf{y}) \gamma_5 \frac{\tau^3}{2} \zeta'(\mathbf{z}) \tag{11}$$

from which we construct the correlation function:

$$f_1 = -\langle \mathcal{O}_0 \mathcal{O}_T \rangle . \tag{12}$$

The boundary wave-function contribution can then be taken out by considering the ratio $f_{O_{12}}(x_0/a)/\sqrt{f_1}$.

There are several physical scales in our problem which all have to be given in units of L, which is the only scale that is to be changed. Therefore, in order to arrive at a definition of the renormalization constant, we have to specify the spatial momentum $\mathbf{p}$, the quark mass m_q and the time x_0 where we read off the expectation value of the operator between SF states from the correlation function. The final physical result does, of course, not depend on our choice of these quantities, but we choose them solely for convenience. In particular we select

$$m_q = 0 , \quad x_0 = T/4 , \tag{13}$$
$$\mathbf{p} \equiv (p_1, p_2, p_3) = (p_1 = 2\pi/L, 0, 0) .$$

The choice of a zero quark mass results in using a massless renormalization scheme and the choice of the smallest available momentum on the lattice minimizes lattice artefacts. With the above choice, it is indeed only the physical box length L (assuming $T = L$) that we identify with the inverse scale, which is varied in the problem.

We are now in a position to give the precise definition of the renormalization constant

$$Z(L) = \bar{Z}(L)/Z_1(L) , \tag{14}$$

with

$$\bar{Z}(L) = f_{O_{12}}(T/4)/f_{O_{12}}^{\mathrm{tree}}(T/4) ,$$
$$Z_1(L) = \sqrt{f_1(L)}/\sqrt{f_1^{\mathrm{tree}}(L)} , \tag{15}$$

where we divide by the corresponding tree level expression as required by the renormalization condition, eq. (7). Instead of computing the Z-factors, we concentrate on the step scaling functions

$$\sigma_{\bar{Z}} = \frac{\bar{Z}(2L)}{\bar{Z}(L)}, \ \sigma_{f_1} = \frac{Z_1(2L)}{Z_1(L)}, \ \sigma_Z = \frac{Z(2L)}{Z(L)} , \tag{16}$$

because, in contrast to the Z-factors, the step scaling functions have a well-defined continuum limit. The strategy is now to compute the step scaling functions at various values of the lattice spacing while keeping fixed the conditions in eqs. (13) and the physical scale $\mu = L^{-1}$ (determined by the running coupling $\bar{g}(\mu)$) and to extrapolate the results thus obtained to $a = 0$.

It is one of the basic ingredients and characteristics of our work that almost all simulation results at non-zero lattice spacings have been obtained by employing the standard Wilson action and the non-perturbatively improved clover action. Since these two formulations lead to different lattice artefacts, it is a very crucial test of our results that their continuum extrapolations give consistent results. That this is indeed the case is demonstrated in fig. 1. It shows that for the two step scaling functions σ_{f_1} and $\sigma_{\bar{Z}}$ the continuum limit of both discretizations agree within the error bars. We note that in the case of $\sigma_{\bar{Z}}$ a quadratic extrapolation in the lattice spacing a is necessary while for σ_{f_1} a linear extrapolation is sufficient. After checking that a similar behaviour is found at all values of the coupling we have simulated, we performed
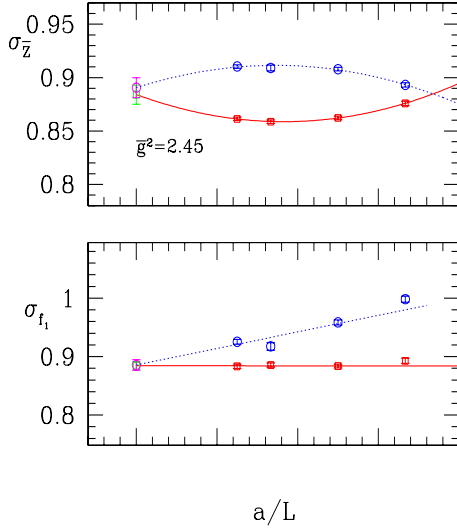
Figure 1. Continuum extrapolation of the step scaling functions $\sigma_{\bar{Z}}$ and σ_{f_1} performed separately for the Wilson action (circles and dotted lines) and the non-perturbatively improved action (squares and full lines) at a fixed value of the running coupling $\bar{g}^2 = 2.45$.

constraint fits, demanding that the continuum value of the step scaling functions be the same from both actions. A summary of our results for $\sigma_{\bar{Z}}$ is shown in fig. 2.

At this point, having performed all necessary continuum extrapolations, we end our detour on a lattice with non-zero lattice spacing and come back to the discussion in the continuum. With the results on the step scaling function, extrapolated to the continuum limit, which were obtained at 9 values of the running coupling constant, we can now compute the (ultraviolet) invariant step scaling function

$$\mathfrak{S}_{\mathrm{INV}}^{\mathrm{UV}}(\mu_0) = \sigma(\mu/\mu_0, \bar{g}^2(\mu_0)) \cdot f(\bar{g}^2(\mu)) , \tag{17}$$

which still depends on the infrared scale μ_0. This scale dependence will only be cancelled when multiplying with the matrix element, renormalized at the scale μ_0. The function
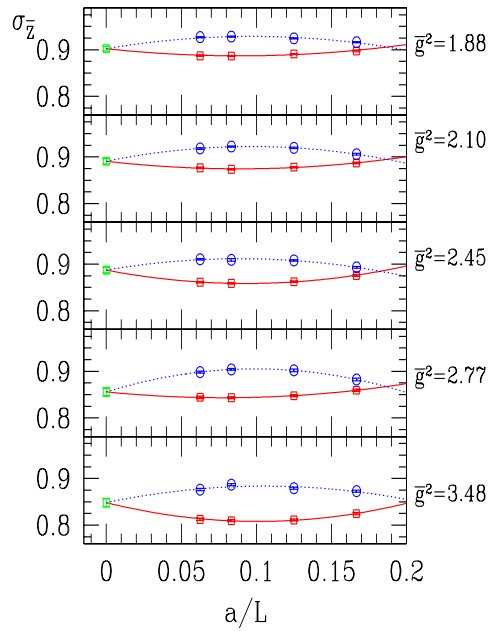

Figure 2. Constraint continuum extrapolation of $\sigma_{\bar{Z}}$ (notation as in fig. 1).
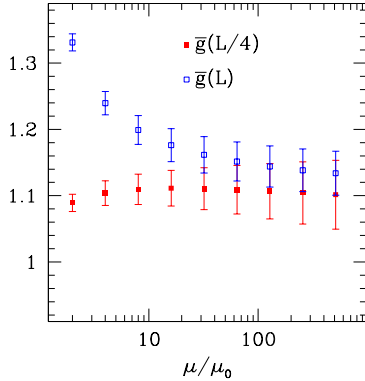
Figure 3. The values of $\mathfrak{S}_{\mathrm{INV}}^{\mathrm{UV}}(\mu_0)$ for two choices of the running coupling.

$f(\bar{g}^2(\mu))$ is the same as in eq. (3) and the β and γ functions are taken up to 3 loops in the SF scheme[a].

In fig. 3 we show $\mathfrak{S}_{\mathrm{INV}}^{\mathrm{UV}}(\mu_0)$ as a function of μ/μ_0. For large enough energy scales, $\mu/\mu_0 > 100$, $\mathfrak{S}_{\mathrm{INV}}^{\mathrm{UV}}(\mu_0)$ does not change within the errors and we can determine a value for it by fitting the last, say, 4 points to a constant. Although there still is a scheme dependence in the invariant step scaling function through the remaining dependence on μ_0, the value should be independent of the choice of coupling used in the analysis. This is nicely illustrated in fig. 3, where the choices of $\bar{g}(L/4)$ and $\bar{g}(L)$ give consistent values for the invariant step scaling function.

We can therefore now give the first piece of information for the invariant matrix element itself, as needed in eq. (6), and quote

$$\mathfrak{S}_{\mathrm{INV}}^{\mathrm{UV}}(\mu_0) = 1.11(4) . \qquad (18)$$

3 Renormalized matrix element

As a next step we have to compute the renormalized matrix element itself. Again, we will

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[a]For the γ function we have taken an effective 3-loop parametrization as obtained by fitting our data to an effective 3-loop form [2].

always use the SF scheme and we remark that this is the first time that a calculation of a 2-quark matrix element is attempted in this set-up. We first tried to compute the matrix element (within pion states) of the operator of eq. (9). However, since this operator needs a non-vanishing momentum, we found, similar as in [6], that the signal is very noisy. We therefore decided to switch to the operator

$$\mathcal{O}_{00}(x) = \bar{\psi}(x) \left[\gamma_0 \overleftrightarrow{D}_0 - \frac{1}{3} \sum_{k=1}^{3} \gamma_k \overleftrightarrow{D}_k \right] \frac{\tau^3}{8} \psi(x) \qquad (19)$$

which has the advantage that it can be computed at zero momentum. Taking the boundary operators of eq. (11) we construct a correlation function

$$f_{\mathrm{M}}(x_0) = a^3 \sum_{\mathbf{x}} \langle \mathcal{O}_0 \mathcal{O}_{00}(x) \mathcal{O}_T \rangle , \qquad (20)$$

which again is to be normalized by f_1, eq. (12), to take out the boundary wave-function contributions. Performing a transfer matrix decomposition, we find that for large enough values of x_0 and staying far enough from both boundaries

$$f_1 \simeq \rho^2 e^{-m_\pi T},$$
$$f_{\mathrm{M}}(x_0) \simeq \rho^2 e^{-m_\pi T} \langle \pi | \mathcal{O}_{00} | \pi \rangle . \qquad (21)$$

Assuming that there is a plateau region where $f_M(x_0)/f_1 = \text{const} \equiv \langle \pi | \mathcal{O}_{00} | \pi \rangle$, and in which the first excited state gives essentially no contributions, we obtain the physical matrix element $\langle x \rangle$ after a suitable normalization (see [6]):

$$\langle x \rangle \equiv \frac{2\kappa}{m_\pi} \langle \pi | \mathcal{O}_{00} | \pi \rangle . \qquad (22)$$

In order to extract the matrix element of eq. (22) we have chosen lattices with $T = 3$ fm and followed the correlation function $f_{\mathrm{M}}(x_0)$ up to a distance of 1 fm in time direction. At this distance we are sure that we project on the pion states as an inspection of the pseudoscalar and axial-vector correlation functions (from which we also ex-
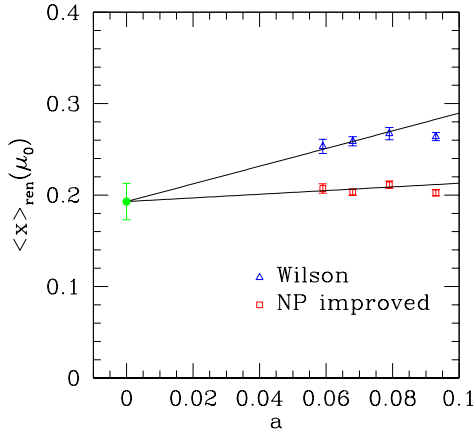
Figure 4. Constrained continuum extrapolation of the renormalized matrix element

tracted the pion masses) showed. Indeed, for 1 fm $< x_0 <$ 2 fm the correlation function exhibits a plateau behaviour as can be seen in fig. 2 of ref. [5].

Once we have the bare matrix element we need to renormalize it. To this end we computed $Z(1/\mu_0)$ with $\mu_0^{-1} = 1.436 r_0$, $r_0 \approx$ 0.5 fm. We repeated such a calculation for various lattice sizes, choosing the values of β such that $L_{\max} = \mu_0^{-1}$ is kept fixed. Interpolating the numerical simulation data we obtain in this way $Z(1/\mu_0)$ in a range of lattice spacing $0.05 \leq a \leq 0.1$, i.e. the range of a where the bare matrix element itself has been computed.

This allowed us to apply the renormalization factor at exactly the same values of β where the matrix element has been computed. We show the matrix element renormalized with $Z_{12}(1/\mu_0)$ in fig. 4. Again all calculations have been performed with two different discretizations, and we find quite convincingly the same continuum limit.

It might, however, not have escaped the reader's attention that we have used the wrong renormalization factor, namely Z_{12},

for renormalizing the operator $\mathcal{O}_{00}(x)$ of eq. (19). The continuum extrapolation of the such renormalized operator needs a correction factor. In ref. [5] we have demonstrated that this correction factor can be taken from perturbation theory and amounts to a shift of the continuum renormalized matrix element by a few per cent. Taking the correction factor to be the same as found in [5], we can finally give our main result:

$$O_{\mathrm{INV}}^{\mathrm{ren}} = 0.222(24) \qquad (23)$$

4 Conclusion

In this contribution we have demonstrated how we can compute on the lattice, in a fully non-perturbative fashion, a renormalization group invariant matrix element $O_{\mathrm{INV}}^{\mathrm{ren}}$ for the second moment of parton distributions of the twist-2 non-singlet operator in the pion. A preliminary value for $O_{\mathrm{INV}}^{\mathrm{ren}}$ is given in eq. (23) and can be used now as integration constant to obtain the renormalized matrix element at any scale in the preferred renormalization scheme.

We want to emphasize that the value of the renormalized matrix element is shifted substantially from a value of about $\langle x \rangle (a = 0.093) = 0.30$ at $\beta = 6$ to $\langle x \rangle (a = 0) = 0.2$ in the continuum limit. Hence we experience strong lattice artefacts in the renormalized matrix element. Still, when the matrix element is run in the $\overline{\mathrm{MS}}$ scheme to a scale of $\mu = 2.4$ GeV, we find $\langle x \rangle (\mu = 2.4 \mathrm{GeV}) \approx 0.3$. Thus we find that, by a conspiration of two effects, our result agrees with the number quoted in the pioneering work of [6]. Therefore, the fact that within the quenched approximation, used here exclusively, the number from the lattice simulations is higher than the experimental value persists.

Acknowledgements

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