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Aspects of semi-classical transport theory for QCD [a]

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We discuss some aspects of a recently proposed semi-classical transport theory for QCD plasmas based on coloured point particles. This includes the derivation of effective transport equations for mean fields and fluctuations which relies on the Gibbs ensemble average. Correlators of fluctuations are interpreted as collision integrals for the effective Boltzmann equation. The approach yields a recipe to integrate-out fluctuations. Systematic approximations (first moment, second moment, polarisation approximation) based on a small plasma parameter are discussed as well. Finally, the application to a hot non-Abelian plasma close to thermal equilibrium is considered and the consistency with the fluctuation-dissipation theorem established.

1 Introduction

Reliable theoretical tools to describe transport phenomena of hot or dense QCD plasmas in- or out-of equilibrium are at the basis for an understanding of the quark-gluon-plasma. This extreme state of matter where quarks and gluons are no longer confined is expected to be produced within the up-coming heavy ion experiments at RHIC and LHC.

In the present contribution a recently proposed semi-classical method to derive effective transport equations for QCD is reviewed, developed in collaboration with Cristina Manuel in ref. [1-4]. The interest in a formalism based on a classical transport theory [5] is that main properties of a hot *quantum* plasma can already be understood within simple *classical* terms. Indeed, the soft non-Abelian gauge fields – those having a huge occupation number – can be treated as classical *fields*, while the hard gauge field modes and the quarks can be treated as classical *particles*. This approach has been known since long for Abelian plasmas [6], but a consistent extension to the non-Abelian case was longtime missing.

Here, we discuss some conceptual aspects of this method, and in particular the step from a microscopic to a macroscopic (kinetic) formulation, the close

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relationship between the ensemble average and the notion of coarse-graining, and systematic expansions in a small plasma parameter or gauge coupling. The latter is at the basis for the integrating-out of fluctuations and the derivation of collision integrals. For a hot plasma close to equilibrium it has been shown that the HTL effective theory follows to leading order in an expansion in moments [7]. We review how Bödeker's effective theory [8] is recovered at second order in this expansion. Finally, the compatibility with the fluctuation-dissipation theorem is established [3].

2 Classical particles and fields

The starting point for a classical transport theory is to consider an ensemble of classical point particles interacting through classical non-Abelian gauge fields. Such an approach is known since long for Coulomb plasmas of charged point particles interacting through photons [6]. The new ingredience in the case of QCD is that the particles have to carry a non-Abelian colour charge Q^a, where the colour index runs from $a = 1$ to $N^2 - 1$ for a SU(N) gauge group. These particles interact self-consistently amongst each others, that is, through the classical gauge fields created by the particles. Their classical equations of motion have first been given by Wong [9],

$$m\frac{d\hat{x}^\mu}{d\tau} = \hat{p}^\mu \ , \quad m\frac{d\hat{p}^\mu}{d\tau} = g\hat{Q}^a F_a^{\mu\nu}\hat{p}_\nu \ , \quad m\frac{d\hat{Q}^a}{d\tau} = -gf^{abc}\hat{p}^\mu A_\mu^b \hat{Q}^c \ . \tag{1}$$

Here, A_μ denotes the gauge field, $F^a_{\mu\nu} = \partial_\mu A^a_\mu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$ the corresponding field strength and $\hat{z} \equiv (\hat{x}, \hat{p}, \hat{Q})(\tau)$ the world line of a particle. The particles interact self-consistently through classical Yang-Mills fields,

$$D_\mu F^{\mu\nu} = J^\nu \ , \tag{2}$$

where the current J is the sum of the currents for the individual particles, $J = g\sum_i \int d\tau\delta(x - \hat{x}_i)d\hat{x}_i/d\tau$.

To further a kinetic description it is useful to introduce a microscopic one-particle distribution function $f(z) \sim \sum_i \int d\tau\delta[z - \hat{z}_i(\tau)]$ which describes an ensemble of such particles. Making use of (1), its kinetic equation is [5]

$$p^\mu\left(\frac{\partial}{\partial x^\mu} - gf^{abc}A^b_\mu Q^c\frac{\partial}{\partial Q^a} - gQ_a F^a_{\mu\nu}\frac{\partial}{\partial p_\nu}\right) f(x, p, Q) = 0 \ . \tag{3}$$

This Boltzmann equation is collisionless, since $df/d\tau = 0$ (Liouville's theorem). However, it contains *effectively* collisions due to the long range interactions as shall become clear in the sequel. The Wong equations transform covariantly under gauge transformations which entails that f and (3) are gauge invariant [7]. Eq. (3) is completed by the Yang-Mills equation (2), where the current J becomes now a functional $J[f]$ of the one-particle distribution function. These equations constitute the basis for the construction of a semi-classical kinetic theory.

3 Kinetic theory

Within the semi-classical approach all informations about properties of the QCD plasma are given by the microscopic dynamical equations (2) and (3). However, for most situations not all microscopic informations are of relevance. Of main physical interest are the characteristics of the QCD plasma at macroscopic scales. This includes quantities like damping rates, colour conductivities or screening lengths within the kinetic regime, or transport coefficients like shear or bulk viscosities within the hydrodynamic regime. The microscopic length scales, like typical inter-particle distances, are much smaller than such macroscopic scales.

Given the statistical ensemble – representing the state of the system – its macroscopic properties follow as functions of the fundamental parameters and the interactions between the particles. Within kinetic theory, the basic 'macroscopic' quantity is the one-particle distribution function from which all further macroscopic observables can be derived. The aim of a kinetic theory is to construct – with as little restrictions or assumptions as possible – a closed set of transport equations for this distribution function. Such an approach assumes implicitly that the 'medium', described by the distribution function, is continuous. If the medium is *not* continuous, stochastic fluctuations due to the particles should be taken into account as well, and their consistent inclusion leads to a coupled set of *effective* transport equations for correlators of fluctuations and the one-particle distribution function [6]. In this classical picture, f is a deterministic quantity once all initial conditions have been fixed. Unfortunately, this is not possible, and f has to be considered as a *stochastic* (fluctuating) quantity instead. The random fluctuations of the distribution function are at the root of the dissipative character of the effective transport theory.

4 Coarse graining and ensemble average

The step from a microscopic to a macroscopic (kinetic) description requires an appropriate definition of macroscopic quantities in the first place. There are (at least) two possible options. The first one consists in taking both volume and time averages of the microscopic distribution function f, the non-Abelian fields and the Boltzmann equation (3) over characteristic physical length and time scales [3]. Clearly, the notion of a characteristic physical scale depends on the particular problem studied. A reasonable choice for a coarse graining scale $r_{\rm ph}$ has to met some few criteria. The coarse graining scale $r_{\rm ph}$ should be sufficiently large such that the number of particles $N_{\rm ph}$ contained in $r^3_{\rm ph}$ is $\gg 1$. This is at the basis for a plasma description in the first place, because $N_{\rm ph} \gg 1$ ensures that many particles interact coherently within a coarse-graining volume (quasi-particle behaviour). The inverse of $N_{\rm ph}$ is also known as the *plasma parameter* ϵ. In addition, $r_{\rm ph}$ should be larger than a typical two-particle correlation length $r_{\rm corr}$. This ensures that two-particle correlators g_2 can be neglected. Finally, $r_{\rm ph}$ should be smaller than the macroscopic scales of the plasma to be investigated, like typical relaxation lengths $r_{\rm rel}$. This way it is guaranteed that the scales of interest are not washed-out. Such coarse grainings remove (irrelevant) microscopic information and thereby modify the transport equation which is expected to become of the Boltzmann-Langevin type,

$$p^\mu \left(\bar{D}_\mu - gQ_a\bar{F}^a_{\mu\nu}\partial^\nu_p\right)\bar{f} = C[\bar{f}] + \zeta \ . \tag{4}$$

Here, we introduced $D_\mu[A]f \equiv [\partial_\mu - gf^{abc}Q_cA_{\mu,b}\partial^Q_a]f$ and the shorthands $\partial_\mu \equiv \partial/\partial x^\mu$, $\partial^p_\mu \equiv \partial/\partial p^\mu$ and $\partial^Q_a \equiv \partial/\partial Q^a$. The new terms on the r.h.s. are the collision integral $C[\bar{f}]$ and a related source for stochastic noise ζ on the r.h.s. of (4) are due to the fact that the Boltzmann equation (3) is quadratic and cubic in the fields. Physically speaking, these terms arise because the coarse-graining integrates-out the short range modes within a coarse-graining volume. The interactions of such modes can result into additional effective interactions for the remaining long range modes.

A second route for obtaining kinetic equations consists in taking an ensemble average of the microscopic transport equation. In the present context we consider particles in a phase space, hence the appropriate average is the Gibbs ensemble average henceforth denoted by $\langle\cdots\rangle$. All statistical informations of the system are then contained in the correlators $\langle\delta f\cdots\delta f\rangle$ with $\delta f = f - \langle f\rangle \equiv f - \bar{f}$ denoting the statistical fluctuations of f about its mean value. The most important equal-time correlator is the quadratic one given by

$$\langle\delta f_{\mathbf{x},p,Q}\,\delta f_{\mathbf{x}',p',Q'}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta(Q-Q')\,\bar{f} + \mathcal{O}(g_2) \tag{5}$$

where we have neglected higher order corrections due to the two-particle correlation function g_2. Similar formula hold for higher order equal-time correlation function. Notice that (5) does not require further informations about the state of the system and applies equally to systems in- or out-of equilibrium. While (5) has been proven rigorously for particles with classical statistics, its generalisation to the case of quantum statistics is known only for weakly interacting particles and consists in replacing

$$\bar{f} \to \bar{f}(1 \pm \bar{f}) \tag{6}$$

in (5) for Bose-Einstein (+) or Fermi-Dirac (−) statistics.

5 Effective transport equations

We now apply the Gibbs ensemble average to the Boltzmann equation (3) in order to derive an effective kinetic equation of the form (4). To that end, we split f into its mean value and a fluctuating part

$$f(x, p, Q) = \bar{f}(x, p, Q) + \delta f(x, p, Q) \ , \tag{7}$$

where $\bar{f} = \langle f \rangle$. The mean value of the statistical fluctuations vanish by definition, $\langle \delta f \rangle = 0$. This separation into mean fields and statistical, random fluctuations corresponds effectively to a split into long vs. short wavelength modes associated to the mean fields vs. fluctuations. In addition to (7) we split the gauge field into a mean field part and fluctuations,

$$A^\mu(x) = \bar{A}^\mu(x) + a^\mu(x) \ , \tag{8}$$

with $\bar{A} = \langle A \rangle$ and $\langle a \rangle = 0$. This split should been seen on a different footing as the split (7) because the Gibbs ensemble average is defined for the particle-like degrees of freedom in the first place. One may wonder whether (8) – and the subsequent split of the dynamical equations – is compatible with gauge symmetry because the gauge field transforms non-linearly. The situation is very similar to the background field method used in path integral formulations of QFT and the compatibility of (8) with gauge symmetry follows along the same lines. For more details, see ref. [2].

Coming back to the initial set of dynamical equations (2) and (3), we perform the ensemble average to find

$$p^\mu \left(\bar{D}_\mu - gQ_a \bar{F}^a_{\mu\nu} \partial^\nu_p \right) f = \langle \eta \rangle + \langle \xi \rangle \ . \tag{9}$$

for the effective transport equation, and

$$\bar{D}_\mu \bar{F}^{\mu\nu} + \langle J^\nu_{\text{fluc}} \rangle = \bar{J}^\nu \ . \tag{10}$$

for the effective Yang-Mills equations. In (9) and (10), we collected all terms quadratic or cubic in the fluctuations into the functions

$$\eta \equiv gQ_a \, p^\mu \left[(\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu)^a + g f^{abc} a^b_\mu a^c_\nu \right] \partial^\nu_p \delta f(x, p, Q) \ , \tag{11}$$

$$\xi \equiv g p^\mu f^{abc} Q^c \left[\partial^Q_a a^b_\mu \ \delta f(x, p, Q) \ + g a^a_\mu a^b_\nu \partial^\nu_p \bar{f}(x, p, Q) \right] , \tag{12}$$

$$J^{a,\nu}_{\text{fluc}} \equiv g \left[f^{dbc} \bar{D}^\mu_{ad} a_{b,\mu} a^\nu_c + f^{abc} a_{b,\mu} \left((\bar{D}^\mu a^\nu - \bar{D}^\nu a^\mu)_c + g f^{cde} a^\mu_d a^\nu_e \right) \right] . \tag{13}$$

The set of equations (9) – (13) is not yet closed as they still involve correlators of fluctuations. Hence, we need their dynamical equations as well. Subtracting (3) from (9) we find for the dynamics of δf

$$p^\mu \left(\bar{D}_\mu - gQ_a \bar{F}^a_{\mu\nu}\partial^\nu_p\right)\delta f = gQ_a(\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu)^a p^\mu \partial^p_\nu \bar{f} \\ + gp^\mu a_{b,\mu} f^{abc} Q_c \partial^Q_a \bar{f} + \eta + \xi - \langle \eta + \xi \rangle \ . \quad (14)$$

Subtracting (2) from (10) yields the corresponding equation for the gauge field fluctuations, to wit

$$\left(\bar{D}^2 a^\mu - \bar{D}^\mu(\bar{D}_\nu a^\nu)\right)^a + 2g f^{abc}\bar{F}^{\mu\nu}_b a_{c,\nu} + J^{a,\mu}_{\text{fluc}} - \langle J^{a,\mu}_{\text{fluc}}\rangle = \delta J^{a,\mu} \ . \quad (15)$$

Eqs. (14) and (15) are the master equations for all higher order correlation functions of fluctuations, e.g. multiplying (14) by δf and taking the ensemble average yields the Boltzmann equation for the correlator $\langle \delta f \delta f\rangle$. This hierarchy of dynamical equations is similar to the BBGKY hierarchy. All the equal-time correlators of δf can be derived from the basic definition of the Gibbs ensemble average in phase space and serve as initial conditions [2].

We remark that the set of dynamical equations (9) – (15) is equivalent to the original set of microscopic equations. It is exact in that no further approximations – apart from the assumptions leading to (1) and (2) – have been performed so far. Furthermore, it can be applied to out-of-equilibrium situations simply because the Gibbs ensemble average is not bound to a plasma close-to-equilibrium.

6 Collision integrals

Let us comment on the terms η, ξ and J_{fluc} in the dynamical equations. In the effective Boltzmann equation, the functions $\langle\eta\rangle$ and $\langle\xi\rangle$ appear only after the splitting (7) and (8) has been performed. These terms are qualitatively different from those already present in the transport equation. The correlators $\langle\eta\rangle$ and $\langle\xi\rangle$ are interpreted as effective collision integrals of the macroscopic Boltzmann equation. The fluctuations in the distribution function of the particles induce fluctuations in the gauge fields, while the gauge field fluctuations, in turn, induce fluctuations in the motion of the quasi-particles. In the present formalism, the correlators of statistical fluctuations have the same effect as collisions. This yields a precise recipe for obtaining collision integrals within semi-classical transport theory.

In the Abelian limit the transport equations reduce to the known set of kinetic equations for Abelian plasmas and only the collision integral $\langle\eta\rangle$ survives. Here it is known that $\langle\eta\rangle$ can be explicitly expressed as the Balescu-Lenard collision integral[6]. This substantiates the correspondence between fluctuations and collisions in an Abelian plasma.

At the same time we observe the presence of stochastic noise in the effective equations. The noise originates in the source fluctuations of the particle distributions and induces *field-independent* fluctuations to the gauge fields. The corresponding terms in the effective Boltzmann equations are therefore η, ξ and $J_{\rm fluc}$ at vanishing mean field or mean current.

Finally, we observe the presence of a fluctuation-induced current $\langle J_{\rm fluc}\rangle$ in the effective Yang-Mills equation for the mean fields. This current, due to its very nature, stems from the induced correlations of gauge field fluctuations. It vanishes identically in the Abelian case. While the collision integrals are linear in the quasi-particle fluctuations, the induced current only contains the gauge field fluctuations. As the fluctuations of the one-particle distribution function are the basic source for fluctuations, we expect that a non-vanishing induced current will appear as a subleading effect.

In order to find explicitly the collision integrals, noise sources or the fluctuation-induced currents for non-Abelian plasmas, one has to solve first the dynamical equations for the fluctuations in the background of the mean fields. This step amounts to 'integrating-out' fluctuations. In general, this is a difficult task, in particular due to the non-linear terms present in η, ξ and $J_{\rm fluc}$. This will only be possible within some approximations.

7 Plasma parameter and moment expansions

When it comes to solving the kinetic equations – and in particular the fluctuation dynamics – it is necessary to identify a small expansion parameter which allows for systematic approximations of the hierarchy of dynamical equations for correlation functions. The non-Abelian plasma is characterised by two dimensionless parameters, the gauge coupling g and the plasma parameter ϵ. A small plasma parameter is mandatory for a kinetic description to be viable. For a classical plasma, ϵ is an independent parameter related to the mean particle number and the gauge coupling. For a quantum plasma, the mean particle number cannot be fixed arbitrarily, and ϵ scales proportional to the gauge coupling as $\epsilon \sim g^3 \ll 1$, explaining why a kinetic description for quantum plasmas is intimately linked to the weak coupling limit.

Here, two systematic approximation schemes are outlined: an expansion in moments of the fluctuations and an expansion in a small gauge coupling. Although they have distinct origins in the first place they are intimately linked due to the requirements of gauge invariance.

A systematic perturbative expansion in powers of g can be done because the differential operator appearing in the effective Boltzmann equation (9) admits such an expansion. The force term $g\,p^\mu Q_a \bar{F}^a_{\mu\nu}\partial^\nu_p$ is suppressed by a power of g as compared to the leading order term $p^\mu \bar{D}_\mu$. In this spirit, we expand

$$\bar{f} = \bar{f}^{(0)} + g\,\bar{f}^{(1)} + \dots \tag{16}$$

This is at the basis for a systematic organisation of the dynamical equations in powers of g.[7] To leading order, this concerns in particular the cubic correlators appearing in $\langle\eta\rangle$ and $\langle J_{\rm fluc}\rangle$. They are suppressed by a power of g as compared to the quadratic ones. At the same time, the quadratic correlator $\sim f_{abc}\langle a^b\, a^c\,\rangle$ within $\langle\xi\rangle$ is also suppressed by an additional power of g and should be suppressed to leading order.

An expansion in moments of the fluctuations has its origin in the framework of kinetic equations, which describe the coherent behaviour of the particles within some physically relevant volume, described by the plasma parameter ϵ. The fluctuations in the number of particles $\sim N_{\rm ph}^{-1/2}$ become arbitrarily small if the physical volume – or the number of particles contained in it – can be made arbitrarily large. For realistic situations, both of them are finite. Still, the fluctuations remain at least parametrically small and suppressed by the plasma parameter. Hence, the underlying expansion parameter for an expansion in moments of fluctuations is a small plasma parameter

$$\epsilon \ll 1\ . \tag{17}$$

The leading order approximation in an expansion in moments assumes $\epsilon \equiv 0$. This is the *first moment approximation* which consists in imposing

$$f = \bar{f}\ , \quad \text{or} \quad \delta f \equiv 0\ , \tag{18}$$

and neglects fluctuations throughout. Sometimes it is referred to as the *mean field* or *Vlasov approximation*. It leads to a closed system of equations for the mean one-particle distribution function and the gauge fields. In particular, the corresponding Boltzmann equation is dissipationless.

Beyond leading order, the *second moment approximation* takes into account the corrections due to correlators up to quadratic order in the fluctuations $\langle\delta f\delta f\rangle$. All higher order correlators like $\langle\delta f_1\delta f_2\dots\delta f_n\rangle = 0$ for $n > 2$ are neglected within the dynamical equations for the mean fields and the quadratic correlators. This approximation is viable if the fluctuations remain sufficiently small. We remark that the second moment approximation no longer yields a closed system of dynamical equations for the one-particle distribution function and quadratic correlators because the initial conditions for the evolution of correlators, which are given by the equal-time correlation functions as derived from the Gibbs ensemble average, involve the two-particle correlation functions g_2. Hence, we have to require that two-particle correlators remain sufficiently small as compared to products of one-particle distribution functions. This approximation is known as the *approximation of second correlation functions*, sometimes also referred to as the *polarisation approximation* [6].

For the dynamical equations of the fluctuations these approximations imply that the terms non-linear in the fluctuations should be neglected to leading order, setting

$$\eta - \langle \eta \rangle = 0 \ , \quad \xi - \langle \xi \rangle = 0 \ , \quad J_{\rm fluc} - \langle J_{\rm fluc} \rangle = 0 \ . \tag{19}$$

It permits truncating the infinite hierarchy of equations for the mean fields and the correlators of fluctuations down to a closed system of differential equations for both mean quantities and quadratic correlators. The polarisation approximation is the minimal choice necessary to genuinely describe dissipative processes, because it takes into account the feed-back of stochastic fluctuations within the particle distribution function.

Finally we notice that small fluctuations δf about $\bar{f}$ in (16) are already of the same order of magnitude as the small corrections from $g\,\bar{f}^{(1)}$ once $g\bar{f}^{(1)}/\bar{f}^{(0)} \sim N_{\rm ph}^{-1/2} \ll 1$. This means that fluctuations do contribute to the effective kinetic equations even if only small departures from the mean value $g\bar{f}^{(1)} \ll \bar{f}^{(0)}$ are considered. This shall become more explicit at the example of a plasma close to thermal equilibrium.

8 Thermal plasmas

We now turn to the specific example of a hot plasma close to thermal equilibrium. 'Hot' implies that particle masses can be neglected to leading order $m \ll T$, and that the gauge coupling, as a function of temperature, is very small $g(T) \ll 1$. To simplify the analysis we shall perform some approximations, all of which can be understood as a systematic expansion in g (and ϵ). After ensemble averaging, the distribution function f is effectively coarse-grained over a Debye volume. Fluctuations of $\bar{f}$ within a Debye volume are parametrically suppressed by powers of g. We expand f about the equilibrium distribution function $\bar{f}^{\rm eq}$ to leading order in g as

$$f = \bar{f}^{\rm eq} + g\bar{f}^{(1)} + \delta f \ . \tag{20}$$

Solving (9) in the *first moment approximation*, that is using (20) for $\delta f \equiv 0$, has been shown in ref. [7] to reproduce the HTL effective theory.

Beyond the HTL level we employ the *polarisation approximation* where $\delta f \neq 0$. It consists in discarding cubic correlator terms in (9) in favour of quadratic ones. Employing (19) means that the effects of collisions are neglected for the dynamics of the fluctuations. All these approximations can be systematically improved to higher order.

Let us consider colour excitations, described by the colour current density

$$\mathcal{J}_a^\mu(x,\mathbf{v}) = \frac{g\,v^\mu}{\pi^2}\int dQ\,dp_0\,d|\mathbf{p}|\,|\mathbf{p}|^3\,\Theta(p_0)\,\delta(p^2)\,Q_a\,f(x,p,Q)\ , \tag{21}$$

and $v^\mu \equiv p^\mu/p_0 = (1,\mathbf{v}), \mathbf{v}^2 = 1$. The current J is obtained after an angle average over the directions of $\mathbf{v}$ as $J(x) = \int \frac{d\Omega_\mathbf{v}}{4\pi}\mathcal{J}(x,\mathbf{v})$. The colour measure dQ is normalised $\int dQ = 1$ and contains the Casimir constraints of the gauge group, such as $\int dQ Q_a Q_b = C_2\delta_{ab}$ with the quadratic group Casimir $C_2 = N$ for particles in the adjoint representation of $SU(N)$ and $C_2 = \frac{1}{2}$ for particles in the fundamental[2,7]. The approximate dynamical equations for the fluctuations become

$$\left(v^\mu \bar{D}_\mu\,\delta\mathcal{J}^\rho\right)_a = -m_D^2 v^\rho v^\mu\left(\bar{D}_\mu a_0 - \bar{D}_0 a_\mu\right)_a - g f_{abc} v^\mu a_\mu^b \bar{\mathcal{J}}^{c,\rho}\ , \tag{22}$$

$$\left(\bar{D}^2 a^\mu - \bar{D}^\mu(\bar{D}a)\right)_a + 2g f_{abc}\bar{F}_b^{\mu\nu} a_{c,\nu} = \delta J_a^\mu\ , \tag{23}$$

where the quantum Debye mass for $SU(N)$ with N_F quarks and anti-quarks

$$\begin{aligned} m_D^2 &= -\frac{g^2}{\pi^2}\int_0^\infty dp p^2\left(N\frac{\bar{f}_B^{\rm eq}}{dp} + N_F\frac{\bar{f}_F^{\rm eq}}{dp}\right) \\ &= g^2T^2(2N+N_F)/6 \end{aligned} \tag{24}$$

obtains from the equilibrium (Bose-Einstein or Fermi-Dirac) distribution function and the group representation of the particles. The classical Debye mass follows by using the Maxwell-Boltzmann distribution, instead. Solving for the fluctuations in the present approximations yields a closed expression for $\delta\mathcal{J}$, and an iterative expansion in powers of the background fields for a (see ref. [2] for the explicit expressions). The seeked-for dynamical equation for the mean quasi-particle colour excitations is

$$v^\mu \bar{D}_\mu \bar{\mathcal{J}}^0 + m_D^2 v^\mu \bar{F}_{\mu 0} = C_{\rm lin}[\bar{\mathcal{J}}^0] + \zeta(x,\mathbf{v})\ , \tag{25}$$

where the linearised collision integral $C_{\rm lin}$ can now be evaluated explicitly, using the expressions for a and $\delta\mathcal{J}$. The Yang-Mills equation remains unchanged at this order. For the collision integral, one finally obtains to linear order in the mean current, and to leading logarithmic order (LLO) in $(\ln 1/g)^{-1} \ll 1$

$$\begin{aligned} C_{\rm lin}[\bar{\mathcal{J}}_a^0](x,\mathbf{v}) &= g f_{abc}\left\langle a_\mu^b(x)\,\delta\mathcal{J}^{c,\mu}(x,\mathbf{v})\right\rangle\Big|_{\bar{A}=0,\ \text{linear in}\ \bar{J},\ \text{LLO}} \\ &= -\gamma\int\frac{d\Omega'}{4\pi} I(\mathbf{v},\mathbf{v}')\bar{\mathcal{J}}_a^0(x,\mathbf{v}') \end{aligned} \tag{26}$$

with the kernel $I(\mathbf{v},\mathbf{v}') = \delta^2(\mathbf{v}-\mathbf{v}') - \frac{4}{\pi}(\mathbf{v}.\mathbf{v}')^2/\sqrt{1-(\mathbf{v}.\mathbf{v}')^2}$. Notice that the collision integral is local in coordinate space, but non-local in the angle variables. The rate $\gamma = \frac{g^2 T}{4\pi}\ln 1/g$ is (twice) the hard gluon damping rate. In obtaining (26), we introduced an infra-red cut-off $\sim gm_D$ for the elsewise unscreened magnetic sector. This result has been first obtained by Bödeker in ref. [8]. For alternative derivations, see ref. [10,11]. In addition, the source for stochastic noise ζ in (25) can be identified as

$$\zeta_a(x,\mathbf{v}) = g f_{abc} a^b_\mu(x)\,\delta\mathcal{J}^{c,\mu}(x,\mathbf{v})\big|_{\bar{A}=0,\,\bar{J}=0}\ . \tag{27}$$

Making use of the basic correlator (5), its self-correlator follows to LLO as

$$\langle\zeta^a(x,\mathbf{v})\zeta^b(y,\mathbf{v}')\rangle = 2\gamma\, m_D^2\, T\, I(\mathbf{v},\mathbf{v}')\,\delta^{ab}\,\delta(x-y)\ . \tag{28}$$

Notice that the strength of the correlator is determined by the kernel of the collision integral, e.g. by the dissipative process. Next, we show the close relationship to the fluctuation-dissipation theorem (FDT).

9 Fluctuation-dissipation theorem

It is well-known that dissipation in a quasi-stationary plasma (= no entropy production) is intimately linked to the fluctuations, a link which is given by the FDT. We shall argue that the FDT holds true for the above set of equations. To that end, following ref. [3], we consider the coarse-grained kinetic equation (4) and ask how the spectral functions of the noise source ζ and of f have to be related to $C[f]$ in order to satisfy the FDT.

Within classical transport theory, the FDT is implemented in a straightforward way[12]. The pivotal element is the kinetic entropy $S[f]$. We consider small deviations of f from the equilibrium, $f = f^{\rm eq} + \Delta f$. The entropy, stationary at equilibrium, is expanded to quadratic order in Δf, $S = S_{\rm eq} + \Delta S$. Defining the thermodynamical force in the usual way as $F(z) = -\delta(\Delta S)/\delta(\Delta f(z))$, where $z \equiv (x,\mathbf{p},Q)$, we obtain

$$F(x,\mathbf{p},Q) = \Delta f(x,\mathbf{p},Q)/\bar{f}_{\rm eq} \tag{29}$$

for a classical plasma. The thermodynamical force for the quantum case follows from (29) using (6). Given $C[f](z)$, it can be shown[3,12] that

$$\langle\zeta(z)\zeta(z')\rangle = -\left(\frac{\delta C(z)}{\delta F(z')} + \frac{\delta C(z')}{\delta F(z)}\right) \tag{30}$$

is the required noise-noise self-correlator for ζ compatible with the FDT. For the particular example studied above, (28) follows from inserting (26) into

(30), proving that Bödeker's effective theory is compatible with the FDT. Furthermore, the correlator $\langle \Delta f \Delta f \rangle|_{t=t'}$ can be derived along the same lines and agrees, to leading order, with $\langle \delta f \delta f \rangle|_{t=t'}$ derived from the Gibbs ensemble average. This guarantees that the formalism is consistent with FDT.

10 Summary

We have reviewed a semi-classical approach to derive effective transport equations for QCD based on 'integrating-out' fluctuations about some mean fields. Most interestingly, the approach is applicable for in- and out-of-equilibrium situations, opening a door for future applications to out-of-equilibrium plasmas. The formalism is consistent with the underlying non-Abelian mean field symmetry, and allows for systematic expansions controlled by a small plasma parameter which is at the basis for any reliable computation.

For thermal plasmas we have discussed how the collision integral as well as the necessary noise source follow explicitly from the microscopic theory to leading logarithmic accuracy, reproducing Bödeker's effective theory, and establishing the consistency with the fluctuation-dissipation theorem.

As a final comment we point out that the effective theory is the same for a classical or a quantum plasma, differing only in the equilibrium distribution function, and hence in the corresponding value for the Debye mass. The sole 'quantum' effect which entered the computation resides in the non-classical statistics of the particles, which is all that is needed to correctly describe a hot *quantum* non-Abelian plasma close to equilibrium at the present order of accuracy.

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