CERN-TH/2000-126
hep-th/0004144

Extra Force in Brane Worlds

Donam Youm[1]

Theory Division, CERN, CH-1211, Geneva 23, Switzerland

Abstract

By carefully analyzing the geodesic motion of a test particle in the bulk of brane worlds, we identify an extra force which is recognized in spacetime of one lower dimensions as a non-gravitational force acting on the particle. Such extra force acts on the particle in such a way that the conventional particle mechanics in one lower dimensions is violated, thereby hinting at the higher-dimensional origin of embedded spacetime in the brane world scenario. We obtain the explicit equations describing the motion of the bulk test particle as observed in one lower dimensions for general gravitating configurations in brane worlds and identify the extra non-gravitational force acting on the particle measured in one lower dimensions.

CERN-TH/2000-126
April, 2000

[1]E-mail: Donam.Youm@cern.ch

1 Introduction

There has been renewed interest in compactification through the non-factorizable Kaluza-Klein (KK) metric Ansatz with the warp factor, after Randall and Sundrum (RS) showed [1, 2, 3] that such unconventional compactification of the extra spatial dimensions provides with a simple solution to the hierarchy problem of particle physics. A novel and surprising feature of the RS scenario is that even if the extra spatial dimension is infinite in size, the Newton's $1/r^2$ law of four-dimensional gravity is recovered with negligible correction from the massive KK modes of graviton. The previous works on gravitational aspects of the RS scenario have attempted to reproduce physics of four-dimensional gravity up to small corrections beyond the current experimental precision in an effort to provide with an evidence that the RS scenario may be a true description of our nature.

Especially, in Refs. [4, 5, 6] it is shown that the geodesic motion of a massless test particle in the bulk of a gravitating configuration in brane world reproduces the geodesic motion of a massive test particle in the corresponding gravitating configuration in one lower dimensions. Such result appears to be an evidence that the RS scenario reproduces physics of our four-dimensional spacetime. However, through the careful analysis of the geodesic motion of a test particle in the bulk spacetime of brane worlds, we find in general that laws of physics governing the motion of a particle in four-dimensional spacetime is violated. Namely, we find that the equation describing the trajectory of a particle as observed in one lower dimensions of the brane world scenario has an extra force term which is parallel to the velocity of the particle. According to the so-far known four-dimensional physics, only the component of the extra non-gravitational four-force F^μ which is perpendicular to the particle's four-velocity $\frac{dx^\mu}{d\tau}$ can influence the particle's motion. Due to such unusual property which cannot be explained by the physics of four-dimensional spacetime, such extra force was dubbed in the previous literature [7, 8, 9, 10, 11, 12, 13, 14] as the fifth force. [Such extra abnormal force is observed also in Refs. [15, 16, 17] by analyzing the geodesic motion of a test particle in the five-dimensional Kaluza-Klein theory.] The so-called fifth force generically exists in the KK theories (with constant moduli scalar fields of the extra space) when the spacetime metric depends on the extra spatial coordinates and the velocity of the particle has nonzero components in both the extra spatial direction and the direction of our three dimensional space. Therefore, it is inevitable that if our four-dimensional world is described by the RS scenario then we should observe the violation of four-dimensional law of particle mechanics, since the RS scenario allows dependence of the spacetime metric on the extra spatial coordinate and physical process in the RS scenario is generally higher-dimensional in nature.

The paper is organized as follows. In section 2, we survey relevant aspects of domain wall solutions in and geodesic motion of a test particle in the bulk of the RS scenario. In this section, we also discuss well-known facts of particle mechanics in curved spacetime for the purpose of understanding the physical implication of the extra force observed in one lower dimensions. Although some aspects of the fifth force were already studied in the previous literature, we feel that its relation to the four-dimensional particle mechanics has not been clearly presented. We hope that the present paper will clarify some of confusing issues in the fifth force. We study the bulk geodesic motion of a test particle moving in general gravitating configurations in brane worlds as observed in one lower dimensions for the case corresponding to the KK zero mode bulk graviton in section 3 and for the case of general bulk graviton including the massive KK modes in section 4. In these sections, we also identify the extra force on the particle which is observed in one lower dimensions as non-gravitational. Conclusions are given in section 5.

2 Preliminaries and General Setup

In this section, we prepare for the main topic of this paper by surveying aspects of domain wall solutions and dynamics of a particle in brane worlds and in general relativity.

Generally, the spacetime metric [2] for a D-dimensional domain wall can be put into the following form:

$$G_{MN}dx^M dx^N = \mathcal{W}(y)\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{1}$$

where $M, N = 0, 1, ..., D-1$, $\mu, \nu = 0, 1, ..., D-2$ and $\mathcal{W}(y)$ is the warp factor. In particular, for the domain wall solution to the field equations of the following bulk action:

$$S = \frac{1}{2\kappa_D^2}\int d^D x\sqrt{-G}\left[\mathcal{R}_G - \frac{4}{D-2}\partial_M\phi\partial^M\phi + e^{-2a\phi}\Lambda\right], \tag{2}$$

the warp factor is given by [1, 18, 19, 20]

$$\mathcal{W}(y) = \left(1 - \frac{(D-2)a^2}{2}\sqrt{\frac{D-2}{4(D-1)-a^2(D-2)^2}\Lambda}|y|\right)^{\frac{8}{(D-2)^2a^2}}, \tag{3}$$

for $a \neq 0$, and

$$\mathcal{W}(y) = \exp\left(-2\sqrt{\frac{\Lambda}{(D-1)(D-2)}}|y|\right), \tag{4}$$

for $a = 0$. Here, we have imposed the invariance under the $\mathbf{Z}_2$ transformation $y \to -y$ and chosen the warp to decrease so that the bulk graviton can be localized.

[2] In this paper, we use the mostly positive convention $(- + \cdots +)$ for the metric signature.

It is the purpose of this paper to study the dynamics of a test particle in the bulk spacetime with the following metric:

$$G_{MN}dx^M dx^N = \mathcal{W} g_{\mu\nu} dx^\mu dx^\nu + dy^2. \tag{5}$$

This metric generically describes any gravitational configuration in brane worlds. The geodesic motion of a test particle, i.e., the motion of a particle which is acted on by the gravitational force only, is described by the following geodesic equations:

$$\frac{d^2 x^R}{d\lambda^2} + \hat{\Gamma}^R_{MN}\frac{dx^M}{d\lambda}\frac{dx^N}{d\lambda} = 0, \tag{6}$$

where $\hat{\Gamma}^R_{MN}$ is the Christoffel symbol (of the second kind) for the metric G_{MN} and λ is an affine parameter for the geodesic path $x^M(\lambda)$. In addition, the metric compatibility along the geodesic path requires that

$$-\epsilon_D = G_{MN}\frac{dx^M}{d\lambda}\frac{dx^N}{d\lambda} = \mathcal{W} g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + \left(\frac{dy}{d\lambda}\right)^2, \tag{7}$$

where $\epsilon_D = 1, 0$ respectively for a massive test particle (i.e., a timelike geodesic) and a massless test particle (i.e., a null geodesic). For a timelike geodesic, ϵ_D actually can take any positive values, but one can always apply the affine transformation $\lambda \to a\lambda + b$ ($a, b \in \mathbf{R}$), which leaves the geodesic equations (6) invariant, to bring $\epsilon_D = 1$. In this paper, we shall not consider the spacelike geodesics, i.e. the $\epsilon_D = -1$ case.

In this paper, we shall re-express the geodesic equations (6) for the bulk geodesic motion of a test particle in terms of quantities of the hypersurface spacetime of one lower dimensions, for the purpose of learning how the bulk geodesic motion of the test particle is observed in one lower dimensions. However, in such study we face some ambiguity and conceptual difficulties. In the previous related works [4, 5, 6], it is assumed that the canonical metric for our four-dimensional world, embedded in a five-dimensional domain wall, is given by $g_{\mu\nu}$. Accordingly, the affine parameter $\tilde{\lambda}$ for the motion observed in one lower dimensions is defined by the relation

$$-\epsilon_{D-1} = g_{\mu\nu}\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}}, \tag{8}$$

where $\epsilon_{D-1} = 1, 0$ respectively for a timelike and lightlike motion observed in one lower dimensions. On the other hand, as the bulk test particle follows its geodesic path, it generally passes through one hypersurface to another. We note that the induced metric on the hypersurface at a specific value of y is given by $\tilde{g}_{\mu\nu} \equiv \mathcal{W}(y) g_{\mu\nu}$. Therefore, it appears that the natural choice for the affine parameter $\tilde{\lambda}$ (or the standard clock on the hypersurface that follows the particle in its motion) for the motion observed on the (comoving) hypersurface is the one satisfying the following

$$-\epsilon_{D-1} = \mathcal{W}(y) g_{\mu\nu}\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}}. \tag{9}$$

However, one might argue that $\tilde{\lambda}$ satisfying Eq. (8) actually corresponds to the affine parameter for the motion as observed on the hypersurface $y = y_0 =$ constant, say the TeV brane of our world, at *fixed* distance from the Planck brane at $y = 0$. Namely, the affine parameter at $y = y_0$ is defined by

$$-\epsilon_{D-1} = \mathcal{W}(y_0) g_{\mu\nu}(x^\rho, y_0) \frac{dx^\mu}{d\tilde{\lambda}} \frac{dx^\nu}{d\tilde{\lambda}}, \tag{10}$$

and by applying the affine transformation $\tilde{\lambda} \to \mathcal{W}(y_0)^{-1/2}\tilde{\lambda}$, one can bring this to the form (8) in the case when $g_{\mu\nu}$ is independent of the extra spatial coordinate y. The objection to this viewpoint is that it does not seem obvious whether we, as beings adapted to sense only the four- or lower-dimensional phenomena, will be able to look into the extra spatial direction to observe objects in different four-dimensional hypersurface. To give an extreme example, it is obviously contradictory that an observer can see an object located at the same three-dimensional spatial coordinates but at different extra spatial coordinate. Any object on different four-dimensional hypersurface cannot be observed; it simply exists in different four-dimensional universe which cannot be observed. So, it seems nonsensical to argue for example that the motion of planets in our solar system (which is governed by the geodesic equations with the Schwarzshield metric) has non-trivial motion along the extra spatial direction, unless we assume that our observed four-dimensional universe as a whole moves along the extra spatial direction. On the other hand, this is also contradictory to the current RS scenario, because the current RS scenario assumes that our universe, i.e., the TeV brane, is at *fixed* distance from the Planck brane. So, to avoid this inconsistency, the current RS scenario has to either explain why objects in our universe cannot move along the extra spatial direction (despite, for example, the repulsive force by the Planck brane) or assume that the TeV brane moves freely along the extra spatial direction. A possible explanation for the former possibility would be that the bulk spacetime exists only between the Plank and the TeV branes and therefore objects on the TeV brane cannot move in the direction away from the Planck brane and the repulsive force by the Planck brane keeps them from moving towards the Planck brane. If this would be the case, then we should just consider the dynamics of a free particle, massive or massless, whose motion is confined within the *fixed* hypersurface $y =$ constant of the TeV brane, unlike the previous studies [4, 5, 6]. If we choose the latter possibility, then the mass scale of Standard Model physics has to change with time, because of the warp factor $\mathcal{W}(y)$ variation with time due to the change in y with time. Because of this conceptual ambiguity, in the following sections we shall consider two cases associated with the two choices of affine parameters defined in Eqs. (8) and (9).

In obtaining the equations for the particle motion observed in one lower dimensions, we assume that the affine parameter $\tilde{\lambda}$ for the spacetime in one lower dimensions is a smooth function of the affine parameter λ for the bulk geodesic motion: $\tilde{\lambda} = f(\lambda)$.

First, for $\tilde{\lambda}$ defined in Eq. (8), which we refer to as 'case 1', the relation (7) for the bulk geodesic motion is rewritten in terms of the new parameter $\tilde{\lambda}$ as

$$g_{\mu\nu}\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}} = -\mathcal{W}^{-1}\left[\epsilon_D\left(\frac{d\lambda}{d\tilde{\lambda}}\right)^2 + \left(\frac{dy}{d\tilde{\lambda}}\right)^2\right]. \tag{11}$$

Second, for $\tilde{\lambda}$ defined in Eq. (9), which we refer to as 'case 2', Eq. (7) is rewritten in terms of $\tilde{\lambda}$ as

$$\mathcal{W}g_{\mu\nu}\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}} = -\left[\epsilon_D\left(\frac{d\lambda}{d\tilde{\lambda}}\right)^2 + \left(\frac{dy}{d\tilde{\lambda}}\right)^2\right]. \tag{12}$$

The parameter $\tilde{\lambda}$ is an affine parameter for the motion observed in one lower dimensions, if Eqs. (11) and (12) respectively take the forms (8) and (9), i.e., the RHS's are either 0 or -1.

We now discuss the condition for $\tilde{\lambda}$ being an affine parameter. First, for a massless test particle ($\epsilon_D = 0$), the RHS's of Eqs. (11) and (12) can be 0 or -1, depending on the motion of a test particle along the y-direction. When the test particle is confined to move along the longitudinal directions of the domain wall [3] (i.e., $\frac{dy}{d\lambda} = 0$ and therefore $\frac{dy}{d\tilde{\lambda}} = \frac{d\lambda}{d\tilde{\lambda}}\frac{dy}{d\lambda} = 0$), the RHS's of Eqs. (11) and (12) are zero, corresponding to the lightlike motion as observed in one lower dimensions. In this case, one can choose $\tilde{\lambda} = \lambda$ as an affine parameter for the motion observed in one lower dimensions, as can be seen from Eqs. (11) and (12). When the y-component of the velocity of the massless test particle is nonzero (i.e., $\frac{dy}{d\lambda} \neq 0$), its motion is observed in one lower dimensions as timelike if the following is satisfied:

$$\left(\frac{dy}{d\tilde{\lambda}}\right)^2 = \mathcal{W}, \tag{13}$$

for the case 1, and

$$\left(\frac{dy}{d\tilde{\lambda}}\right)^2 = 1, \tag{14}$$

for the case 2, for which cases the RHS's of Eqs. (11) and (12) are -1. Second, for a massive test particle ($\epsilon_D = 1$), the parameter $\tilde{\lambda}$ is an affine parameter for the timelike motion as observed in one lower dimensions, if $\tilde{\lambda}$ is related to the bulk affine parameter λ as

$$\left(\frac{d\lambda}{d\tilde{\lambda}}\right)^2 = \mathcal{W} - \left(\frac{dy}{d\tilde{\lambda}}\right)^2, \tag{15}$$

for the case 1, and

$$\left(\frac{d\lambda}{d\tilde{\lambda}}\right)^2 = 1 - \left(\frac{dy}{d\tilde{\lambda}}\right)^2, \tag{16}$$

[3]From Eq. (28) with $\epsilon_D = 0$, one can see that such bulk geodesic motion is possible for a massless test particle.

for the case 2.

In the following sections, we will see that from the perspective of an observer in one lower dimensions a bulk test particle appears to be under the influence of an abnormal non-gravitational force. So, it would be useful to discuss some aspects of dynamics of particles in general relativity to better understand physical implication of such extra force. Although we assume the spacetime to be four-dimensional in the discussion in the following paragraphs, our discussion holds for arbitrary spacetime dimensions without modification of equations.

In terms of the relativistic four-vector notation, the following classical Newton's second law of mechanics plus the work-energy relation (the law of conservation of energy):

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \qquad \frac{dT}{dt} = \mathbf{F}\cdot\mathbf{v}, \tag{17}$$

where $\mathbf{F}$ is the force on the particle, $\mathbf{p} = m_0\mathbf{v} = m_0\frac{d\mathbf{x}}{dt}$ is the momentum of the particle with the inertial mass m_0 and T is the kinetic energy of the particle, is written as

$$\frac{dp_\mu}{d\tau} = F_\mu, \tag{18}$$

where $p^\mu = m_0\frac{dx^\mu}{d\tau}$ is the contravariant components of the four-momentum of a particle with the rest mass m_0, τ is the proper time defined from $d\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$, $F_\mu = (-\mathbf{F}\cdot\frac{d\mathbf{x}}{d\tau}, \mathbf{F}\frac{dt}{d\tau})$, and $(x^\mu) = (t, \mathbf{x})$. In curved spacetime with metric $g_{\mu\nu}$, Eq. (18) is modified to:

$$\frac{Dp^\mu}{d\tau} \equiv \frac{dp^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}p^\sigma = F^\mu, \tag{19}$$

in the contravariant notation, or

$$\frac{Dp_\mu}{d\tau} \equiv \frac{dp_\mu}{d\tau} - \frac{1}{2}\frac{\partial g_{\rho\sigma}}{\partial x^\mu}\frac{dx^\rho}{d\tau}p^\sigma = F_\mu, \tag{20}$$

in the covariant notation, where $\Gamma^\mu_{\rho\sigma}$ is the Christoffel symbol of the second kind for the metric $g_{\mu\nu}$. Here, F_μ and p^μ are defined in the same way as in the flat spacetime case except that the proper time τ is now defined from $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$.

First, we show that a purely mechanical non-gravitational four-force F^μ acts on a particle perpendicularly to its four-velocity $\frac{dx^\mu}{d\tau}$. By taking the covariant derivative of $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1$ along the particle trajectory $x^\mu(\tau)$, one can see that $g_{\mu\nu}\frac{D}{d\tau}\left(\frac{dx^\mu}{d\tau}\right)\frac{dx^\nu}{d\tau} = 0$. This implies that $g_{\mu\nu}F^\mu\frac{dx^\nu}{d\tau} = g_{\mu\nu}\frac{Dp^\mu}{d\tau}\frac{dx^\nu}{d\tau} = m_0 g_{\mu\nu}\frac{D}{d\tau}\left(\frac{dx^\mu}{d\tau}\right)\frac{dx^\nu}{d\tau} = 0$.

Next, we consider the possibility that the four-force F^μ has nonzero component parallel to the four-velocity $\frac{dx^\mu}{d\tau}$. In deriving the relation $g_{\mu\nu}F^\mu\frac{dx^\nu}{d\tau} = 0$ in the previous paragraph, we assumed that the proper mass m_0 of the particle is constant in time. Had we considered the possibility that m_0 changes with time, we would instead have obtained

$$g_{\mu\nu}F^{\mu}\frac{dx^{\nu}}{d\tau} = g_{\mu\nu}\frac{Dp^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = m_0 g_{\mu\nu}\frac{D}{d\tau}\left(\frac{dx^{\mu}}{d\tau}\right)\frac{dx^{\nu}}{d\tau} + \frac{dm_0}{d\tau}g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = -\frac{dm_0}{d\tau}. \quad (21)$$

Therefore, the existence of parallel component of the force F^{μ} implies non-conservation of the proper mass m_0 of a particle. Generally, the change in the proper mass of a particle occurs when there exist some non-mechanical external forces which cause such change. To take into account of the additional non-mechanical forces, one has to modify F_0 in the following way:

$$F_0 = -\mathbf{F}\cdot\frac{d\mathbf{x}}{d\tau} - Q\frac{dt}{d\tau}, \quad (22)$$

where the first term on the RHS is the mechanical work done by the mechanical force $\mathbf{F}$ per unit time and the second term is the heat or non-mechanical energy developed per unit time. The extra term is added to take into account of the contribution from the non-mechanical energy so that the energy can be conserved in the system under consideration. Meanwhile, the remaining components of F_{μ} take the same form as the above, i.e., (the curved space analog of) the Newton's second law of mechanics $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ continues to hold. So, from Eqs. (21) and (22), we obtain

$$\frac{dm_0}{d\tau} = Q\left(\frac{dt}{d\tau}\right)^2, \quad (23)$$

namely, the proper mass is converted into the non-mechanical energy and vice versa.

Finally, we discuss the motion of a particle under the influence of an extra non-gravitational force F^{μ} in curved spacetime. First, when F^{μ} acts on the particle perpendicularly to its four-velocity $\frac{dx^{\mu}}{d\tau}$, i.e., the proper mass m_0 of the particle is constant in time, from Eqs. (19) and (20) we obtain the following equations for the particle trajectory $x^{\mu}(\tau)$:

$$\begin{aligned}\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^{\rho}}{d\tau}\frac{dx^{\sigma}}{d\tau} &= \frac{F^{\mu}}{m_0},\\ \frac{d^2x_{\mu}}{d\tau^2} - \frac{1}{2}\frac{\partial g_{\rho\sigma}}{\partial x^{\mu}}\frac{dx^{\rho}}{d\tau}\frac{dx^{\sigma}}{d\tau} &= \frac{F_{\mu}}{m_0}.\end{aligned} \quad (24)$$

As expected, when the particle is free, i.e., when only gravitational force is acted on the particle ($F^{\mu} = 0$), the particle will execute geodesic motion. Second, when the force F^{μ} has nonzero parallel component, i.e., when m_0 is not conserved, the equation for the particle trajectory $x^{\mu}(\tau)$ takes the following form:

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^{\rho}}{d\tau}\frac{dx^{\sigma}}{d\tau} = \frac{F^{\mu}}{m_0} + g_{\rho\sigma}\frac{F^{\rho}}{m_0}\frac{dx^{\sigma}}{d\tau}\frac{dx^{\mu}}{d\tau}, \quad (25)$$

which can be obtained from Eqs. (19) and (21). We note that the RHS of Eq. (25) is perpendicular to the four-velocity $\frac{dx^\mu}{d\tau}$, implying that only the perpendicular component of F^μ influences the motion of the particle. In other words, if F^μ is parallel to $\frac{dx^\mu}{d\tau}$, then the particle will just follow the geodesic path (since the RHS of Eq. (25) vanishes for this case) while its mass changing with time according to the relation (23). This result also shows that according to the particle mechanics of four-dimensional general relativity the extra non-gravitational force term in the equation for the particle trajectory cannot have non-zero component parallel to the four-velocity of the particle.

3 Dynamics in the Kaluza-Klein Zero Mode Spacetime

In this section, we consider the case when $g_{\mu\nu}$ in the bulk metric (5) does not depend on the extra spatial coordinate y. In this case, the bulk test particle is regarded as being under the influence of the Kaluza-Klein zero mode of graviton, only.

The geodesic equations (6), with the bulk metric given by Eq. (5) with $g_{\mu\nu} = g_{\mu\nu}(x^\rho)$, take the following forms:

$$\frac{d^2x^\rho}{d\lambda^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + \frac{\mathcal{W}'}{\mathcal{W}}\frac{dx^\rho}{d\lambda}\frac{dy}{d\lambda} = 0, \tag{26}$$

$$\frac{d^2y}{d\lambda^2} - \frac{1}{2}\mathcal{W}' g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{27}$$

where $\Gamma^\rho_{\mu\nu}$ is the Christoffel symbol for the metric $g_{\mu\nu}$ and $\mathcal{W}' \equiv d\mathcal{W}/dy$. By using the relation (7), one can put the y-component geodesic equation (27) into the following form:

$$\frac{d^2y}{d\lambda^2} + \frac{1}{2}\frac{\mathcal{W}'}{\mathcal{W}}\left[\epsilon_D + \left(\frac{dy}{d\lambda}\right)^2\right] = 0. \tag{28}$$

3.1 Case 1

In this subsection, we study the motion of the bulk test particle as observed in one lower dimensions with the spacetime metric $g_{\mu\nu} = g_{\mu\nu}(x^\rho)$.

First, we consider the geodesic motion of a massless particle in the bulk spacetime, i.e. the $\epsilon_D = 0$ case. When $\frac{dy}{d\lambda} = 0$, by choosing $\tilde{\lambda} = \lambda$ as an affine parameter, one can put the x^ρ-component bulk geodesic equation (26) into the following form:

$$\frac{d^2x^\rho}{d\tilde{\lambda}^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}} = 0. \tag{29}$$

So, the test particle's motion is observed in one lower dimensions with the metric $g_{\mu\nu}$ as the lightlike geodesic motion. Next, when the y-component of the velocity of the test particle is nonzero (i.e., $\frac{dy}{d\lambda} \neq 0$), from Eqs. (13) and (28) with $\epsilon_D = 0$, one can see that the parameters λ and $\tilde{\lambda}$ are related as

$$\left(\frac{d\tilde{\lambda}}{d\lambda}\right)^{-1} \frac{d}{d\tilde{\lambda}}\left(\frac{d\tilde{\lambda}}{d\lambda}\right) = -\mathcal{W}^{-\frac{1}{2}}\mathcal{W}', \tag{30}$$

which can be solved by

$$\frac{d\tilde{\lambda}}{d\lambda} = \mathcal{W}^{-1}. \tag{31}$$

By using Eqs. (13) and (30), one can express the x^ρ-component bulk geodesic equation (26) in terms of the $(D-1)$-dimensional affine parameter $\tilde{\lambda}$. The resulting equation also takes the form (29). To summarize, a massless test particle moving in the bulk spacetime with the metric (5) is observed in one lower dimensions with the metric $g_{\mu\nu}(x^\rho)$ to be (i) a *free massless* particle if the motion of the test particle is confined along the longitudinal directions of the domain wall and (ii) a *free massive* particle if the y-component of its velocity is nonzero. This is just a generalization of the result in Ref. [4] to the case of an arbitrary warp factor $\mathcal{W}$ and an arbitrary gravitating configuration with the metric $g_{\mu\nu}(x^\rho)$ within the brane world. (See also Ref. [6] for the generalization to the case of multi-codimensional brane world.)

Second, we consider the geodesic motion of a massive bulk test particle, i.e., the $\epsilon_D = 1$ case. By using the relation (15), one can re-express the y-component geodesic equation (28) (with $\epsilon_D = 1$) in terms of the new parameter $\tilde{\lambda}$ as follows:

$$\frac{d^2y}{d\tilde{\lambda}^2} - \frac{\mathcal{W}'}{\mathcal{W}}\left(\frac{dy}{d\tilde{\lambda}}\right)^2 + \frac{1}{2}\mathcal{W}' = 0. \tag{32}$$

Making use of Eqs. (15) and (32), one can again put the x^ρ-component bulk geodesic equation (26) into the form (29). So, we see that a massive test particle moving in the bulk spacetime with the metric (5) is observed in one lower dimensions with the metric $g_{\mu\nu}(x^\rho)$ to be a *free* massive particle. This result extends the previous studies [4, 5, 6] on the geodesic motion in brane worlds to include the case of a massive test particle. [Note, even if some aspects of the geodesic motion of a massive test particle in brane worlds have been previously studied, it has never been shown that the geodesic motion of a *massive* test particle in the bulk spacetime is observed in one lower dimensions as the motion of a *free massive* particle.]

Just by looking at Eq. (29), it appears from the lower-dimensional perspective that the particle is under the influence of the gravitational field $g_{\mu\nu}(x^\rho)$, only. However, this is not the case, as we explain in the following. A test particle of mass m_0 in

D-dimensional bulk spacetime with the metric (5) appears in $(D-1)$-dimensional embedded spacetime with the metric $g_{\mu\nu}$ to have mass given by [15, 21]:

$$\tilde{m}_0 = m_0 \frac{d\tilde{\lambda}}{d\lambda} = m_0 \left[\mathcal{W} - \left(\frac{dy}{d\tilde{\lambda}} \right)^2 \right]^{-\frac{1}{2}}, \tag{33}$$

where we used Eq. (15). So, as the test particle executes its geodesic motion with the nonzero y-component velocity $\frac{dy}{d\tilde{\lambda}}$, its mass appears to change with the following rate from the $(D-1)$-dimensional perspective:

$$\frac{d\tilde{m}_0}{d\tilde{\lambda}} = -\tilde{m}_0 \frac{\mathcal{W}'}{\mathcal{W}} \frac{dy}{d\tilde{\lambda}}. \tag{34}$$

From Eqs. (29) and (34), we obtain the following equation describing the conservation of energy and the Newton's second law of mechanics in $(D-1)$-dimensions:

$$\frac{Dp^\mu}{d\tilde{\lambda}} \equiv \frac{dp^\mu}{d\tilde{\lambda}} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tilde{\lambda}} p^\sigma = -\tilde{m}_0 \frac{\mathcal{W}'}{\mathcal{W}} \frac{dy}{d\tilde{\lambda}} \frac{dx^\mu}{d\tilde{\lambda}}, \tag{35}$$

where $p^\mu = \tilde{m}_0 \frac{dx^\mu}{d\tilde{\lambda}}$ is the $(D-1)$-momentum of the particle. This implies that from the $(D-1)$-dimensional perspective the particle is under the influence of the extra non-gravitational velocity dependent force $F^\mu = -\tilde{m}_0 \frac{\mathcal{W}'}{\mathcal{W}} \frac{dy}{d\tilde{\lambda}} \frac{dx^\mu}{d\tilde{\lambda}}$. This extra force does not influence the motion of the particle, because it acts parallelly to the particle's $(D-1)$-velocity $\frac{dx^\mu}{d\tilde{\lambda}}$ (Cf. Eq. (25)), but is responsible for the change of the inertial mass $\tilde{m}_0$ of the particle (from the $(D-1)$-dimensional perspective). The intuitive reason is that the effect of the force $\mathbf{F}$ on the particle velocity $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ is canceled by the effect of the inertial mass change with time on $\mathbf{v}$. Although the extra force is gravitational in nature from the D-dimensional perspective, a $(D-1)$-dimensional observer will sense that some non-mechanical non-gravitational force causes the inertial mass $\tilde{m}_0$ of the particle to be converted into the heat energy and vice versa. Such heat energy generated per unit time is

$$Q \frac{dt}{d\tilde{\lambda}} = -\tilde{m}_0 \frac{\mathcal{W}'}{\mathcal{W}} \frac{dy}{dt}, \tag{36}$$

according to Eq. (23).

3.2 Case 2

In this subsection, we study the motion of a bulk test particle as observed on the (comoving) hypersurface with the spacetime metric $\tilde{g}_{\mu\nu} = \mathcal{W}(y) g_{\mu\nu}(x^\rho)$.

First, we consider the bulk geodesic motion of a massless particle ($\epsilon_D = 0$). When the motion of the test particle is confined along the longitudinal directions of the domain wall (i.e., $\frac{dy}{d\lambda} = 0$), one can choose $\tilde{\lambda} = \lambda$ as an affine parameter on the hypersurface $y =$ constant to express the x^ρ-component bulk geodesic equations (26) in the following form:

$$\frac{d^2 x^\rho}{d\tilde{\lambda}^2} + \tilde{\Gamma}^\rho_{\mu\nu} \frac{dx^\mu}{d\tilde{\lambda}} \frac{dx^\nu}{d\tilde{\lambda}} = 0, \tag{37}$$

where $\tilde{\Gamma}^\rho_{\mu\nu}$ is the Christoffel symbol for the metric $\tilde{g}_{\mu\nu} = \mathcal{W} g_{\mu\nu}$ and we used the fact that $\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}$. So, its motion observed on the hypersurface $y =$ constant is that of a massless free particle. Nontrivial result arises when the massless test particle has nonzero y-component for its velocity. From Eqs. (14) and (28) with $\epsilon_D = 0$, one can see that the parameters λ and $\tilde{\lambda}$ are related as

$$\left(\frac{d\tilde{\lambda}}{d\lambda}\right)^{-1} \frac{d}{d\tilde{\lambda}} \left(\frac{d\tilde{\lambda}}{d\lambda}\right) = -\frac{1}{2} \frac{\mathcal{W}'}{\mathcal{W}}, \tag{38}$$

which can be solved by

$$\frac{d\tilde{\lambda}}{d\lambda} = \mathcal{W}^{-\frac{1}{2}}. \tag{39}$$

So, the bulk geodesic equation (26) for the x^ρ-component motion is rewritten in terms of the new parameter $\tilde{\lambda}$ as

$$\frac{d^2 x^\rho}{d\tilde{\lambda}^2} + \tilde{\Gamma}^\rho_{\mu\nu} \frac{dx^\mu}{d\tilde{\lambda}} \frac{dx^\nu}{d\tilde{\lambda}} = -\frac{1}{2} \frac{\mathcal{W}'}{\mathcal{W}} \frac{dx^\rho}{d\tilde{\lambda}}. \tag{40}$$

The bulk geodesic motion of a free massless particle with nonzero y-component for its velocity is therefore observed on the hypersurface $y = y(\tilde{\lambda})$ with the metric $\tilde{g}_{\mu\nu}$ as the motion of a massive particle which is under the influence of the extra non-gravitational force as well as the gravitational field $\tilde{g}_{\mu\nu}$.

Second, we consider the massive bulk test particle ($\epsilon_D = 1$). By using Eq. (16), one can re-express the y-component geodesic equation (28) (with $\epsilon_D = 1$) in terms of the new parameter $\tilde{\lambda}$ as

$$\frac{d^2 y}{d\tilde{\lambda}^2} + \frac{1}{2} \frac{\mathcal{W}'}{\mathcal{W}} \left[1 - \left(\frac{dy}{d\tilde{\lambda}}\right)^2\right] = 0. \tag{41}$$

The x^ρ-component geodesic equations (26) can be put into the following simplified form in terms of the new parameter $\tilde{\lambda}$ by applying Eqs. (16) and (41):

$$\frac{d^2 x^\rho}{d\tilde{\lambda}^2} + \tilde{\Gamma}^\rho_{\mu\nu} \frac{dx^\mu}{d\tilde{\lambda}} \frac{dx^\nu}{d\tilde{\lambda}} = -\frac{1}{2} \frac{\mathcal{W}'}{\mathcal{W}} \frac{dy}{d\tilde{\lambda}} \frac{dx^\rho}{d\tilde{\lambda}}. \tag{42}$$

So, the geodesic motion of a free massive particle in the bulk spacetime is observed on the hypersurface $y = y(\tilde{\lambda})$ with the metric $\tilde{g}_{\mu\nu}$ as the motion of a massive particle under the additional influence of an extra non-gravitational force. For both the massive and massless test particle cases, the extra force term exists in the equations (40) and (42) for the particle motion, if the x^ρ-component of its velocity is nonzero, and acts on the particle parallelly to its four velocity $\frac{dx^\rho}{d\tilde{\lambda}}$.

We have seen in the previous section that the Newton's second law of mechanics and the conservation of energy (and their curved space generalization) in four-dimensional spacetime imply that only orthogonal (to the four-velocity $\frac{dx^\mu}{d\tau}$ of the particle) component of non-gravitational force F^μ influences the motion of the particle. However, as can be seen from the equations (40) and (42) for the trajectory $x^\rho(\tilde{\lambda})$ of the particle (as observed on the comoving hypersurface $y = y(\tilde{\lambda})$), the particle's motion is additionally influenced by the extra force term (the RHS's of Eqs. (40) and (42)) which is parallel to its velocity $\frac{dx^\rho}{d\tilde{\lambda}}$. Existence of such abnormal force term implies violation of four-dimensional physics and therefore can be an implication of existence of extra spatial dimensions (since such force term cannot be explained by the known four-dimensional physics). Or maybe it is due to the wrong choice of frame, since we have seen in the previous subsection that in the metric frame of the case 1 such abnormal force term does not exist. The author does not have yet clear understanding of which metric frame is the correct choice for the spacetime in one lower dimensions. However, as we will see in the following section, when $g_{\mu\nu}$ depends on the extra spatial coordinate y, the x^ρ-component equation for the particle trajectory (expressed in terms of $\tilde{\lambda}$) has the extra abnormal force term for both case 1 and case 2. So, it seems inevitable that the abnormal force term in general exists for natural choices of metric frame, i.e., those associated with $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu} = \mathcal{W} g_{\mu\nu}$. Due to extraordinary property of such abnormal force term, the previous literature [7, 8, 9, 10, 11, 12, 13, 14] dubbed the extra force as the fifth force. Actually, it should not be regarded as the violation of the four-dimensional physics, since such contradiction arises because we attempt to interpret the phenomenon which is higher-dimensional in nature from the perspective of lower dimensional physics. We, as beings incapable of sensing higher-dimensional spacetime, is apt to regard the higher-dimensional physical process as violation of four-dimensional physics. Anyhow, in the following we reconstruct the equation for the energy conservation and the Newton's second law of mechanics in the spacetime of one lower dimensions from the equations for the particle trajectory to see any physical implication of the extra force in one lower dimensions.

On the hypersurface $y = y(\tilde{\lambda})$ with the metric $\tilde{g}_{\mu\nu} = \mathcal{W} g_{\mu\nu}$, a bulk test particle with mass m_0 appears to have mass given by

$$\tilde{m}_0 = m_0 \left[1 - \left(\frac{dy}{d\tilde{\lambda}} \right)^2 \right]^{-\frac{1}{2}}, \tag{43}$$

where we used Eq. (16). By applying Eq. (41), we obtain the following mass change rate with $\tilde{\lambda}$ as observed on the (comoving) hypersurface:

$$\frac{d\tilde{m}_0}{d\tilde{\lambda}} = -\frac{\tilde{m}_0}{2}\frac{\mathcal{W}'}{\mathcal{W}}\frac{dy}{d\tilde{\lambda}}. \tag{44}$$

From Eqs. (42) and (44), we obtain the following equation describing the conservation of energy and the Newton's second law on the hypersurface:

$$\frac{Dp^\mu}{d\tilde{\lambda}} \equiv \frac{dp^\mu}{d\tilde{\lambda}} + \tilde{\Gamma}^\mu_{\rho\sigma}\frac{dx^\rho}{d\tilde{\lambda}}p^\sigma = -\tilde{m}_0\frac{\mathcal{W}'}{\mathcal{W}}\frac{dy}{d\tilde{\lambda}}\frac{dx^\mu}{d\tilde{\lambda}}, \tag{45}$$

which has the same form as the equation (35) for the case 1. So, in both cases the particle appears to be under the influence of the extra force of the same form $F^\mu = -\tilde{m}_0\frac{\mathcal{W}'}{\mathcal{W}}\frac{dy}{d\tilde{\lambda}}\frac{dx^\mu}{d\tilde{\lambda}}$. However, the motions of the particle are different for the two cases, because the mass $\tilde{m}_0$ changes with different rates due to different choices of the $(D-1)$-dimensional metric. For the case 2, the particle mechanics of one lower dimensions appears to be violated, whereas for the case 1 it is not.

4 Dynamics in General Spacetime

In this section, we consider the case when $g_{\mu\nu}$ in the bulk metric (5) depends on the extra spatial coordinate y. In this case, a bulk test particle is regarded as being under the influence of both the zero and the massive KK modes of graviton. We will see that the massive KK modes of graviton induce the perpendicular component of the extra force F^μ for both cases 1 and 2, and the parallel component of the force term even for the case 1.

The bulk geodesic equations (6), with the bulk metric given by Eq. (5) with $g_{\mu\nu} = g_{\mu\nu}(x^\rho, y)$, take the following forms:

$$\frac{d^2x^\rho}{d\lambda^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + \mathcal{W}^{-1}g^{\rho\sigma}\partial_y(\mathcal{W}g_{\sigma\mu})\frac{dx^\mu}{d\lambda}\frac{dy}{d\lambda} = 0, \tag{46}$$

$$\frac{d^2y}{d\lambda^2} - \frac{1}{2}\partial_y(\mathcal{W}g_{\mu\nu})\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{47}$$

The consistency condition for the geodesic motion in the bulk expressed in terms of an affine parameter λ [a new parameter $\tilde{\lambda} = f(\lambda)$] takes the same forms (7) [(11) and (12)] except that the metric $g_{\mu\nu}$ now depends on y.

4.1 Case 1

In this subsection, we study the geodesic motions of a bulk test particle as observed in one lower dimensions with the spacetime metric $g_{\mu\nu}(x^\rho, y)$.

First, we consider a massless test particle ($\epsilon_D = 0$) in the bulk spacetime. For the trivial case of the geodesic motion with $\frac{dy}{d\lambda} = 0$, the motion is observed in one lower dimensions to be that of massless free particle under the influence of gravitational field $g_{\mu\nu}$, only. When $\frac{dy}{d\lambda} \neq 0$, by using Eqs. (13) and (47), one obtains the following relation between the two parameters λ and $\tilde{\lambda}$:

$$\left(\frac{d\tilde{\lambda}}{d\lambda}\right)^{-1} \frac{d}{d\tilde{\lambda}} \left(\frac{d\tilde{\lambda}}{d\lambda}\right) = -\frac{1}{2}\mathcal{W}^{-\frac{1}{2}}\mathcal{W}' + \frac{1}{2}\mathcal{W}^{-\frac{1}{2}}\partial_y(\mathcal{W}g_{\mu\nu})\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}}. \tag{48}$$

The x^ρ-component geodesic equation (46) takes the following form after Eqs. (13) and (48) are applied:

$$\frac{d^2x^\rho}{d\tilde{\lambda}^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}} = \frac{1}{2}\mathcal{W}^{-\frac{1}{2}}\mathcal{W}'\frac{dx^\rho}{d\tilde{\lambda}} - \left[\mathcal{W}^{-\frac{1}{2}}g^{\rho\sigma} + \frac{1}{2}\mathcal{W}^{-\frac{1}{2}}\frac{dx^\rho}{d\tilde{\lambda}}\frac{dx^\sigma}{d\tilde{\lambda}}\right]\frac{dx^\mu}{d\tilde{\lambda}}\partial_y(\mathcal{W}g_{\rho\mu}). \tag{49}$$

The bulk geodesic motion of the massless particle with $\frac{dy}{d\lambda} \neq 0$ is therefore observed in one lower dimensions as the motion of a massive particle under the additional influence of the extra non-gravitational force, unlike the case of the KK zero mode metric $g_{\mu\nu}$ as discussed in the previous section. The extra force term on the RHS of Eq. (49) has both the parallel and the perpendicular components given by

$$\begin{aligned} f^\rho_\parallel &= \frac{1}{2}\left[\mathcal{W}^{-\frac{1}{2}}\mathcal{W}' + \mathcal{W}^{-\frac{1}{2}}\partial_y(\mathcal{W}g_{\mu\nu})\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\mu}{d\tilde{\lambda}}\right]\frac{dy}{d\tilde{\lambda}}\frac{dx^\rho}{d\tilde{\lambda}}, \\ f^\rho_\perp &= -\left[\mathcal{W}^{-\frac{1}{2}}g^{\rho\sigma} + \mathcal{W}^{-\frac{1}{2}}\frac{dx^\rho}{d\tilde{\lambda}}\frac{dx^\sigma}{d\tilde{\lambda}}\right]\frac{dx^\mu}{d\tilde{\lambda}}\partial_y(\mathcal{W}g_{\sigma\mu}). \end{aligned} \tag{50}$$

These vanish when $g_{\mu\nu}$ is independent of the extra spatial coordinate y, implying that the extra force is due to the massive KK modes of graviton.

Second, for a massive bulk test particle ($\epsilon_D = 1$), by using (15) one can express the y-component bulk geodesic equation (47) in terms of $\tilde{\lambda}$ as

$$\frac{d^2y}{d\tilde{\lambda}^2} - \frac{1}{2}\frac{\mathcal{W}'}{\mathcal{W}}\left(\frac{dy}{d\tilde{\lambda}}\right)^2 - \frac{1}{2}\left[\mathcal{W} - \left(\frac{dy}{d\tilde{\lambda}}\right)^2\right]\mathcal{W}^{-1}\partial_y(\mathcal{W}g_{\mu\nu})\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}} = 0. \tag{51}$$

The x^ρ-component bulk geodesic equation (46) can be expressed in terms of $\tilde{\lambda}$ as follows by applying Eqs. (15) and (51):

$$\frac{d^2x^\rho}{d\tilde{\lambda}^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}} = \frac{1}{2}\frac{\mathcal{W}'}{\mathcal{W}}\frac{dy}{d\tilde{\lambda}}\frac{dx^\rho}{d\tilde{\lambda}} - \mathcal{W}^{-1}\left[g^{\rho\sigma} + \frac{1}{2}\frac{dx^\rho}{d\tilde{\lambda}}\frac{dx^\sigma}{d\tilde{\lambda}}\right]\frac{dy}{d\tilde{\lambda}}\frac{dx^\mu}{d\tilde{\lambda}}\partial_y\left(\mathcal{W}g_{\sigma\mu}\right). \tag{52}$$

So, the bulk geodesic motion of a massive test particle is observed in one lower dimensions as the motion of a massive particle under the additional influence of the extra non-gravitational force, also unlike the case of y-independent $g_{\mu\nu}$. The extra force term on the RHS of Eq. (52) has both the parallel and the perpendicular components given by

$$
\begin{aligned}
f_{\parallel}^{\rho} &= \frac{1}{2}\left[\frac{\mathcal{W}'}{\mathcal{W}} + \mathcal{W}^{-1}\partial_y(\mathcal{W}g_{\mu\nu})\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\mu}{d\tilde{\lambda}}\right]\frac{dy}{d\tilde{\lambda}}\frac{dx^\rho}{d\tilde{\lambda}}, \\
f_{\perp}^{\rho} &= -\mathcal{W}^{-1}\left[g^{\rho\sigma} + \frac{dx^\rho}{d\tilde{\lambda}}\frac{dx^\sigma}{d\tilde{\lambda}}\right]\frac{dy}{d\tilde{\lambda}}\frac{dx^\mu}{d\tilde{\lambda}}\partial_y(\mathcal{W}g_{\sigma\mu}).
\end{aligned} \tag{53}
$$

As expected, these vanish when $g_{\mu\nu}$ is independent of y.

We now obtain the expression for the extra $(D-1)$-force F^μ acting on the particle from the $(D-1)$-dimensional perspective. The bulk test particle of mass m_0 is observed to have mass $\tilde{m}_0$ given by Eq. (33) from the perspective of $(D-1)$-dimensional spacetime with the metric $g_{\mu\nu}$. By using Eq. (51), one obtains the following rate of mass change with $\tilde{\lambda}$ as observed in the embedded $(D-1)$-dimensional spacetime:

$$
\frac{d\tilde{m}_0}{d\tilde{\lambda}} = -\frac{\tilde{m}_0}{2}\left[\mathcal{W}^{-1}\mathcal{W}' - \mathcal{W}^{-1}\partial_y(\mathcal{W}g_{\mu\nu})\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}}\right]\frac{dy}{d\tilde{\lambda}}. \tag{54}
$$

From Eqs. (52) and (54), we obtain the following equation describing the conservation of energy and the Newton's second law of mechanics in $(D-1)$-dimensions:

$$
\frac{Dp^\mu}{d\tilde{\lambda}} \equiv \frac{dp^\mu}{d\tilde{\lambda}} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tilde{\lambda}}p^\sigma = -\tilde{m}_0\mathcal{W}^{-1}g^{\mu\nu}\frac{dx^\rho}{d\tilde{\lambda}}\partial_y(\mathcal{W}g_{\nu\rho})\frac{dy}{d\tilde{\lambda}}. \tag{55}
$$

The extra force F^μ on the RHS of Eq. (55) has both the parallel and orthogonal components given by

$$
\begin{aligned}
F_{\parallel}^{\mu} &= \tilde{m}_0\mathcal{W}^{-1}\frac{dx^\rho}{d\tilde{\lambda}}\frac{dx^\sigma}{d\tilde{\lambda}}\partial_y(\mathcal{W}g_{\rho\sigma})\frac{dy}{d\tilde{\lambda}}\frac{dx^\mu}{d\tilde{\lambda}}, \\
F_{\perp}^{\mu} &= -\tilde{m}_0\mathcal{W}^{-1}\left[g^{\mu\nu} + \frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}}\right]\frac{dy}{d\tilde{\lambda}}\frac{dx^\rho}{d\tilde{\lambda}}\partial_y(\mathcal{W}g_{\nu\rho}).
\end{aligned} \tag{56}
$$

As expected, when $g_{\mu\nu}$ does not depend on y, $F_{\perp}^{\mu}$ vanishes and $F_{\parallel}^{\mu}$ takes the form of the RHS of Eq. (35).

Even with the choice of the metric by $g_{\mu\nu}$ for the $(D-1)$-dimensional spacetime, which it has been previously regarded as the natural canonical choice for the $(D-1)$-dimensional spacetime embedded in brane worlds, the motion of the particle observed in the $(D-1)$-dimensional spacetime appears to be under the additional influence of the abnormal force term, which cannot be explained by laws of physics in $(D-1)$-dimensions, if the metric $g_{\mu\nu}$ depends on the extra spatial coordinate y. Namely, the equations (49) and (52) describing the particle trajectory $x^\mu(\tilde{\lambda})$ observed in one lower dimensions have nonzero parallel component force term $f_{\parallel}^{\rho}$ and the the mass change (54) with $\tilde{\lambda}$ is not in accordance with the conventional formula (21) for a given extra non-gravitational force $F^\mu = -\tilde{m}_0 \mathcal{W}^{-1} g^{\mu\nu} \frac{dx^\rho}{d\tilde{\lambda}} \partial_y (\mathcal{W} g_{\nu\rho}) \frac{dy}{d\tilde{\lambda}}$. Also, with a choice of the metric frame $\tilde{g}_{\mu\nu}$, the same holds true as expected, as we will see in the following subsection. Note, the abnormal force term is not due to the wrong choice [4] of parameter $\tilde{\lambda}$ describing motion observed in $(D-1)$-dimensions, since we have fixed (up to affine transformations) the parameter $\tilde{\lambda}$ through the $(D-1)$-dimensional affine conditions (8) and (9). So, the massive KK modes of graviton not only give a small correction to Newton's $1/r^2$ law of four-dimensional gravity but also causes violation of four-dimensional laws of physics, which can be an indication that our four-dimensional world is embedded in higher-dimensional spacetime.

If we take the viewpoint that laws of physics in $(D-1)$-dimensional spacetime should not be violated, then we have to take the metric $\bar{g}_{\mu\nu}$ for which the equation for the particle trajectory does not have abnormal force term as the physical metric of the $(D-1)$-dimensional spacetime. With a choice of such metric, according to Eq. (21), the $(D-1)$-dimensional mass $\bar{m}_0$ should change with an affine parameter $\bar{\lambda}$ as

$$\frac{d\bar{m}_0}{d\bar{\lambda}} = -\bar{g}_{\mu\nu} F^\mu \frac{dx^\nu}{d\bar{\lambda}}, \tag{59}$$

where $\bar{m}_0 = m_0 \frac{d\bar{\lambda}}{d\lambda}$, the parameter $\bar{\lambda}$ is defined through the relation $\bar{g}_{\mu\nu} \frac{dx^\mu}{d\bar{\lambda}} \frac{dx^\nu}{d\bar{\lambda}} = -1$, F^μ is the extra non-gravitational force observed in $(D-1)$-dimensions, and of course the parameter λ is defined through Eq. (7) with $\epsilon_D = 1$. In the case where $g_{\mu\nu}$ is independent of y, such physical metric is given by $\bar{g}_{\mu\nu} = g_{\mu\nu}$. For a general y-dependent $g_{\mu\nu}$, it seems not clear whether a simple and natural form of the physical metric $\bar{g}_{\mu\nu}$ that satisfies this equation exists.

---

[4]If one chooses a non-affine parameter to describe the motion of a particle, the abnormal force term also occurs in the equation for particle trajectory. To see this, we consider the following geodesic equation for a free particle, whose motion is under the influence of the gravitational force, only:

$$\frac{d^2x^\rho}{ds^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0, \tag{57}$$

where s is an affine parameter. If we take a new parameter $\tilde{s} = f(s)$ to parameterize the motion of the particle, then the above geodesic equations transform to the following form:

$$\frac{d^2x^\rho}{d\tilde{s}^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{d\tilde{s}} \frac{dx^\nu}{d\tilde{s}} = -\frac{d^2\tilde{s}/ds^2}{(d\tilde{s}/ds)^2} \frac{dx^\rho}{d\tilde{s}}. \tag{58}$$

So, through non-affine transformation one induces an extra velocity dependent fictitious force term $-\frac{d^2\tilde{s}/ds^2}{(d\tilde{s}/ds)^2} \frac{dx^\rho}{d\tilde{s}}$ parallel to the four-velocity $\frac{dx^\rho}{d\tilde{s}}$ of the particle. Eq. (58) also shows that the geodesic equations (57) are invariant only under the affine transformations $s \to \tilde{s} = as + b$, $a, b \in \mathbf{R}$.

4.2 Case 2

In this subsection, we study the geodesic motion of a bulk test particle as observed in the (comoving) hypersurface $y = y(\tilde{\lambda})$ with the metric $\tilde{g}_{\mu\nu} = \mathcal{W}(y)g_{\mu\nu}(x^\rho, y)$.

First, we consider the case of a free massless bulk particle ($\epsilon_D = 0$). When $\frac{dy}{d\lambda} = 0$, one can put the x^ρ-component geodesic equation (46) to the form (37). When $\frac{dy}{d\lambda} \neq 0$, by using Eqs. (14) and (47), one can see that the affine parameters λ and $\tilde{\lambda}$ are related as

$$\left(\frac{d\tilde{\lambda}}{d\lambda}\right)^{-1} \frac{d}{d\tilde{\lambda}}\left(\frac{d\tilde{\lambda}}{d\lambda}\right) = \frac{1}{2}\partial_y\left(\mathcal{W}g_{\mu\nu}\right)\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}}. \tag{60}$$

Applying Eqs. (14) and (60), one can put the x^ρ-component bulk geodesic equation (46) into the following form in terms of the new parameter $\tilde{\lambda}$:

$$\frac{d^2x^\rho}{d\tilde{\lambda}^2} + \tilde{\Gamma}^\rho_{\mu\nu}\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}} = -\left[\mathcal{W}^{-1}g^{\rho\sigma} + \frac{1}{2}\frac{dx^\rho}{d\tilde{\lambda}}\frac{dx^\sigma}{d\tilde{\lambda}}\right]\frac{dx^\mu}{d\tilde{\lambda}}\partial_y\left(\mathcal{W}g_{\sigma\mu}\right), \tag{61}$$

where $\tilde{\Gamma}^\rho_{\mu\nu}$ is the Christoffel symbol for the metric $\tilde{g}_{\mu\nu} = \mathcal{W}(y)g_{\mu\nu}(x^\rho, y)$ and we used the fact that $\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}$. So, the geodesic motion of a massless free particle in the bulk spacetime with the metric (5) is observed on the (comoving) hypersurface $y = y(\tilde{\lambda})$ as the motion of a massive particle under the additional influence of the extra non-gravitational force. Unlike the case of the y-independent $g_{\mu\nu}$, the extra force term on the RHS of Eq. (61) is no longer parallel to the $(D-1)$-velocity $\frac{dx^\rho}{d\tilde{\lambda}}$ of the test particle. The parallel and the perpendicular components of the extra force term are given by

$$\begin{aligned} f^\rho_\parallel &= \frac{1}{2}\partial_y\left(\mathcal{W}g_{\mu\nu}\right)\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}}\frac{dx^\rho}{d\tilde{\lambda}}, \\ f^\rho_\perp &= -\left[\frac{dx^\rho}{d\tilde{\lambda}}\frac{dx^\sigma}{d\tilde{\lambda}} + \mathcal{W}^{-1}g^{\rho\sigma}\right]\frac{dx^\mu}{d\tilde{\lambda}}\partial_y\left(\mathcal{W}g_{\sigma\mu}\right). \end{aligned} \tag{62}$$

As expected, the perpendicular component $f^\rho_\perp$ vanishes and the parallel component $f^\rho_\parallel$ takes the form of the RHS of Eq. (40), when $g_{\mu\nu}$ is independent of y, implying that the perpendicular component is due to the massive KK modes of graviton.

Second, we consider the case of a massive bulk test particle ($\epsilon_D = 1$). By using Eq. (16), one can put the y-component geodesic equations (47) into the following form in terms of a new parameter $\tilde{\lambda}$:

$$\frac{d^2y}{d\tilde{\lambda}^2} - \frac{1}{2}\left[1 - \left(\frac{dy}{d\tilde{\lambda}}\right)^2\right]\partial_y\left(\mathcal{W}g_{\mu\nu}\right)\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}} = 0. \tag{63}$$

The x^ρ-component bulk geodesic equation (46) takes the following form after Eqs. (16) and (63) are applied:

$$\frac{d^2x^\rho}{d\tilde{\lambda}^2} + \tilde{\Gamma}^\rho_{\mu\nu}\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}} = -\left[\mathcal{W}^{-1}g^{\rho\sigma} + \frac{1}{2}\frac{dx^\rho}{d\tilde{\lambda}}\frac{dx^\sigma}{d\tilde{\lambda}}\right]\frac{dy}{d\tilde{\lambda}}\frac{dx^\mu}{d\tilde{\lambda}}\partial_y\left(\mathcal{W}g_{\sigma\mu}\right). \tag{64}$$

So, the geodesic motion of a massive particle in the bulk spacetime with the metric (5) is observed on the (comoving) hypersurface $y = y(\tilde{\lambda})$ as the motion of a massive particle under the additional influence of the extra non-gravitational force. As in the massless test particle case, the extra force term on the RHS of Eq. (64) has both parallel and perpendicular components given by

$$
\begin{aligned}
f^{\rho}_{\parallel} &= \frac{1}{2}\partial_y\left(\mathcal{W}g_{\mu\nu}\right)\frac{dy}{d\tilde{\lambda}}\frac{dx^{\mu}}{d\tilde{\lambda}}\frac{dx^{\nu}}{d\tilde{\lambda}}\frac{dx^{\rho}}{d\tilde{\lambda}}, \\
f^{\rho}_{\perp} &= -\left[\frac{dx^{\rho}}{d\tilde{\lambda}}\frac{dx^{\sigma}}{d\tilde{\lambda}} + \mathcal{W}^{-1}g^{\rho\sigma}\right]\frac{dy}{d\tilde{\lambda}}\frac{dx^{\mu}}{d\tilde{\lambda}}\partial_y\left(\mathcal{W}g_{\sigma\mu}\right). \qquad (65)
\end{aligned}
$$

As expected, when $g_{\mu\nu}$ is independent of y, $f^{\rho}_{\parallel}$ takes the form of the RHS of Eq. (42) and $f^{\rho}_{\perp}$ vanishes.

We now obtain the expression for the extra non-gravitational $(D-1)$-force F^{μ} observed on the (comoving) hypersurface $y = y(\tilde{\lambda})$. A bulk test particle with mass m_0 is measured on the hypersurface $y = y(\tilde{\lambda})$ to have mass $\tilde{m}_0$ given by Eq. (43). By using Eq. (63), we obtain the following mass change with $\tilde{\lambda}$ as observed on the hypersurface:

$$
\frac{d\tilde{m}_0}{d\tilde{\lambda}} = \frac{\tilde{m}_0}{2}\partial_y(\mathcal{W}g_{\mu\nu})\frac{dx^{\mu}}{d\tilde{\lambda}}\frac{dx^{\nu}}{d\tilde{\lambda}}\frac{dy}{d\tilde{\lambda}}. \qquad (66)
$$

So, from Eqs. (64) and (66), we obtain the equation $\frac{Dp^{\mu}}{d\tilde{\lambda}} = F^{\mu}$ which has the same form (55) as the case 1. However, since we have chosen different $(D-1)$-dimensional metric, the expressions for the parallel and the perpendicular components of F^{μ} are instead given by

$$
\begin{aligned}
F^{\mu}_{\parallel} &= \tilde{m}_0\frac{dx^{\rho}}{d\tilde{\lambda}}\frac{dx^{\sigma}}{d\tilde{\lambda}}\partial_y(\mathcal{W}g_{\rho\sigma})\frac{dy}{d\tilde{\lambda}}\frac{dx^{\mu}}{d\tilde{\lambda}}, \\
F^{\mu}_{\perp} &= -\tilde{m}_0\left[\mathcal{W}^{-1}g^{\mu\nu} + \frac{dx^{\mu}}{d\tilde{\lambda}}\frac{dx^{\nu}}{d\tilde{\lambda}}\right]\frac{dy}{d\tilde{\lambda}}\frac{dx^{\rho}}{d\tilde{\lambda}}\partial_y(\mathcal{W}g_{\nu\rho}). \qquad (67)
\end{aligned}
$$

As expected, when $g_{\mu\nu}$ is independent of y, $F^{\mu}_{\perp}$ vanishes and $F^{\mu}_{\parallel}$ takes the form of the RHS of Eq. (45).

5 Conclusions

In this paper, we carefully studied the geodesic motions of a test particle in the bulk spacetime of general gravitating configurations in the RS scenario as observed in the embedded spacetime of one lower dimensions. We presented the explicit equations describing such particle motion perceived by an observer in one lower dimensions and the explicit forms of the extra force on the particle measured in one lower dimensions. Such equations and extra forces are inconsistent with laws of particle mechanics in one lower dimensions. Such inconsistency does not mean the violation of physics in one lower dimensions, but results from our effort to interpret the physical process which is higher-dimensional in nature with physics of one lower dimensions. The RS model assumes that the extra spatial dimension is noncompact, and therefore generically physical phenomena in the RS model have to show higher-dimensional character, which is observed to be inconsistent with physics of our four-dimensional world. So, one can test the RS scenario by detecting inconsistency with the four-dimensional physics such as the one present in this paper. It would be interesting to further explore implications of violation of four-dimensional particle mechanics in the RS scenario that we presented in this paper.

References

[1] L. Randall and R. Sundrum, "A large mass hierarchy from a small extra dimension," Phys. Rev. Lett. **83** (1999) 3370, hep-ph/9905221.

[2] L. Randall and R. Sundrum, "An alternative to compactification," Phys. Rev. Lett. **83** (1999) 4690, hep-th/9906064.

[3] J. Lykken and L. Randall, "The shape of gravity," hep-th/9908076.

[4] A. Chamblin, S.W. Hawking and H.S. Reall, "Brane-world black holes," Phys. Rev. **D61** (2000) 065007, hep-th/9909205.

[5] D. Youm, "Solitons in brane worlds II," hep-th/0001018.

[6] A. Chamblin, C. Csaki, J. Erlich and T.J. Hollowood, "Black diamonds at brane junctions," hep-th/0002076.

[7] D.J. Gross and M.J. Perry, "Magnetic monopoles in Kaluza-Klein theories," Nucl. Phys. **B226** (1983) 29.

[8] Y.M. Cho and D.H. Park, "Fifth force from Kaluza-Klein unification," Gen. Rel. Grav. **23** (1991) 741.

[9] D. Kalligas, P.S. Wesson and C.W. Everitt, "The classical tests in Kaluza-Klein gravity," Astrophys. J. **439** (1994) 548.

[10] B. Mashhoon, H. Liu and P.S. Wesson, "Particle masses and the cosmological constant in Kaluza Klein theory," Phys. Lett. **B331** (1994) 305.

[11] P.S. Wesson and J. Ponce de Leon, "The equation of motion in Kaluza-Klein cosmology and its implication for astrophysics," Astron. Astrophys. **294** (1995) 1.

[12] P.S. Wesson, B. Mashhoon and H. Liu, "The (im)possibility of detecting a fifth dimension," Mod. Phys. Lett. **A12** (1997) 2309.

[13] B. Mashhoon, P.S. Wesson and H. Liu, "Dynamics in Kaluza-Klein gravity and a fifth force," Gen. Rel. Grav. **30** (1998) 555.

[14] P.S. Wesson, B. Mashhoon, H. Liu and W.N. Sajko, "Fifth force from fifth dimension," Phys. Lett. **B456** (1999) 34.

[15] E. Schmutzer, "Beitrag zur projektiven Relativitätstheorie," Z. Phys. **149** (1957) 329; "Beitrag zur projektiven Relativitätstheorie. II," Z. Phys. **154** (1959) 312.

[16] D. Kovacs, "The geodesic equation in five-dimensional relativity theory of Kaluza-Klein," Gen. Rel. Grav. **16** (1984) 645.

[17] J. Gegenberg and G. Kunstatter, "The motion of charged particles in Kaluza-Klein space-time," Phys. Lett. **A106** (1984) 410.

[18] D. Youm, "Solitons in brane worlds," hep-th/9911218.

[19] S. Kachru, M. Schulz and E. Silverstein, "Self-tuning flat domain walls in 5d gravity and string theory," hep-th/0001206.

[20] D. Youm, "Bulk fields in dilatonic and self-tuning flat domain walls," hep-th/0002147.

[21] D. Schmutzer, "Relativistische Physik: klassische Theorie," *Leipzig, Germany: Teubner (1968) 974 p.*.