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# Representations of (1,0) and (2,0) superconformal algebras in six dimensions: massless and short superfields

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**Abstract**

We construct unitary representations of (1,0) and (2,0) superconformal algebras in six dimensions by using superfields defined on harmonic superspaces with coset manifolds $USp(2n)/[U(1)]^n$, $n = 1, 2$.
In the spirit of the $AdS_7/CFT_6$ correspondence massless conformal fields correspond to "supersingletons" in $AdS_7$. By tensoring them we produce all short representations corresponding to 1/2 and 1/4 BPS anti-de Sitter bulk states of which "massless bulk" representations are particular cases.

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# 1 Introduction

Superconformal field theories in space-time dimensions $d \leq 6$ have received a lot of attention in recent time because of their connection to $(d-1)$-branes and their near-horizon $AdS_{d+1}$ geometries [1]. The most popular examples are IIB string theory $D3$-branes and M-theory five- and two-branes related to $d = 4, 6$ and 3 dimensional superconformal field theories.

A classification of a certain type of UIR's (called highest-weight UIR's) of their algebras has been made in the literature in a variety of ways, either by using the oscillator method [2]-[5] which is directly linked to the $AdS$ interpretation or by using superconformal fields defined on $\tilde{M}_d = \partial AdS_{d+1}$, i.e. "Minkowski space" regarded as the boundary at infinity of anti-de Sitter space.

The second approach has recently been employed [6, 7] for all four-dimensional superconformal algebras $SU(2,2/N)$ to construct "massless" and "short" representations. The latter are the generalization of the "chiral superfields" of $N = 1$ supersymmetry [8]. The generalization of "chirality" to $N > 1$ theories is made transparent by using superfields augmented with "harmonic variables" [9], i.e. coordinates of coset spaces obtained as quotients $G/H$ where $G$ is the R-symmetry group and $H$ is a maximal subgroup of $G$ (with rank $G =$ rank $H$). The most convenient choice is to take $H$ to be the maximal torus, i.e. the group related to the Cartan subalgebra $[U(1)]^{\text{rank } G}$ of $G$. Such cosets are called "flag manifolds" [10] and an important property is that highest-weight UIR's of $G$ defined on such manifolds correspond to "analytic functions" with some degree of homogeneity. For the algebras $SU(2,2/N)$ such manifolds are the cosets $SU(N)/[U(1)]^{N-1}$.

In the present paper we extend the analysis to the $N = (n, 0)$ ($n = 1, 2$) conformal supersymmetry in six dimensions, i.e. to the superalgebras $OSp(8^*/2n)$. In this case the flag manifolds in question are $USp(2n)/[U(1)]^n$. Accordingly, superconformal fields will be defined on "harmonic superspaces" with space-time coordinates $x^\mu$ ($\mu = 0, 1, \ldots, 5$), odd coordinates $\theta^\alpha_i$ (left-handed spinors of $SU^*(4) \sim SO(5,1)$ and spinors of $USp(2n)$) and coordinates $u^I_i$ on the flag manifold. A particular output of our investigation will be the superfields describing massless conformal fields ("supersingletons" in the $AdS$ language [11]) which satisfy Dirac-type equations

$$\partial^{\alpha\alpha_1}\omega_{(\alpha_1\alpha_2\ldots\alpha_k)} = 0 \tag{1}$$

(or $\Box\omega = 0$ if $k = 0$) with $\partial^{\alpha\beta} = \partial_\mu(\gamma^\mu)^{\alpha\beta}$ and $\omega$ totally symmetric in their spinor indices, i.e. belonging to the $(0,0,k)$ representation of $SU^*(4)$.

We will show that supersingleton superfields are of two kinds: (i) those whose first component carries any spin label of the above type but is an $USp(2n)$ singlet; (ii) those which have an analytic structure in harmonic superspace and are "ultrashort"; their first component is a Lorentz scalar but carries $USp(2n)$ indices. By tensoring the second kind of superfields we are able to produce "short representations" which do not depend on one half or on one quarter of the odd variables. In the AdS bulk language [12], these states correspond to 1/2 or 1/4 BPS states, respectively. A particular example of such states are the so-called "massless bulk states" which correspond to tensoring two massless multiplets. Agreement is found for the classification of such states as compared to the "oscillator method" [13].

Massive towers corresponding to 1/2 BPS states are the K-K states [5] coming from compactification of M theory on $AdS_7 \times S_4$ [14]-[16]. The description of such K-K states in terms of the $(2,0)$ superconformal field theory was considered by several authors [17]-[21]. Extension of the analysis to $(1,0)$ theories was also investigated [22].

The paper is organized as follows. In section 2 we list the six-dimensional notations and conventions. In section 3 massless conformal supermultiplets (supersingletons) are described by constrained superfields in ordinary superspace. In section 4 most of these multiplets are reformulated in harmonic superspace and it is shown that two of them are "ultrashort". In section 5 the "short representations" of the $N=(1,0)$ and $(2,0)$ superalgebras are constructed by tensoring the basic multiplets.

# 2 Notations

We use the six-dimensional notations of Ref. [23] with some minor modifications. The six-dimensional superspace has coordinates

$$x^{\alpha\beta} = -x^{\beta\alpha} = x^\mu \gamma_\mu^{\alpha\beta} \, , \qquad \theta^\alpha_i \, . \tag{2}$$

Here $\alpha,\beta$ are left-handed[1] chiral spinor indices of $SU^*(4) \sim SO(5,1)$ and $i$ is a spinor index of $USp(2n) \sim SO(2n+1)$ in the case of $N=(n,0)$ supersymmetry, $n=1$ or 2. The latter can be raised and lowered with the

[1]Right-handed spinors are denoted, e.g., $\psi_\alpha$ which makes transparent the meaning of the contraction $\theta^\alpha\psi_\alpha$.

help of the $USp(2n)$ matrix:

$$\lambda_i = \lambda^j \Omega_{ji} \ , \qquad \lambda^i = \Omega^{ij} \lambda_j \ , \qquad \Omega_{ij} \Omega^{jk} = -\delta_i^k \ . \tag{3}$$

We choose $\Omega$ in the following standard form:

$$\begin{pmatrix} 0 & \mathcal{I} \\ -\mathcal{I} & 0 \end{pmatrix} \tag{4}$$

where $\mathcal{I}$ is the $n \times n$ identity matrix. The odd coordinates satisfy a Majorana-Weyl pseudoreality condition:

$$\bar{\theta}^i_\alpha = \Omega^{ij} \theta^\beta_j c_{\beta\alpha} \tag{5}$$

where $c$ is a $4 \times 4$ unitary "charge conjugation" matrix.

The spinor covariant derivatives

$$D^i_\alpha = \frac{\partial}{\partial \theta^\alpha_i} - \frac{i}{2} \theta^{\beta i} \partial_{\beta\alpha} \tag{6}$$

satisfy the supersymmetry algebra

$$\{D^i_\alpha, D^j_\beta\} = i \Omega^{ij} \gamma^\mu_{\alpha\beta} \partial_\mu \ . \tag{7}$$

The generators of $USp(2n)$ $L_{ij} = L_{ji}$ form the algebra

$$[L^{ij}, L^{kl}] = \Omega^{i(k} L^{l)j} + \Omega^{j(k} L^{l)i} \tag{8}$$

and commute with an $USp(2n)$ spinor as follows:

$$[L^{ij}, D^k_\alpha] = -\Omega^{k(i} D^{j)}_\alpha \tag{9}$$

where the symmetrization has weight 1.

# 3 Massless supermultiplets (supersingletons)

There exist several types of massless multiplets in six dimensions corresponding to UIR's of $OSp(8^*/2n)$, $n = 1, 2$. All of them can be described by constrained superfields following closely the four-dimensional case [24].

*(i)* The first type only exists in the case $N = (2,0)$ since the corresponding superfield $W^{\{ij\}}(x,\theta)$ is antisymmetric and *traceless* in the external indices ($\underline{5}$ of $USp(4)$). It satisfies the constraint

$$D_\alpha^{(k} W^{\{i)j\}} = 0 \ . \tag{10}$$

One can also impose the reality condition

$$\overline{W}_{\{ij\}} = \Omega_{ik}\Omega_{jl} W^{\{kl\}} \ . \tag{11}$$

Using the spinor derivative algebra (7), it is not hard to show that this superfield has the following $\theta$ expansion:

$$W^{\{ij\}} = \phi^{\{ij\}} + \theta^{\alpha\{i}\psi_\alpha^{j\}} + \theta^{\alpha\{i}\theta^{\beta j\}} F_{(\alpha\beta)} + \text{derivative terms} \ . \tag{12}$$

Here one finds 5 scalars $\phi^{\{ij\}}$, 4 right-handed spinors $\psi_\alpha^i$ and a $\underline{10}$ of $SU^*(4)$ $F_{(\alpha\beta)} = \gamma^{\mu\nu\lambda}_{(\alpha\beta)} F_{\mu\nu\lambda}$ (a self-dual three-form), as well as a few more terms containing derivatives of the above fields. These fields satisfy massless equations:

$$\Box\phi^{\{ij\}} = 0 \ , \quad \partial^{\alpha\beta}\psi_\beta^i = 0 \ , \quad \partial^{\alpha\beta} F_{(\beta\gamma)} = 0 \ . \tag{13}$$

The latter equation implies that the three-form $F_{\mu\nu\lambda}$ is the curl of a two-form,

$$F_{\mu\nu\lambda} = \partial_{[\mu} B_{\nu\lambda]} \ . \tag{14}$$

One recognizes the content of the on-shell tensor $N = (2,0)$ multiplet in six dimensions [25].

It is instructive to give the on-shell counting of degrees of freedom in a six-dimensional massless multiplet. A massless field of non-vanishing spin $\omega_{(\alpha_1\alpha_2\ldots\alpha_n)}(x)$ describes an irrep of $SU^*(4)$ with Dynkin labels $(0,0,n)$. It is subject to the Dirac-type field equation

$$\partial^{\alpha\alpha_1}\omega_{(\alpha_1\alpha_2\ldots\alpha_n)} = 0 \tag{15}$$

which clearly implies $\Box\omega_{(\alpha_1\alpha_2\ldots\alpha_n)} = 0$, i.e. the field is indeed massless. Thus one can go to the Lorentz frame in which the momentum takes the form $p^\mu = (p^0, 0, 0, 0, 0, p^0)$. There the little group is $SO(4) \sim SU(2) \times SU(2)'$ and an $SU^*(4)$ spinor index is decomposed in a pair of $SU(2)$ indices, $\alpha = (a, a')$. Then, in the appropriate basis for the gamma matrices the operator $p^{\alpha\alpha_1} = p^\mu\gamma_\mu^{\alpha\alpha_1}$ in eq. (15) becomes a projector onto, e.g., the indices $a'$. This means that the only $SU(2) \times SU(2)'$ irreducible part of the multispinor $\omega_{(\alpha_1\alpha_2\ldots\alpha_n)}$

surviving in eq. (15) is $\omega_{(a_1 \dots a_n)}$, i.e. an $n+1$-plet of the first $SU(2)$. The conclusion is that such a field describes $n+1$ massless degrees of freedom (in general complex, unless a reality condition can be imposed on the field). Applied to eqs. (13), this counting results in 8 bosons (5 from the scalars $\phi^{\{ij\}}$ and 3 from the tensor $F_{(\alpha\beta)}$) and 8 fermions (from the four spinors $\psi^i_\alpha$). Note that these degrees of freedom are real if the reality condition (11) is imposed.

Concluding the discussion of the tensor multiplet we note that the $SU^*(4)$ and $USp(4)$ quantum numbers and the dimensions (relative to the first component) of the components found in the on-shell superfield expansion (12) exactly match those of the states of the "doubleton" supermultiplet listed in Table 1 of Ref. [13] [2]

All other types of massless multiplets exist in both cases $N=(n,0)$, $n=1,2$.

*(ii)* The second type is described by a superfield $W^i(x,\theta)$ which is in the fundamental UIR of $USp(2n)$. The corresponding constraint is

$$D^{(k}_\alpha W^{i)} = 0 \ . \tag{16}$$

In the case $N=(1,0)$ the superfield has a very short expansion

$$N=(1,0): \qquad W^i = \phi^i + \theta^{\alpha i}\psi_\alpha + \text{derivative terms} \ . \tag{17}$$

The doublet of scalars $\phi^i$ and the spinor $\psi_\alpha$ satisfy the field equations

$$\Box\phi^i = 0 \ , \quad \partial^{\alpha\beta}\psi_\beta = 0 \ . \tag{18}$$

This is the $N=(1,0)$ hypermultiplet [26] in six dimensions with 2+2 complex on-shell degrees of freedom (note that one cannot impose a reality condition on the superfield $W^i$).

In the case $N=(2,0)$ the expansion of $W^i$ becomes

$$\begin{aligned} N=(2,0): \qquad W^i \ &= \ \phi^i + \theta^\alpha_j \psi^{[ij]}_\alpha + \theta^\alpha_k \theta^\beta_l \epsilon^{klij} F_{(\alpha\beta)j} \\ &\quad + \theta^\alpha_j \theta^\beta_k \theta^\gamma_l \epsilon^{ijkl} \chi_{(\alpha\beta\gamma)} + \text{d. t.} \end{aligned} \tag{19}$$

The components are scalars $\phi^i$ ($\underline{4}$ of $USp(4)$), spinors $\psi^{[ij]}_\alpha$ ($\underline{5}+\underline{1}$ of $USp(4)$), three-forms $F^i_{(\alpha\beta)}$ ($\underline{4}$ of $USp(4)$) and a $\underline{20}$ of $SU^*(4)$ $\chi_{(\alpha\beta\gamma)}$. These fields satisfy the massless equations

$$\Box\phi^i = 0 \ , \quad \partial^{\alpha\beta}\psi^{[ij]}_\beta = \partial^{\alpha\beta}F^i_{(\beta\gamma)} = \partial^{\alpha\beta}\chi_{(\beta\gamma\delta)} = 0 \tag{20}$$

[2] Note that, compared to Ref. [13], we use the Dynkin labels of the conjugate representations.

and thus describe 16 bosons (4 from $\phi^i$ and 12 from $F^i_{(\alpha\beta)}$) and 16 fermions (12 in $\psi^{[ij]}_\alpha$ and 4 in $\chi_{(\alpha\beta\gamma)}$) (all complex). The fields in (19) match the states of the "doubleton" supermultiplet listed in Table 2 of Ref. [13].

*(iii)* The next multiplet stands apart since it is the only one described by a superfield without external indices, $W(x,\theta)$ (it can be made real, $\overline{W} = W$). The corresponding constraint is second-order in the spinor derivatives:

$$D^{(i}_\alpha D^{j)}_\beta W = 0 \ . \tag{21}$$

In the case $N = (1,0)$ the superfield expansion is

$$N = (1,0): \qquad W = \phi + \theta^\alpha_i \psi^i_\alpha + \theta^{\alpha i}\theta^\beta_i F_{(\alpha\beta)} + \text{d. t.} \tag{22}$$

where the fields satisfy the massless equations

$$\Box\phi = 0 \ , \quad \partial^{\alpha\beta}\psi^i_\beta = \partial^{\alpha\beta}F_{(\beta\gamma)} = 0 \ . \tag{23}$$

This is the so-called "linear multiplet" of Ref. [25] describing $4+4$ real (if $W$ is real) degrees of freedom.

In the case $N = (2,0)$ the components of the superfield are

$$\begin{aligned} N = (2,0): \qquad W \ &= \ \phi + \theta^\alpha_i \psi^i_\alpha + \theta^\alpha_i \theta^\beta_j F^{[ij]}_{(\alpha\beta)} + \theta^\alpha_i \theta^\beta_i \theta^\gamma_k \epsilon^{ijkl}\chi_{(\alpha\beta\gamma)l} \\ &\quad + \theta^\alpha_i \theta^\beta_j \theta^\gamma_k \theta^\delta_l \epsilon^{ijkl}\sigma_{(\alpha\beta\gamma\delta)} + \text{d. t.} \end{aligned} \tag{24}$$

and they obey the massless field equations

$$\Box\phi = 0 \ , \quad \partial^{\alpha\beta}\psi^i_\beta = \partial^{\alpha\beta}F^{[ij]}_{(\beta\gamma)} = \partial^{\alpha\beta}\chi^i_{(\beta\gamma\delta)} = \partial^{\alpha\beta}\sigma_{(\beta\gamma\delta\varepsilon)} = 0 \ . \tag{25}$$

This amounts to $24+24$ real (if $W$ is real) degrees of freedom. The corresponding "doubleton" supermultiplet of Ref. [13] is listed in Table 3 for $j = 1$.

*(iv)* Finally, there exists a series of multiplets described by superfields with external Lorentz indices, $W_{(\alpha_1\ldots\alpha_n)}(x,\theta)$ in the $SU^*(4)$ UIR with Dynkin labels $(0,0,n)$. These superfields can be made real in the case $n = 2k$. Now the constraint takes the form

$$D^i_{[\beta} W_{(\alpha_1]\ldots\alpha_n)} = 0 \ . \tag{26}$$

The resulting expansion is

$$N = (1,0): \quad W_{(\alpha_1\ldots\alpha_n)} = \phi_{(\alpha_1\ldots\alpha_n)} + \theta^\beta_i \psi^i_{(\beta\alpha_1\ldots\alpha_n)} + \theta^{\beta i}\theta^\gamma_i F_{(\beta\gamma\alpha_1\ldots\alpha_n)} + \text{d. t.} \tag{27}$$

describing $(2n+4)+(2n+4)$ massless degrees of freedom or

$$N=(2,0): \qquad W_{(\alpha_1\ldots\alpha_n)} = \phi_{(\alpha_1\ldots\alpha_n)} + \theta^\beta_i \psi^i_{(\beta\alpha_1\ldots\alpha_n)} + \theta^\beta_i\theta^\gamma_j F^{[ij]}_{(\beta\gamma\alpha_1\ldots\alpha_n)} \\ +\theta^\beta_i\theta^\gamma_j\theta^\delta_k\epsilon^{ijkl}\chi_{l(\beta\gamma\delta\alpha_1\ldots\alpha_n)} + \theta^\beta_i\theta^\gamma_j\theta^\delta_k\theta^\varepsilon_l\epsilon^{ijkl}\sigma_{(\beta\gamma\delta\varepsilon\alpha_1\ldots\alpha_n)} + \text{d. t.} \tag{28}$$

describing $(8n+24)+(8n+24)$ massless degrees of freedom. The corresponding "doubleton" supermultiplet of Ref. [13] is listed in Table 3 for $j>1$.

The highest-weight UIR's of the $OSp(8^*/2n)$ algebras will be denoted by

$$\mathcal{D}(\ell, J_1, J_2, J_3; a_1, \ldots, a_n)$$

where $\ell$ is the conformal dimension, $J_1, J_2, J_3$ are the $SU^*(4)$ Dynkin labels and $a_1,\ldots,a_n$ are the $USp(2n)$ Dynkin labels of the first component. The analytic supersingletons for $n=2$ correspond to $\mathcal{D}(2,0,0,0;1,0)$ and $\mathcal{D}(2,0,0,0;0,1)$. The other supersingletons correspond to $\mathcal{D}(2+k,0,0,k;0,0)$ for $k=0,1,2,\ldots$. In Section 5.2 we will show that the short representations corresponding to analytic superfields are given by $\mathcal{D}(2p+4q,0,0,0;p,2q)$ for $p,q=0,1,2,\ldots$.

# 4 Harmonic superspace formulation

The massless multiplets *(i-iii)* admit an alternative formulation in harmonic superspace [9]. The advantage of this formulation is that the constraints (10), (16) become simply conditions for Grassmann analyticity (i.e., independence of the superfield of part of the odd coordinates) whereas the constraint (21) becomes a linearity condition. This new simple form of the constraints greatly simplifies the tensor multiplication of the multiplets.

Harmonic superspace is obtained by augmenting ordinary superspace $x^\mu, \theta^\alpha_i$ by an internal space (a coset of the automorphism group of the supersymmetry algebra). In our case the relevant cosets are

$$N=(n,0): \qquad \frac{USp(2n)}{[U(1)]^n}\ . \tag{29}$$

These cosets can be parametrized by harmonic variables forming a matrix of $USp(2n)$:

$$u\in USp(2n): \qquad u^I_i u^i_J = \delta^I_J\ , \quad u^I_i\Omega^{ij}u^J_j = \Omega^{IJ}\ , \quad u^I_i = (u^i_I)^*\ . \tag{30}$$

Here the indices $i,j$ belong to the fundamental representation of $USp(2n)$ and $I,J$ are two (four) labels corresponding to the $U(1)$ charge(s). The harmonic derivatives (the covariant derivatives on the coset (29)) are the operators

$$D^{IJ} = \Omega^{K(I}u_i^{J)}\frac{\partial}{\partial u_i^K} \ . \tag{31}$$

They are clearly compatible with the defining conditions (30) and act on the harmonics as follows:

$$D^{IJ}u_i^K = -\Omega^{K(I}u_i^{J)} \ . \tag{32}$$

In fact, it is easy to see that these derivatives form the algebra of $USp(2n)$ (see (8)) realised on the indices $I,J$ of the harmonics. In particular, the operators $H=-2D^{12}$ in the case $USp(2)$ and $H'=-2D^{14}$, $H''=-2D^{23}$ in the case $USp(4)$ correspond to the $U(1)$ charges:

$$USp(2): \qquad Hu_i^1 = u_i^1 \ , \ Hu_i^2 = -u_i^2 \ ; \tag{33}$$

$$\begin{aligned} USp(4): \qquad & H'u_i^1 = u_i^1 \ , \ H'u_i^2 = 0 \ , \ H'u_i^3 = 0 \ , \ H'u_i^4 = -u_i^4 \ , \\ & H''u_i^1 = 0 \ , \ H''u_i^2 = u_i^2 \ , \ H''u_i^3 = -u_i^3 \ , \ H''u_i^4 = 0 \ . \end{aligned} \tag{34}$$

A key ingredient of the harmonic superspace approach of Refs. [9] is the coordinateless realization of cosets like (29) on harmonic functions homogeneous under the action of the charge operators. Such functions are defined by their $USp(2n)$ invariant "harmonic" expansions. For instance, in the case $USp(2)$ the function $f^1(u)$ carries charge $+1$ (like $u_i^1$, see (33)) and has the expansion

$$f^1(u) \equiv f^{(+1)}(u) = f^i u_i^1 + f^{(ijk)}u_i^1u_j^1u_k^2 + f^{(ijklm)}u_i^1u_j^1u_k^1u_l^2u_m^2 + \ldots \ . \tag{35}$$

In the case $USp(4)$ the same function $f^1(u)$ carries charges $(+1,0)$ (like $u_i^1$, see (34)) and has the expansion

$$f^1(u) \equiv f^{(+1,0)}(u) = f^i u_i^1 + g^{(ij)k}u_i^1u_j^1u_k^4 + h^{ijk}u_1^1u_j^2u_k^3 + \ldots \ . \tag{36}$$

In other words, the expansion is formed by the various products of harmonics carrying an overall index 1 (i.e., charge $+1$ in the case $USp(2)$ or $(+1,0)$ in the case $USp(4)$). The crucial point about this coordinateless parametrization of the coset is that the coefficients in the harmonic expansion are manifestly $USp(2n)$ covariant. Another example of a $USp(4)$ harmonic function is

$$f^{12}(u) \equiv f^{(+1,+1)}(u) = f^{\{ij\}}u_i^1u_j^2 + g^{(ij)}u_i^1u_j^2 + \ldots \ . \tag{37}$$

Note the absence of a singlet part (trace) in the coefficient $f^{\{ij\}}$, since $\Omega^{ij}u_i^1 u_j^2 = 0$ (see (30), (4)).

As one can see from the above examples, the harmonic functions are infinitely reducible under $USp(2n)$. An important point is that the "step-up" operators (the positive roots) of $USp(2n)$ can be used to impose irreducibility conditions on the harmonic functions. In the case $USp(2)$ this is the harmonic derivative $D^{11}$ and in the case $USp(4)$ these are the harmonic derivatives $D^{11}$, $D^{12}$, $D^{13}$, $D^{22}$. So, for example,

$$\begin{aligned}
USp(2): \quad & D^{11} f^1(u) = 0 \\
& \Rightarrow \ f^1(u) = f^i u_i^1 \ ; && (38) \\
USp(4): \quad & D^{11} f^1(u) = D^{12} f^1(u) = D^{13} f^1(u) = D^{22} f^1(u) = 0 \\
& \Rightarrow \ f^1(u) = f^i u_i^1 \ ; && (39) \\
& D^{11} f^{12}(u) = D^{12} f^{12}(u) = D^{13} f^{12}(u) = D^{22} f^{12}(u) = 0 \\
& \Rightarrow \ f^{12}(u) = f^{\{ij\}} u_i^1 u_j^2 \ . && (40)
\end{aligned}$$

In fact, not all of the conditions (39), (40) are independent, since $D^{11} = 2[D^{12}, D^{13}]$ and $D^{12} = [D^{22}, D^{13}]$ (see (8)).

Let us now use the $USp(4)$ harmonics to project the defining constraint (10) of the $N = (2,0)$ tensor multiplet:

$$D_\alpha^{(k} W^{\{i)j\}} = 0 \ \times \left\{ \begin{array}{lcl} u_k^1 u_i^1 u_j^2 & \Rightarrow & D_\alpha^1 W^{12} = 0 \\ u_k^2 u_i^2 u_j^1 & \Rightarrow & D_\alpha^2 W^{12} = 0 \end{array} \right. . \tag{41}$$

Here $D_\alpha^{1,2} = D_\alpha^i u_i^{1,2}$ and $W^{12} = W^{\{ij\}} u_i^1 u_j^2$. In other words, the constraint (10) now takes the form of a Grassmann analyticity condition:

$$D_\alpha^1 W^{12} = D_\alpha^2 W^{12} = 0 \ . \tag{42}$$

In addition, the projected superfield $W^{12}$ clearly satisfies the $USp(4)$ irreducibility conditions (40). The equivalence between the two forms of the constraint follows from the obvious properties of the harmonic products $u_{[k}^1 u_{i]}^1 = u_{[k}^2 u_{i]}^2 = 0$ and $\Omega^{ij} u_i^1 u_j^2 = 0$.

Now it becomes clear that the constraints (42) can be solved in an appropriate basis in superspace:

$$D_\alpha^1 W^{12} = D_\alpha^2 W^{12} = 0 \ \Rightarrow \ W^{12} = W^{12}(x_A, \theta^1, \theta^2, u) \tag{43}$$

where

$$x_A^{\alpha\beta} = x^{\alpha\beta} - i\theta^{\alpha(i}\theta^{\beta j)}(u^1_i u^4_j + u^2_i u^3_j) \ , \quad \theta^{1\alpha} = \theta^\alpha_4 = \theta^\alpha_i u^i_4 \ , \ \theta^{2\alpha} = \theta^\alpha_3 = \theta^\alpha_i u^i_3 \ . \tag{44}$$

We see that the superfield $W^{12}$ is independent of half of the odd coordinates, $\theta^3 = -\theta_2$ and $\theta^4 = -\theta_1$ (hence the name "Grassmann analytic"). We call such superfields "short" (compared to a generic $N = (2,0)$ superfield).

Having solved the constraints (42), we should not forget that the equivalence with the initial form (10) is only achieved if the superfield $W^{12}$ satisfies the $USp(4)$ irreducibility conditions (40). This is not so trivial now, since in the basis (44) the harmonic derivatives acquire space-time derivative terms:

$$\begin{aligned}
D^{11} &= \partial^{11} + \frac{i}{4}\theta^{1\alpha}\theta^{1\beta}\partial_{\alpha\beta} \\
D^{12} &= \partial^{12} + \frac{i}{4}\theta^{1\alpha}\theta^{2\beta}\partial_{\alpha\beta} \\
D^{22} &= \partial^{22} + \frac{i}{4}\theta^{2\alpha}\theta^{2\beta}\partial_{\alpha\beta} \\
D^{13} &= \partial^{13}
\end{aligned} \tag{45}$$

where $\partial^{IJ}$ include the harmonic and $\theta$ partial derivatives. Thus, in this basis the $USp(4)$ irreducibility conditions (40),

$$D^{11}W^{12} = D^{12}W^{12} = D^{13}W^{12} = D^{22}W^{12} = 0 \tag{46}$$

not only eliminate the infinite towers of components in the harmonic expansion of $W^{12}$ but also yield the field equations on the remaining physical fields. As a result, the analytic superfield $W^{12}$ becomes "ultrashort" (i.e., shorter than a generic analytic superfield):

$$W^{12} = \phi^{12} + \frac{1}{2}(\theta^{1\alpha}\psi^2_\alpha - \theta^{2\alpha}\psi^1_\alpha) + \theta^{1\alpha}\theta^{2\beta}F_{(\alpha\beta)} + \text{d.t.} \tag{47}$$

where $\phi^{12} = \phi^{\{ij\}}(x)u^1_i u^2_j$ , $\psi^{1,2}_\alpha = \psi^i_\alpha(x)u^{1,2}_i$ and $F_{(\alpha\beta)} = F_{(\alpha\beta)}(x)$ are the massless fields. Thus we recover the component content of eq. (12).

One can treat the case *(ii)* in the same way. Projecting the constraint (16) with $u^1_k u^1_i$ we obtain

$$D^{(k}_\alpha W^{i)} = 0 \quad \Leftrightarrow \quad D^1_\alpha W^1 = 0 \ . \tag{48}$$

This constraint of Grassmann analyticity is solved in the appropriate basis in $N = (1,0)$ or $N = (2,0)$ superspace and yields

$$D^1_\alpha W^1 = 0 \quad \Rightarrow \quad \begin{cases} W^1 = W^1(\theta^1) \ , & N = (1,0) \\ W^1 = W^1(\theta^1, \theta^2, \theta^3) \ , & N = (2,0) \end{cases} \quad . \tag{49}$$

In addition, one has to impose the conditions of $USp(2n)$ irreducibility (recall (38) and (39))

$$N = (1,0): \qquad D^{11} W^1 = 0 \ ; \tag{50}$$

$$N = (2,0): \qquad D^{11} W^1 = D^{12} W^1 = D^{13} W^1 = D^{22} W^1 = 0 \ . \tag{51}$$

The resulting superfield has the following "ultrashort" expansion:

$$N = (1,0): \quad W^1 = \phi^1 + \theta^{1\alpha} \psi_\alpha + \text{d.t.} \tag{52}$$

with $\phi^1 = \phi^i(x) u^1_i$ and $\psi_\alpha = \psi_\alpha(x)$;

$$\begin{aligned} N = (2,0): \qquad W^1 = & \ \phi^1 + \theta^{1\alpha}\psi_\alpha - (\theta^{1\alpha}\psi^{23}_\alpha + \text{cycle } 123) \\ & -(\theta^{1\alpha}\theta^{2\beta}F^3_{(\alpha\beta)} + \text{cycle } 123) + 6\theta^{1\alpha}\theta^{2\beta}\theta^{3\gamma}\chi_{(\alpha\beta\gamma)} + \text{d.t.} \end{aligned} \tag{53}$$

with $\psi^{23}_\alpha = \psi^{\{ij\}}_\alpha(x) u^2_i u^3_j$, $F^3_{(\alpha\beta)} = F^i_{(\alpha\beta)}(x) u^3_i$, etc.

Finally, we turn to the case *(iii)* which is different since the constraint (21) is second-order in the spinor derivatives. After projection with $u^I_i u^I_j$ (no summation over $I$) it becomes

$$D^I_\alpha D^I_\beta W = 0 \tag{54}$$

where $I = 1,2$ in the case $N = (1,0)$ and $I = 1,2,3,4$ in the case $N = (2,0)$. This time we do not have Grassmann analyticity but just linearity in each of the $\theta^I$. As usual, the superfield $W$ also satisfies the $USp(2n)$ irreducibility conditions. Here we only give the expansion of $W$ in the case $N = (1,0)$:

$$N = (1,0): \qquad W = \phi + \frac{1}{2}(\theta^{1\alpha}\psi^2_\alpha - \theta^{2\alpha}\psi^1_\alpha) + \theta^{1\alpha}\theta^{2\beta}F_{(\alpha\beta)} + \text{d.t.} \tag{55}$$

# 5 Tensoring massless multiplets

Usually tensoring UIR's is a very non-trivial procedure. The problem is to decompose the reducible product into irreps. The interpretation of some of the massless six-dimensional multiplets as analytic ($W^{12}$ and $W^1$) or linear ($W$) superfields we gave above greatly facilitates this task. We are able to single out the principal irreducible part of the various tensor products by just imposing our usual harmonic conditions. We shall treat the cases $N=(1,0)$ and $N=(2,0)$ separately.

## 5.1 The case $N=(1,0)$

As we have shown in eq. (52), the superfield $W^1$ is ultra short in the sense that its expansion ends at the top "spin" (i.e., $SU^*(4)$ irrep) $(0,0,1)$, as compared to the top spin $(0,2,0)$ of a generic $N=(1,0)$ "long" superfield. Its square $(W^1)^2$ still satisfies the same Grassmann and harmonic conditions,

$$D^1_\alpha (W^1)^2 = D^{11}(W^1)^2 = 0 \ , \tag{56}$$

but the content is now different. It is not hard to derive from (56) that the superfield $(W^1)^2$ has the following expansion:

$$(W^1)^2 = \phi^{11} + \theta^{1\alpha}\psi^1_\alpha + \theta^{1\alpha}\theta^{1\beta}A_{[\alpha\beta]} + \text{d.t.} \tag{57}$$

Here we find a triplet of scalars $\phi^{11} = \phi^{(ij)}(x)u^1_i u^1_j$, a doublet of spinors $\psi^1_\alpha = \psi^i_\alpha(x)u^1_i$ and a vector (i.e., top "spin" $(0,1,0)$). All of these fields are off shell and the vector is conserved,

$$\partial^{\alpha\beta}A_{[\alpha\beta]} = 0 \ . \tag{58}$$

This amounts to $8+8$ *off-shell* degrees of freedom. Note that unlike $W^1$ itself, the composite superfield $W^{11} = (W^1)^2$ can be made real. In the $AdS$ interpretation this is the bulk multiplet of massless gauge fields.

All higher powers of $W^1$, $(W^1)^p$, $p \geq 3$ are short superfields depending on half of the odd variables. Their first component is a scalar in the $(p+1)$-plet UIR of $USp(2)$ and the expansion reaches the same top spin $(0,1,0)$. This time, however, there are no space-time constraints on the components. In the bulk language these states are massive short vector multiplets.

The short superfield $W$ (54) is linear in $\theta^{1,2}$, therefore its expansion (55) terminates at the top spin $(0,0,2)$. The square of $W$, $(W)^2$ satisfies a weaker constraint:

$$(D^I_\alpha)^3(W)^2 = 0 \ , \qquad I = 1,2 \tag{59}$$

which implies that it is bilinear in each $\theta^I$. Consequently, the top spin appears in the term $\theta^{1\alpha}\theta^{1\beta}\theta^{2\gamma}\theta^{2\delta}A_{[\alpha\beta][\gamma\delta]}$, so it is $(0,2,0)$ (and a $USp(2)$ singlet). In fact, this is the maximal spin one can have in a generic $N = (1, 0)$ "long" superfield, so in this sense $(W)^2$ is not "short". Note, however, that the top spin in $(W)^2$ is conserved,

$$\partial^{\alpha\beta} A_{[\alpha\beta][\gamma\delta]} = 0 \; , \tag{60}$$

whereas this is not the case for any higher power of $W$. The state in (60) is a massless bulk graviton while higher powers of $W$ correspond to the massive graviton recurrences.

Finally, we have the possibility to tensor $W$ with $W^1$. Comparing eqs. (49) and (54), we see that the product $W(W^1)^p$ satisfies the linearity constraint

$$D^1_\alpha D^1_\beta (W(W^1)^p) = 0 \tag{61}$$

as well as the usual harmonic condition

$$D^{11}(W(W^1)^p) = 0 \; . \tag{62}$$

This means that it is linear in $\theta^2$ but the dependence in $\theta^1$ is not restricted. Consequently, the top spin appears in the term $\theta^{1\alpha}\theta^{1\beta}\theta^{2\gamma}\psi^{(p-1)}_{[\alpha\beta]\gamma}$, so it is $(0, 1, 1)$ (and it also is a $p$-plet of $USp(2)$). The case $p = 1$ (i.e., the bilinear product $WW^1$) is again special, since the condition (62) implies that the top spin is conserved,

$$\partial^{\alpha\beta}\psi_{[\alpha\beta]\gamma} = 0 \; . \tag{63}$$

In the bulk language the above state corresponds to a massless "gravitino".

### 5.2 The case $N = (2, 0)$

In this case we shall restrict ourselves to the products of analytic superfields of the type $W^{12}$ and $W^1$. The superfield $W^{12}$ is ultrashort (recall (47)), its top spin being $(0, 0, 2)$. The square $(W^{12})^2$ still satisfies the analyticity constraints (43) (as well as the harmonic conditions (46)), so it only depends on $\theta^{1,2}_\alpha$ and its expansion goes up to the top spin $(0, 2, 0)$ found in the term

$$(W^{12})^2 = \phi^{1122} + \ldots + \theta^{1\alpha}\theta^{1\beta}\theta^{2\gamma}\theta^{2\delta} A_{[\alpha\beta][\gamma\delta]} + \text{d.t.} \tag{64}$$

This is the maximal spin in an analytic $N = (2, 0)$ superfield depending on two $\theta$'s only. However, this top spin satisfies a conservation condition, as is always the case with bilinear (current-like) products. As to the higher powers $(W^{12})^p$, $p \geq 3$, the top spin there still is $(0, 2, 0)$ but it is unconstrained.

To summarize, tensoring $W^{12}$'s we obtain the following series of UIR's of $OSp(8^*/4)$:

$$(W^{12})^p = \phi^{\overbrace{1\ldots1}^{p}\overbrace{2\ldots2}^{p}} + \ldots + \theta^{1\alpha}\theta^{1\beta}\theta^{2\gamma}\theta^{2\delta}A_{[\alpha\beta][\gamma\delta]}^{\overbrace{1\ldots1}^{p-2}\overbrace{2\ldots2}^{p-2}} + \text{d.t.} \tag{65}$$

having the first component in the $(p,0)$ and the top spin $(0,2,0)$ in the $(p-2,0)$ UIR's of $USp(4)$. [3] These superfields do not depend on one half of the odd variables and thus correspond to 1/2 BPS states in the $AdS$ language.

Besides $W^{12}$, we have another analytic superfield $W^1$ which is "intermediate short" since it depends on $\theta^{1,2,3}_\alpha$ (recall (49)) (or, to put it differently, it does not depend on 1/4 of the odd variables). It is clear that by multiplying the "short" $W^{12}$'s by "intermediate short" $W^1$'s we obtain "intermediate short" composite objects. However, there exists an alternative way of constructing such objects [6, 7]. The choice of harmonic projections in eq. (41) is not unique. Exchanging, e.g., 2 with 3 we could obtain another analytic superfield $W^{13}(\theta^1,\theta^3)$ which provides an equivalent description of the on-shell tensor multiplet. Now, consider the product $W^{12}(\theta^1,\theta^2)W^{13}(\theta^1,\theta^3)$. It depends on $\theta^{1,2,3}_\alpha$, just like $W^1$. In addition, we can impose on it the same harmonic conditions (46) as on $W^{12}$ alone. The result is an "intermediate short" superfield with top spin $(0,3,0)$:

$$W^{12}W^{13} = \phi^{11} + \ldots + \theta^{1\alpha}\theta^{1\beta}\theta^{2\gamma}\theta^{2\delta}\theta^{3\kappa}\theta^{3\sigma}A_{[\alpha\beta][\gamma\delta][\kappa\sigma]} + \text{d.t.} \tag{66}$$

It should be noted that the composite object $(W^1)^2$ has exactly the same content. Indeed, it depends on the same $\theta$'s and has the same first component (a scalar in the $(0,2)$ of $USp(4)$), so

$$W^{12}W^{13} \simeq (W^1)^2\ . \tag{67}$$

Generalizing the above tensor product we can construct a two-parameter series:

$$(W^{12})^{p+q}(W^{13})^q = \phi^{\overbrace{1\ldots1}^{p+2q}\overbrace{2\ldots2}^{p}} + \ldots + \theta^{1\alpha}\theta^{1\beta}\theta^{2\gamma}\theta^{2\delta}\theta^{3\kappa}\theta^{3\sigma}A_{[\alpha\beta][\gamma\delta][\kappa\sigma]}^{\overbrace{1\ldots1}^{p+2q-2}\overbrace{2\ldots2}^{p}} + \text{d.t.} \tag{68}$$

[3] In the harmonic formalism the representation of $USp(4)$ to which belongs each component is identified by just counting the 1's (2's) which gives the number of cells in the top (bottom) row of the corresponding Young tableau.

having the first component in the $(p, 2q)$ and the top spin $(0, 3, 0)$ in the $(p, 2q-2)$ UIR's of $USp(4)$. Note that, as usual, the top spin in the bilinear combinations $W^{12}W^{13}$ is conserved. These superfields do not depend on one quarter of the odd variables and thus correspond to 1/4 BPS states in the $AdS$ language.

We remark that the three bilinear cases $(W^{12})^2$, $W^{12}W^1$ and $W^{12}W^{13} \simeq (W^1)^2$ can be identified with the massless (in the $AdS_7$ sense) supermultiplets in Tables 4, 5 and 6, respectively from Ref. [13].

In conclusion we can say that the analytic series $(W^{12})^p$ correspond to 1/2 BPS states in the sense that they preserve 1/2 of the original supersymmetry. These are the operators which classify the K-K states of M-theory on $AdS_7 \times S^4$ [5]. In superfield language these are the analytic superfields in harmonic superspace which do not depend on two of the four odd variables in $N = (2, 0)$ superspace. The other short representations $(W^{12})^{p+q}(W^{13})^q$ with $q > 0$ correspond to 1/4 BPS states. The one with $p = 0$, $q = 2$ is contained in the two-graviton state.

The above results should be relevant for the analysis of $n$-point functions in $D = 6$ and the correspondence with $n$-graviton amplitudes in M-theory on $AdS_7 \times S^4$ [27].

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