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CLASSIFICATION OF IRREGULAR ALGEBRAIC FIBER SURFACES OF GENUS 3

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Abstract

A classification of algebraic fiber surfaces of genus 3 with relative irregularity 2 is given in this note.

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Introduction

Let S be a smooth complex projective surface, and $f: S \to B$ be a (relatively minimal) fibration of genus $g \geq 2$. The relative irregularity q_f of f, defined by $q_f = h^0(R^1 f_* \mathcal{O}_S)$, plays a special important role in the classification theory of fiber surfaces. It is well known that $q_f \leq g$, and the equality holds iff f is a trivial fibration. This fact suggests that fiber surfaces with large relative irregularity (with respective to g) can be hopefully classified. Indeed, the fiber surfaces of genus 2 with $q_f = 1$ were completely classified by Xiao [X1]. The aim of this note is to give a classification of fiber surfaces of genus 3 with $q_f = 2$.

Let $f: S \to B$ be a fiber surface of genus 3 of variable moduli with $q_f = 2$. The constant part A_f of the Jacobian fibration of f with the natural polarization Θ_{A_f} induced by the Jacobian of the fiber is a point (A_f, Θ_{A_f}) of $\mathcal{A}_{1,d}$, for some $d \geq 2$, where $\mathcal{A}_{1,d}$ is the moduli space for polarized Abelian surfaces of type $(1, d)$. Let $(X, \Theta_X) \in \mathcal{A}_{1,d}$. We say that f is *of type* (X, Θ_X) if $(A_f, \Theta_{A_f}) \simeq (X, \Theta_X)$. Fix a symplectic isomorphism $\beta: K(\Theta_{A_f}) \to \mathbb{Z}_d \oplus \mathbb{Z}_d$, where $K(\Theta_{A_f})$ is the kernel of the natural homomorphism from A_f to its dual $\hat{A}_f$ defined by Θ_{A_f}, then there is a unique morphism $\nu_{f,\beta}: B \to \mathcal{C}_d$, where $\mathcal{C}_d$ is the principal modular curve of level d (see Proposition 1.3). Our main result is the following

Theorem. *Let $d \geq 3$, and $(X, \Theta_X) \in \mathcal{A}_{1,d}$. Given (and fix) a symplectic isomorphism*

$$\beta_d: K(\Theta_X) \to \mathbb{Z}_d \oplus \mathbb{Z}_d.$$

Assume that X is simple, and $\mathrm{End}_{\mathbb{Q}}(X) \not\supset \mathbb{Q}(\sqrt{-3})$ when $d = 3$. Then there exists a unique fiber surface $f_{X,\Theta_X}: S_{X,\Theta_X} \to C_d$, of genus 3 of variable moduli with the following properities:

(i) For any base change $b: B \to \mathcal{C}_d$, let $f: S \to B$ be the pullback of f_{X,Θ_X} under b. If f has a section, then f is of type (X, Θ_X).

(ii) (The universal property of f_{X,Θ_X}) Any fiber surface $f: S \to B$ of type (X, Θ_X), with given $\beta = \beta_d$, is the pullback of $f_{(X,\Theta_X)}$ under the unique base change $\nu_{f,\beta}: B \to \mathcal{C}_d$.

The key point of the proof is to construct a family of principally polarized Abelian threefolds containing the given X as subvariety. The construction is inspired by Xiao's work on the classification of fiber surfaces of genus 2 with $q_f = 1$ (cf. [X1, Chap. 3; or X3, Chap. 4].

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1. Proof of the theorem

A fiber surface of genus g is a smooth projective complex surface with a relatively minimal fibration whose general fiber is a connected curve of genus g.

Let $f: S \to B$ be a fiber surface of genus g, F a general fiber of f. We define the *relative irregularity* of f by

$$q_f = h^0(R^1 f_* \mathcal{O}_S).$$

By Leray spectral sequence, one has $q_f = q(S) - g(B)$. We have that $q_f \leq g(F)$, and equality holds iff f is trivial. We say that f is of variable moduli if $m_f: B \to \mathcal{M}_g$, induced by f, is not a constant map; otherwise, we say that f is of constant moduli.

Now let $f: S \to B$ be a fiber surface of genus 3 of variable moduli with $q_f = 2$. We set

$$\mathcal{E} = B \times_{\mathrm{Jac}(B)} \mathrm{Alb}(S),$$

where Alb(S) is the Albanese variety of S. Let $p: \mathcal{E} \to B$ be the projection. Then p is a locally trivial fibration of Abelian surfaces. Denote by A the fiber of p. We have a natural commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\tau} & \mathcal{E} \\ f\downarrow & & p\downarrow \\ B & \xrightarrow{=} & B, \end{array}$$

where $\tau = (f, \text{ alb})$, alb: $S \to$ Alb(S) is the Albanese map. Let F be a general fiber of f. Denote by $(\mathrm{J}(F), \Theta_{\mathrm{J}(C)})$ the Jacobian of F. We have a natural commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\tau_F} & A \\ \downarrow & & h_F\uparrow \\ \mathrm{J}(F) & \xrightarrow{=} & \mathrm{J}(F), \end{array}$$

where τ_F is the restriction of τ on F, and h_F is obtained by the universal property of $\mathrm{J}(F)$.

Let $E_F = \mathrm{Ker}(h_F)$. Then E_F is an elliptic curve (by the universal property of the Albanese map, we have that $\mathrm{Ker}(h_F)$ is connected). Let

$$d = \deg \iota_{E_F}^* \Theta_{\mathrm{J}(F)},$$

where (and in the sequel) we denote by ι_Y the embedding morphism of a variety Y into a variety which is clear from the content. Clearly, d is independent of the choice of the general fiber F. We have that $d \geq 2$.

Let A_F be the Abelian subvariety of $\mathrm{J}(F)$ complementary to E_F (cf. e.g. [L-B, Chap. 5] for the definition and properties of the complementary Abelian subvariety.) Then A_F is isogenous to A. Since f is of variable moduli by assumption, we have that A_F is independent of the choice of the general fiber F. Denote A_F by A_f. A_f is the constant part of the Jacobian fibration of f. Let $\Theta_{A_f} = \iota_{A_f}^* \Theta_{\mathrm{J}(F)}$ be the induced polarization of $\Theta_{\mathrm{J}(F)}$ on A_f. One has that Θ_{A_f} is of type $(1, d)$ (cf. [L-B, Corollary 12.1.5]).

Let $\mathcal{A}_{1,d}$ be the moduli space for polarized Abelian surfaces of type $(1, d)$.

Definition 1.1. Let $(X, \Theta_X) \in \mathcal{A}_{1,d}$. A fiber surface $f: S \to B$ of genus 3 of variable moduli with $q_f = 2$ is said to be of type (X, Θ_X) if $(A_f, \Theta_{A_f}) \simeq (X, \Theta_X)$.

Notation 1.2. When (Y, Θ_Y) is a polarized Abelian variety, we denote by $\hat{Y}$ its dual, $K(\Theta_Y)$ the kernel of the natural homomorphism $A \to \hat{Y}$ defined by Θ_Y, and e^{Θ_Y} the Weil pairing on $K(\Theta_Y)$.

Let $d \geq 3$, and $(X, \Theta_X) \in \mathcal{A}_{1,d}$. A symplectic isomorphism

$$\beta: K(\Theta_X) \to \mathbb{Z}_d \oplus \mathbb{Z}_d$$

is an isomorphism β satisfying $e^{\Theta_X}(x_1, x_2) = e_d(\beta(x_1), \beta(x_2))$ for any $x_1, x_2 \in K(\Theta_X)$, where we denote by e_d the alternating form on $\mathbb{Z}_d \oplus \mathbb{Z}_d$ defined by

$$e_d((1,0),(0,1)) = \exp(\frac{2\pi i}{d}).$$

Let E be an elliptic curve. Denote by Θ_E the canonical principal polarization. A d-level structure on (E, Θ_E) is a symplectic isomorphism

$$\alpha: E[d] := K(d\Theta_E) \to \mathbb{Z}_d \oplus \mathbb{Z}_d.$$

Denote by $\mathcal{C}'_d$ the moduli space for elliptic curves of level d, and $\mathcal{C}_d$ the compactification of $\mathcal{C}'_d$.

Proposition 1.3. *Let $d \geq 3$, $(X, \Theta_X) \in \mathcal{A}_{1,d}$, and $f: S \to B$ be of type (X, Θ_X). Fix a symplectic isomorphism $\beta: K(\Theta_X) \to \mathbb{Z}_d \oplus \mathbb{Z}_d$. Then there is a unique morphism*

$$\nu_{f,\beta}: B \to \mathcal{C}_d$$

with the following property: for each $t \in B^o := B-$critical points of f, let $\nu_{f,\beta}(t) = [(E_{z(t)}, \alpha_{z(t)})] \in \mathcal{C}'_d$, there is an isogenous of polarized Abelian varieties

$$h_t: (E_{z(t)} \times X, dp_1^*\Theta_{E_{z(t)}} + p_2^*\Theta_X) \to (\mathrm{J}(f^*t), \Theta_{\mathrm{J}(f^*t)}).$$

Proof. By the definition, we have that $X \hookrightarrow \mathrm{J}(f^*t)$, and $\Theta_X = \iota^*\Theta_{\mathrm{J}(f^*t)}$. Let $E_{z(t)}$ be the Abelian subvariety of $\mathrm{J}(f^*t)$ complementary to X. We have an isogenous of polarized Abelian varieties

$$h_t := \iota_{E_{z(t)}} + \iota_X: (E_{z(t)} \times X, dp_1^*\Theta_{E_{z(t)}} + p_2^*\Theta_X) \to (\mathrm{J}(f^*t), \Theta_{\mathrm{J}(f^*t)}).$$

Now the kernel of h_t, regarded as the graph of $E_{z(t)}[d] \times K(\Theta_X)$, defines an isomorphism $\gamma_t: E_{z(t)}[d] \to K(\Theta_X)$. Since $dp_1^*\Theta_{E_{z(t)}} + p_2^*\Theta_X) = h_t^*\Theta_{\mathrm{J}(f^*t)}$, by [Mu, Corollary p. 231], we have that

$$e^{d\Theta_{E_{z(t)}}}(r_1, r_2) e^{\Theta_X}(\gamma_t(r_1), \gamma_t(r_2)) = 1$$

for any $r_1, r_2 \in E_{z(t)}[d]$. So there exists a unique symplectic isomorphism

$$\alpha_{z(t)}: E_{z(t)}[d] \to \mathbb{Z}_d \oplus \mathbb{Z}_d$$

such that

$$e_d(\alpha_{z(t)}(r_1), \alpha_{z(t)}(r_2)) = e_d(\beta\gamma_t(r_1), \beta\gamma_t(r_2))^{-1}.$$

Now we have a unique morphism

$$\nu'_{f,\beta}: B^o \to \mathcal{C}'_d$$

defined by setting $\nu'_{f,\beta}(t) = [(E_{z(t)}, \alpha_{z(t)})]$. Clearly $\nu'_{f,\beta}$ can be extended uniquely to a morphism

$$\nu_{f,\beta}: B \to \mathcal{C}_d.$$

Since f is of variable moduli, $\nu_{f,\beta}$ is surjective. □

Theorem 1.4. *Let $d \geq 3$, and $(X, \Theta_X) \in \mathcal{A}_{1,d}$. Given (and fix) a symplectic isomorphism*

$$\beta_d: K(\Theta_X) \to \mathbb{Z}_d \oplus \mathbb{Z}_d.$$

Assume that X is simple, and $\mathrm{End}_{\mathbb{Q}}(X) \not\supset \mathbb{Q}(\sqrt{-3})$ when $d = 3$. Then there exists a unique fiber surface $f_{X,\Theta_X}: S_{X,\Theta_X} \to C_d$, of genus 3 of variable moduli with the following properities:

(1.4.1) For any base change $b: B \to \mathcal{C}_d$, let $f: S \to B$ be the pullback of f_{X,Θ_X} under b. If f has a section, then f is of type (X, Θ_X).

(1.4.2) (The universal property of f_{X,Θ_X}) Any fiber surface $f: S \to B$ of type (X, Θ_X), with given $\beta = \beta_d$, is the pullback of $f_{(X,\Theta_X)}$ under the unique base change $\nu_{f,\beta}: B \to \mathcal{C}_d$, defined as in Proposition 1.3.

To prove Theorem 1.4, we need the following lemmas.

Lemma 1.5. *Let $(X, \Theta_X) \in \mathcal{A}_{1,d}$, and F be a curve of genus 3. Assume that $X \hookrightarrow \mathrm{J}(F)$, and $\Theta_X = \iota_X^* \Theta_{\mathrm{J}(F)}$. Then there is a natural morphism of degree d*

$$\pi: F \to E,$$

where E is the Abelian subvariety of $\mathrm{J}(F)$ complementary to X. Moreover, π does not factorize via any etale cover $e: E' \to E$, of degree ≥ 2.

Proof. Let π be the composition:

$$F \xrightarrow{\iota_F} \mathrm{J}(F) \xrightarrow{\sim} \widehat{\mathrm{J}(F)} \xrightarrow{\hat{\iota}_E} \hat{E} \xrightarrow{\sim} E.$$

Identifying $\mathrm{J}(F)$ (resp. E) with $\widehat{\mathrm{J}(F)}$ (resp. $\hat{E}$), we have that

$$\pi^* = \iota_F^* \circ (\hat{\iota}_E)^* = \iota_F^* \circ \widehat{\hat{\iota}_E} = \iota_F^* \circ \iota_E: E \to \mathrm{J}(F).$$

Since $\iota_F^*: \mathrm{Pic}^0(\mathrm{J}(F)) \to \mathrm{Pic}^0(F)$ is an isomorphism, the lemma follows from [L-B, Proposition 11.4.3, Lemma 12.3.1]. □

Lemma 1.6. *Let $f: S \to B$ be a fiber surface of genus 3 of variable moduli, and F the general fiber of f. Then we have that $\mathrm{Aut}(F) = \{\mathrm{id}_F\}$ unless F is either hyperelliptic or bielliptic or $3{:}1$ cyclic cover over an elliptic curve.*

Proof. If $\mathrm{Aut}(F) \neq \{\mathrm{id}_F\}$, we can choose a $\sigma \in \mathrm{Aut}(F)$ of prime order. Consider the quotient map $\alpha: F \to C := F/ < \sigma >$. If $g(C) = 2$, then F is hyperelliptic (cf. [Ac]). If $g(C) \leq 1$, using the Hurwitz formula for α, we have that

$$\mathrm{ord}(\sigma) = \frac{4 + r}{2g(C) - 2 + r},$$

where r is the number of branch points of α. Note that since f is of variable moduli, we have $r \geq 4$ if $g(C) = 0$. So we have

$$(\mathrm{ord}(\sigma), g(C), r) = (2, 0, 8), (2, 1, 4), \text{or } (3, 1, 2). \qquad \square$$

Proof of Theorem 1.4. Denote by $\mathbb{H}_n$ the Siegel upper half space of complex symmetric $n \times n$-matrices with positive definite imaginary part. Let $\Lambda_X = H_1(X, \mathbb{Z})$, $V = H^0(X, \Omega_X^1)^*$. Then $X \simeq V/\Lambda_X$. With respect to a symplectic basis of Λ_X for Θ_X, the period matrix of X is of the form

$$P_X = \begin{pmatrix} z_1 & z_2 & 1 & 0 \\ z_2 & z_3 & 0 & d \end{pmatrix}^t$$

for some $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathbb{H}_2$. Now Λ_X is the lattice spanned by the row of P_X, and

$$K(\Theta_X) = (\mathbb{Z}(\frac{z_2}{d}, \frac{z_3}{d}) + \mathbb{Z}(0, 1) + \Lambda_X)/\Lambda_X.$$

Let

$$G_d = \{A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathrm{GL}(4, \mathbb{Z}) \mid A \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} A^t = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}\},$$

where $D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$. G_d acts on $\mathbb{H}_2$ by

$$A(Z) = (A_1 Z + A_2 D)(D^{-1}A_3 Z + D^{-1}A_4 D)^{-1}, \text{ for } Z \in \mathbb{H}_2.$$

For a suitable choice Z in the orbit $G_d \cdot Z$, we can assume that β_d is given by

$$\beta_d(m(\frac{z_2}{d}, \frac{z_3}{d}) + n(0,1) \mod \Lambda_X) = (m,n), \text{where } 0 \le m,\ n < d.$$

Let

$$p\colon\ \mathcal{V} := \mathbb{H}_1 \times \mathbb{C}^3 \to \mathbb{H}_1$$

be the trivial vector bundle over $\mathbb{H}_1$. We define sections u_i $(i = 1, \cdots, 8)$ of p as follows. For each $z \in \mathbb{H}_1$, Let

$$\begin{aligned}
&u_1(z) = (z,\ 0,\ 0),\quad u_2(z) = (0,\ z_1,\ z_2),\\
&u_3(z) = (0,\ z_2,\ z_3),\quad u_4(z) = (1,\ 0,\ 0),\\
&u_5(z) = (0,\ 1,\ 0),\quad u_6(z) = (0,\ 0,\ d),\\
&u_7(z) = u_3(z) + u_4(z) = (\frac{1}{d},\ \frac{z_2}{d},\ \frac{z_3}{d}),\\
&u_8(z) = u_1(z) + u_6(z) = (\frac{z}{d},\ 0,\ 1).
\end{aligned}$$

Let $\mathcal{U}$ be the discrete subgroup of sections of $\mathcal{V}$ generated by u_i $(i = 1, \cdots, 8)$. Then we have that

$$j\colon \mathcal{J} := \mathcal{V}/\mathcal{U} \to \mathbb{H}_1$$

is a family of Abelian 3-folds.

Clearly u_i $(i = 1, \cdots, 6)$ is a basis of $\mathcal{V}$ considered as a real vector bundle. There is a real valued alternating form E on $\mathcal{V}$ satisfying

$$\begin{aligned}
&E(u_1, u_4) = d,\quad E(u_1, u_j) = 0 \text{ for } j \neq 4,\ 1 \le j \le 6,\\
&E(u_2, u_5) = 1,\quad E(u_2, u_j) = 0 \text{ for } j \neq 5,\ 1 \le j \le 6,\\
&E(u_3, u_6) = d,\quad E(u_3, u_j) = 0 \text{ for } j \neq 6,\ 1 \le j \le 6, \text{and}\\
&E(u_k, u_j) = 0 \text{ for } 4 \le k < j \le 6.
\end{aligned}$$

It is easy to verify that

$$\begin{aligned}
&E(u_7, u_1) = -1,\quad E(u_7, u_6) = 1,\quad E(u_7, u_j) = 0 \text{ for } j \neq 1, 6,\ 1 \le j \le 8,\\
&E(u_8, u_3) = -1,\quad E(u_8, u_4) = 1,\quad E(u_8, u_j) = 0 \text{ for } j \neq 3, 4,\ 1 \le j \le 8.
\end{aligned}$$

With respect to the basis u_1, u_2, u_3, u_7, u_5, u_8 of $\mathcal{U}$, E is given by the matrix

$$\begin{pmatrix} 0 & 1_3 \\ -1_3 & 0 \end{pmatrix},$$

where 1_3 is the identity 3×3-matrix. So E is integer-valued and unimodular on $\mathcal{U}$.

The period matrix of $j^{-1}(z)$ ($z \in \mathbb{H}_1$), with respect to the basis u_1, u_2, u_3, u_7, u_5, u_8 of $\mathcal{U}$, is given by the matrix

$$\Pi = \begin{pmatrix} z & 0 & 0 & \frac{1}{d} & 0 & \frac{z}{d} \\ 0 & z_1 & z_2 & \frac{z_2}{d} & 1 & 0 \\ 0 & z_2 & z_3 & \frac{z_3}{d} & 0 & 1 \end{pmatrix}^t .$$

It is easy to verify that

$$\Pi^t \begin{pmatrix} 0 & -1_3 \\ 1_3 & 0 \end{pmatrix} \Pi = 0 \text{ and}$$

$$i\Pi^t \begin{pmatrix} 0 & -1_3 \\ 1_3 & 0 \end{pmatrix} \bar{\Pi} = \begin{pmatrix} 2\operatorname{im}\frac{z}{d} & 0 & 0 \\ 0 & 2\operatorname{im} z_1 & 2\operatorname{im} z_2 \\ 0 & 2\operatorname{im} z_2 & 2\operatorname{im} z_3 \end{pmatrix} > 0.$$

Consequently, there is a relatively ample invertible sheaf $\mathcal{L}$ on $\mathcal{J}$ for j such that $\mathcal{L}|_{j^{-1}(z)}$ is a principal polarization on $j^{-1}(z)$ defined by $E|_{p^{-1}(z)}$.

Now we define an action of $\Gamma(d)$, the principal congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, on $\mathcal{V}$ by setting

$$(1.7) \qquad \gamma(z, t_1, t_2, t_3) = (\gamma(z), \frac{t_1}{a_3 z + a_4}, t_2, t_3)$$

for each $\gamma = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \Gamma(d)$, where $\gamma(z) = \frac{a_1 z + a_2}{a_3 z + a_4}$.

We have that, for each $\gamma = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \Gamma(d)$,

$$(1.8) \qquad \begin{aligned} &(\gamma u_i)(z) = u_i(\gamma(z)) \text{ for } i = 2, 3, 5, 6, \\ &(\gamma u_1)(z) = a_4 u_1(\gamma(z)) - a_2 u_4(\gamma(z)), \\ &(\gamma u_4)(z) = a_1 u_4(\gamma(z)) - a_3 u_1(\gamma(z)), \\ &(\gamma u_7)(z) = u_7(\gamma(z)) - \frac{a_3}{d} u_1(\gamma(z)) + \frac{a_1 - 1}{d} u_4(\gamma(z)), \\ &(\gamma u_8)(z) = u_8(\gamma(z)) + \frac{a_4 - 1}{d} u_1(\gamma(z)) - \frac{a_2}{d} u_4(\gamma(z)). \end{aligned}$$

So $\mathcal{U}$ is invariant under the action of $\Gamma(d)$. Hence (1.7) induces an action of $\Gamma(d)$ on $\mathcal{J}$, which is clearly compatible with the map $j: \mathcal{J} \to \mathbb{H}_1$, with the natural action of $\Gamma(d)$ on $\mathbb{H}_1$. By (1.8), it's easy to verify that E is invariant under the action of $\Gamma(d)$. So

$$\bar{j}: \mathcal{J}/\Gamma(d) \to \mathbb{H}_1/\Gamma(d) = \mathcal{C}'_d$$

with the relatively ample invertible sheaf $\bar{\mathcal{L}} = \mathcal{L}/\Gamma(d)$ is a fiber space of principally polarized Abelian 3-folds.

(1.9) By the construction, we have that, for each $[z] = [(E_z, \alpha_z)] \in C'_d$, where $E_z = \mathbb{C}^1/\Lambda_z, \Lambda_z = \mathbb{Z}z + \mathbb{Z}$, and α_z is given by $\alpha_z(m\frac{z}{d} + n\frac{1}{d} \bmod \Lambda_z) = (m, n)$, there is an isogenous $\phi_z: E_z \times X \to \bar{j}^{-1}([z])$ defined in a natural way. We have that

$$\operatorname{Ker}(\phi_z) = \{(m\frac{z}{d} + n\frac{1}{d} \bmod \Lambda_z, n(\frac{z_2}{d}, \frac{z_3}{d}) + m(0,1) \bmod \Lambda_X) |\ 0 \le n, m < d\}.$$

Clearly, $\mathrm{Ker}(\phi_z)$ is the graph the automorphism $\gamma_z := \beta_d^{-1} \circ \tau \circ \alpha_z : E_z[d] \to K(\Theta_X)$, where τ is the permutation of the two factors of $\mathbb{Z}_d \oplus \mathbb{Z}_d$. It is easy to verify that

$$e_d(\alpha_z(r_1), \alpha_z(r_2)) e_d(\beta_d \gamma_z(r_1), \beta_d \gamma_z(r_2)) = 1,$$

that is,

$$e^{d\Theta_{E_z}}(r_1, r_2) e^{\Theta_X}(\gamma_z(r_1), \gamma_z(r_2)) = 1$$

for any $r_1, r_2 \in E_z[d]$. So there is a polarization Θ_z on $E_z \times X / \mathrm{Ker}(\phi_z) \simeq \bar{j}^{-1}([z])$, such that $\phi_z^* \Theta_z = d p_1^* \Theta_{E_z} + p_2^* \Theta_X$. Since Θ_z and $\bar{\mathcal{L}}_{[z]}$ induce the same polarization on $\phi_z(X)$ and on $\phi_z(E_z)$, we have that $\Theta_z = \bar{\mathcal{L}}_{[z]}$ as polarizations.

Claim 1.10. For each $[z] \in C'_d$, $(\bar{j}^{-1}([z]), \bar{\mathcal{L}}_{[z]} := \bar{\mathcal{L}}|_{\bar{j}^{-1}([z])})$ is isomorphic to the Jacobian of a smooth curve of genus 3.

Proof of the claim. It is well known that $(\bar{j}^{-1}(t), \bar{\mathcal{L}}_{[z]})$ is isomorphic to either the Jacobian of a smooth curve of genus 3 or the principally polarized product of an Abelian surface with an elliptic curve. We show that the latter case does not occur. Otherwise, $((\bar{j}^{-1}(t), \bar{\mathcal{L}}_{[z]}) \simeq (A \times E, p_1^* \Theta_A + p_2^* \Theta_E)$, where (A, Θ_A) (resp. (E, Θ_E)) is a principally polarized Abelian surface (resp. an elliptic curve). Since X is simple, we have that A is also simple. So $p_1(\phi_z(E_z \times \mathrm{pt})) = \mathrm{pt}$, that is, $\phi_z(E_z \times \mathrm{pt})$ is a fiber of $p_1 : j^{-1}(z) \simeq A \times E \to A$. By (1.9), we have that $\phi_z|_{E_z \times \mathrm{pt}} : E_z \times \mathrm{pt} \to \bar{j}^{-1}([z])$ is an embedding. Now by (1.9) and the projection formula, we have that

$$\begin{aligned} d &= (E_z \times \mathrm{pt}) \cdot (p_1^* d\Theta_{E_z} + p_2^* \Theta_X) = (E_z \times \mathrm{pt}) \cdot \phi_z^* \bar{\mathcal{L}}_{[z]} \\ &= (\phi_z(E_z \times \mathrm{pt})) \cdot \bar{\mathcal{L}}_{[z]} = (\phi_z(E_z \times \mathrm{pt})) \cdot (p_1^* \Theta_A + p_2^* \Theta_E) = 1. \end{aligned}$$

This is a contradiction. □

Let $\mathcal{M}_3$ (resp. $\mathcal{A}_3$) be the coarse moduli space for curves of genus 3 (resp. principally polarized Abelian 3-folds). We consider the Torelli map $t : \mathcal{M}_3 \to \mathcal{A}_3$, which is an immersion (cf. [O-S]). By (1.10), the morphism $m_{\bar{j}} : C'_d \to \mathcal{A}_3$, induced by $\bar{j}$, factors through t, i.e., there is a morphism $m : C'_d \to \mathcal{M}_3$ such that $m_{\bar{j}} = t \circ m$.

By the property of the coarse moduli of $\mathcal{M}_3$, we have that there exist a finite base change $b : B' \to C'_d$, and a fiber surface $f' : S' \to B'$, such that the moduli map $m_{f'}$ induced by f' factors through m.

We claim that we can take $B' = C'_d$, and $b =$ the identity map. In fact, it's enough to show that the general fiber F of f' has no nontrivial automorphisms.

(1.11) We note that F is not hyperelliptic. Otherwise, let f'' be (the compactization and desingularization of) the pullback of f' under a suitable base change such that f'' has a section, then we have $q_{f''} = 2$ by the same argument in (1.12) below. By [X2, Theorem 1], f'' is of constant moduli. This is a contradiction.

Now if $\mathrm{Aut}(F) \neq \mathrm{id}_F$, then F is either bielliptic or 3:1 cyclic cover over an elliptic curve by Lemma 1.6 and (1.11). On the other hand, since $(X, \Theta_X) \hookrightarrow \mathrm{J}(F)$ by the construction, F admits a covering over an elliptic curve of degree d by Lemma 1.5. Note that when $d = 3$, the condition $\mathrm{End}_{\mathbb{Q}}(X) \not\supset \mathbb{Q}(\sqrt{-3})$ implies that such a covering is not cyclic. Now F admits two coverings such that neither of them factors through the other (by Lemma 1.5). So $\mathrm{J}(F)$ is isogenous to a product of three elliptic curves. This contradicts the assumption that X is simple.

Now let $f_{X,\Theta_X} : S_{X,\Theta_X} \to C_d$, be the (minimal) fiber surface obtained from f' by compactization and desingularization.

(1.12) For any given surjective morphism $b: B \to \mathcal{C}_d$, let $f: S \to B$ be the pullback of f_{X,Θ_X} under base change b. Assume that f has a section s. Let B^o be the Zariski open subset of B such that f is smooth over B^o. Let $i: B^o \to B$ be the inclusion. For each $t \in B^o$, composing the Albanese map $\lambda_{s(t)}: f^*t \to J(f^*t)$ with the projection $J(f^*t) \to \hat{X}$, we get a map

$$\varphi_t: f^*t \to \hat{X}.$$

Let $V_{\mathbb{Q}}$ be the constant sheaf on B^o with fiber $H^1(\hat{X},\ \mathbb{Q})$. The maps

$$\varphi_t^*: H^1(\hat{X},\ \mathbb{Q}) \to H^1(f^*t,\ \mathbb{Q}),$$

induced by the φ_t fit together to give an injection of $i_*V_{\mathbb{Q}}$ in $R^1f_*\mathbb{Q}$, where $R^1f_*\mathbb{Q}$ is the first direct image sheaf of the constant sheaf $\mathbb{Q}$ over S. The Leray spectral sequence induces an exact sequence

$$0 \to H^1(B,\ \mathbb{Q}) \to H^1(S,\ \mathbb{Q}) \to H^0(B,\ R^1f_*\mathbb{Q}) \to 0.$$

Consequently, we have $q_f \geq 2$. On the other hand, since f is of variable moduli, we have that $q_f < 3$. So f is of type (X, Θ_X). This proves (1.4.1).

Note that the Jacobian of $f^*_{X,\Theta_X}([z])$ $([z] \in \mathcal{C}'_d \subset \mathcal{C}_d)$ is isomorphic to $(\bar{j}^{-1}([z]), \bar{\mathcal{L}}_{[z]})$. (1.4.2) follows by Proposition 1.3 and by the fact that the general fiber F of f_{X,Θ_X} has no nontrivial automorphisms. This completes the proof of Theorem 1.4. $\square$

2. Remarks and examples

Remark 2.1. Notations as in section 1. By the proof of (1.4.1), if f_{X,Θ_X} has a section, then $q_{f_{X,\Theta_X}} = 2$, and f_{X,Θ_X} itself is of type (X, Θ_X). In general, it is difficult to decide whether $q_{f_{X,\Theta_X}} = 2$ or not.

Remark 2.2. Let $f: S \to B$ be a fiber surface of genus 3 of variable moduli with $q_f = 2$ of type (X, Θ_X). By Lemma 1.5, we have that $(X, \Theta_X) \in \mathcal{A}_{1,2}$ iff f is bielliptic (that is, its general fiber is bielliptic). Theorem 0.2 in [Ca] is incorrect as stated. We note that $\tau': F \to A$ (p.285, line 12 from below) is birational and not an embedding in general. The corrected version of Theorem 0.2 in [Ca] should be read as follows

Let $f: S \to B$ be a bielliptic fiber surface of genus 3 with $q_f = 2$. Suppose that f is of variable moduli. Then f is associated to f_Γ for some , $\subset \Phi^{-1}(A)$, and some $A \in \mathcal{A}_2$.

The proof of the above statement is simple: let $\pi: F \to E_F$ be the bielliptic cover given by f, where F is a general fiber of f. We have that $A := \mathrm{J}(F)/\pi^*E_F$ is isogenous to the constant part A_f of the Jacobian fibration of f. So A is independent of the choice of F since f is of variable moduli. Since $\pi: F \to E_F$ is bielliptic, $F \hookrightarrow A$ by [Ba, Prop. 1.8]. So F defines a polarization Θ_A of type $(1,2)$ on A. Now the image of the morphism $v: B^o \to \mathcal{H}_3$, induced by f, is contained in $\Psi^{-1}((A,\ \Theta_A))$. We get the result by copying the last paragraph of section 3 in [Ca].

Now we give examples of fiber surfaces of genus 3 with $q_f = 2$ not covered by Theorem 1.4.

Example 2.3. Examples of fiber surfaces of genus 3 of type (X, Θ_X) with X non-simple.

Given six distinct points p_i $(i = 1, \cdots, 6)$ of $\mathbb{P}^1$. Let $\pi_1: E_1 \to \mathbb{P}^1$ (resp. $\pi_2: E_2 \to \mathbb{P}^1$) be the double cover ramified exactly at points p_1, p_2, p_3 and p_4 (resp. p_3, p_4, p_5 and p_6). Consider the diagram

$$\begin{array}{ccc} C' := E_1 \times_{\mathbb{P}^1} E_2 & \xrightarrow{q_1} & E_1 \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ E_2 & \xrightarrow{\pi_2} & \mathbb{P}^1. \end{array}$$

We have that $C' \hookrightarrow E_1 \times E_2$ is irreducible of arithmetic genus 5 with exactly two nodes. Let C be the normalization of C'. Denote also by q_j the composition $C \to C' \xrightarrow{q_j} E_j$. Since $q_j: C \to E_j$ is a bielliptic cover, one has that $C \hookrightarrow A_j := \mathrm{J}(C)/q_j^* E_j$ (cf.[Ba, Prop. 1.8]). Let $S_j \to A_j$ be the blowing up of the base points of $|C|$ on A_j. We get a fiber surface $f: S_j \to \mathbb{P}_1$, of genus 3 of variable moduli with $q_f = 2$ of type $(\hat{A}_j, \Theta_{\hat{A}_j})$ ($\in \mathcal{A}_{1,2}$). (Note that f is of variable moduli since the Albanese map $S \to \mathrm{Alb}(S)$ is birational.)

Example 2.4. Fiber surfaces of genus 3 of constant moduli with relative irregularity 2 are easily constructed.

Let C_i ($i = 1, 2$) be a curve of genus g_i with an automorphism σ_i of order 2. Assume that $g_1 = 3$, and σ_1 acts freely on C_1. Let $S = C_1 \times C_2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $C_1 \times C_2$ by $(x, y) \mapsto (\sigma_1(x), \sigma_2(y))$. Then the fiber surface $p: S \to C_2/\sigma_2$ is of genus 3 (of constant moduli) with $q_p = 2$.

Since any fiber surface of constant moduli is birationally isomorphic to a product of two curves dividing by a finite group, it's easy to see that any fiber surface $f: S \to B$ of constant moduli of genus 3 with $q_f = 2$ is isomorphic to the one given above.

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