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## Representations of superconformal algebras in the $\text{AdS}_{7/4}/\text{CFT}_{6/3}$ correspondence

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We perform a general analysis of representations of the superconformal algebras  $\text{OSp}(8/4, \mathbb{R})$  and  $\text{OSp}(8^*/2N)$  in harmonic superspace. We present a construction of their highest-weight UIR's by multiplication of the different types of massless conformal superfields ("supersingletons").

In particular, all "short multiplets" are classified. Representations undergoing shortening have "protected dimension" and may correspond to BPS states in the dual supergravity theory in anti-de Sitter space.

These results are relevant for the classification of multitrace operators in boundary conformally invariant theories as well as for the classification of AdS black holes preserving different fractions of supersymmetry.

### 1. Introduction

Superconformal algebras and their representations play a crucial rôle in the AdS/CFT correspondence because of their dual rôle of describing the gauge symmetries of the AdS bulk supergravity theory and the global symmetries of the boundary conformal field theory<sup>1,2,3</sup>.

A special class of configurations which are particularly relevant are the so-called BPS states, i.e. dynamical objects corresponding to representations which undergo "shortening".

These representations can only occur when the conformal dimension of a (super)primary operator is "quantized" in terms of the R symmetry quantum numbers

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and they are at the basis of the so-called “non-renormalization” theorems of supersymmetric quantum theories <sup>4</sup>.

There exist different methods of classifying the UIR’s of superconformal algebras. One is the so-called oscillator construction of the Hilbert space in which a given UIR acts <sup>5</sup>. Another one, more appropriate to describe field theories, is the realization of such representations on superfields defined in superspaces <sup>6,7</sup>. The latter are “supermanifolds” which can be regarded as the quotient of the conformal supergroup by some of its subgroups.

In the case of ordinary superspace the subgroup in question is the supergroup obtained by exponentiating a non-semisimple superalgebra which is the semidirect product of a super-Poincaré graded Lie algebra with dilatation ( $SO(1,1)$ ) and the R symmetry algebra. This is the superspace appropriate for non-BPS states. Such states correspond to bulk massive states which can have “continuous spectrum” of the AdS mass (or, equivalently, of the conformal dimension of the primary fields).

BPS states are naturally associated to superspaces with lower number of “odd” coordinates and, in most cases, with some internal coordinates of a coset space  $G/H$ . Here  $G$  is the R symmetry group of the superconformal algebra, i.e. the subalgebra of the even part which commutes with the conformal algebra of space-time and  $H$  is some subgroup of  $G$  having the same rank as  $G$ .

Such superspaces are called “harmonic” <sup>8</sup> and they are characterized by having a subset of the initial odd coordinates  $\theta$ . The complementary number of odd variables determines the fraction of supersymmetry preserved by the BPS state. If a BPS state preserves  $K$  supersymmetries then the  $\theta$ ’s of the associated harmonic superspace will transform under some UIR of  $H_K$ .

For 1/2 BPS states, i.e. states with maximal supersymmetry, the superspace involves the minimal number of odd coordinates (half of the original one) and  $H_K$  is then a maximal subgroup of  $G$ . On the other hand, for states with the minimal fraction of supersymmetry  $H_K$  reduces to the “maximal torus” whose Lie algebra is the Cartan subalgebra of  $G$ .

It is the aim of the present paper to give a comprehensive treatment of BPS states related to “short representations” of superconformal algebras for the cases which are most relevant in the context of the AdS/CFT correspondence, i.e. the  $d = 3$  ( $N = 8$ ) and  $d = 6$  ( $N = (2,0)$ ). The underlying conformal field theories correspond to world-volume theories of  $N_c$  copies of  $M_2$ ,  $M_5$  and  $D_3$  branes in the large  $N_c$  limit <sup>9-13</sup> which are “dual” to AdS supergravities describing the horizon geometry of the branes <sup>14</sup>.

The present contribution summarizes the results which have already appeared elsewhere <sup>15,16,17</sup>. We first carry out an abstract analysis of the conditions for Grassmann (G-)analyticity <sup>18</sup> (the generalization of the familiar concept of chirality <sup>7</sup>) in a superconformal context. We find the constraints on the conformal dimension and R symmetry quantum numbers of a superfield following from the requirement that it do not depend on one or more Grassmann variables. Introducing G-analyticity in a traditional superspace cannot be done without breaking the R symmetry. The

latter can be restored by extending the superspace by harmonic variables<sup>19,8,20-24</sup> parametrizing the coset  $G/H_K$ . We also consider the massless UIR's ("supersingleton" multiplets)<sup>25,26</sup>, first as constrained superfields in ordinary superspace<sup>27-29</sup> and then, for a part of them, as G-analytic harmonic superfields<sup>8,24,29</sup>. Next we use supersingleton multiplication to construct UIR's of  $OSp(8^*/2N)$  and  $OSp(8/4, \mathbb{R})$ . We show that in this way one can reproduce the complete classification of UIR's of ref.<sup>30</sup>. We also discuss different kinds of shortening which certain superfields (not of the BPS type) may undergo. We conclude the paper by listing the various BPS states in the physically relevant cases of  $M_2$  and  $M_5$  branes horizon geometry where only one type of supersingletons appears.

Massive towers corresponding to 1/2 BPS states are the K-K modes coming from compactification of M-theory on  $AdS_{7/4} \times S_{4/7}$ <sup>31,9</sup>. Short representations of superconformal algebras also play a special rôle in determining  $N$ -point functions from OPE<sup>32,33</sup>.

Another area of interest is the classification of AdS black holes<sup>34-37</sup>, according to the fraction of supersymmetry preserved by the black hole background.

In a parallel analysis with black holes in asymptotically flat background<sup>38</sup>, the AdS/CFT correspondence predicts that such BPS states should be dual to superconformal states undergoing "shortening" of the type discussed here.

## 2. The six-dimensional case

In this section we describe highest-weight UIR's of the superconformal algebras  $OSp(8^*/2N)$  in six dimensions. Although the physical applications refer to  $N = 1$  and  $N = 2$ , it is worthwhile to carry out the analysis for general  $N$ , along the same lines as in the four-dimensional case<sup>39,40</sup>. We first examine the consequences of G-analyticity and conformal supersymmetry and find out the relation to BPS states. Then we will construct UIR's of  $OSp(8^*/2N)$  by multiplying supersingletons. The results exactly match the general classification of UIR's of  $OSp(8^*/2N)$  of Ref.<sup>30</sup>.

### 2.1. The conformal superalgebra $OSp(8^*/2N)$ and Grassmann analyticity

The standard realization of the conformal superalgebra  $OSp(8^*/2N)$  makes use of the superspace

$$\mathbb{R}^{6|8N} = \frac{OSp(8^*/2N)}{\{K, S, M, D, T\}} = (x^\mu, \theta^{\alpha i}) \quad (2.1)$$

where  $\theta^{\alpha i}$  is a left-handed spinor carrying an index  $i = 1, 2, \dots, 2N$  of the fundamental representation of the R symmetry group  $USp(2N)$ . Unlike the four-dimensional case, here chirality is not an option but is already built in. The only way to obtain smaller superspaces is through Grassmann analyticity. We begin by imposing a single condition of G-analyticity on the superfield defined in (2.1):

$$q_\alpha^1 \Phi(x, \theta) = 0 \quad (2.2)$$

which amounts to considering the coset

$$\mathbb{A}^{6|4(2N-1)} = \frac{\text{OSp}(8^*/2N)}{\{K, S, M, D, T, Q^1\}} = (x^\mu, \theta^{\alpha 1, 2, \dots, 2N-1}) \quad (2.3)$$

From the algebra of  $\text{OSp}(8^*/2N)$  we obtain

$$m_{\mu\nu} = 0, \quad (2.4)$$

$$t^{11} = t^{12} = \dots = t^{1 \ 2N-1} = 0, \quad (2.5)$$

$$4t^{1 \ 2N} + \ell = 0. \quad (2.6)$$

Eq. (2.4) implies that the superfield  $\Phi$  must be a Lorentz scalar. In order to interpret eqs. (2.5), (2.6), we need to split the generators of  $\text{USp}(2N)$  into raising operators (corresponding to the positive roots),  $T^{k \ 2N-l}$ ,  $k = 1, \dots, N$ ,  $l = k, \dots, 2N - k$  (simple if  $l = k$ ),  $[\text{U}(1)]^N$  charges  $H_k = -2T^{k \ 2N-k+1}$ ,  $k = 1, \dots, N$  and lowering operators. The Dynkin labels  $a_k$  of a  $\text{USp}(2N)$  irrep are defined as follows:

$$a_k = H_k - H_{k+1}, \quad k = 1, \dots, N-1, \quad a_N = H_N, \quad (2.7)$$

so that, for instance, the generator  $Q^1$  is the HWS of the fundamental irrep  $(1, 0, \dots, 0)$ .

Now it becomes clear that (2.5) is part of the  $\text{USp}(2N)$  irreducibility conditions whereas (2.6) relates the conformal dimension to the sum of the Dynkin labels:

$$\ell = 2 \sum_{k=1}^N a_k. \quad (2.8)$$

Let us denote the highest-weight UIR's of the  $\text{OSp}(8^*/2N)$  algebra by

$$\mathcal{D}(\ell; J_1, J_2, J_3; a_1, \dots, a_N)$$

where  $\ell$  is the conformal dimension,  $J_1, J_2, J_3$  are the  $\text{SU}^*(4)$  Dynkin labels and  $a_k$  are the  $\text{USp}(2N)$  Dynkin labels of the first component. Then the G-analytic superfields defined above are of the type

$$\Phi(\theta^{1, 2, \dots, 2N-1}) \Leftrightarrow \mathcal{D}(2 \sum_{k=1}^N a_k; 0, 0, 0; a_1, \dots, a_N). \quad (2.9)$$

From the superconformal algebra it is clear that we can go on in the same manner until we remove half of the  $\theta$ 's, namely  $\theta^{N+1}, \dots, \theta^{2N}$ . Each time we have to set a new Dynkin label to zero. We can summarize by saying that the superconformal algebra  $\text{OSp}(8^*/2N)$  admits the following short UIR's corresponding to BPS states:

$$\frac{p}{2N} \text{ BPS} : \quad \mathcal{D}(2 \sum_{k=p}^N a_k; 0, 0, 0; 0, \dots, 0, a_p, \dots, a_N), \quad p = 1, \dots, N. \quad (2.10)$$

## 2.2. *Supersingletons*

There exist three types of massless multiplets in six dimensions corresponding to ultrashort UIR's (supersingletons) of  $O\text{Sp}(8^*/2N)$  (see, e.g., <sup>41</sup> for the case  $N = 2$ ). All of them can be formulated in terms of constrained superfields as follows.

(i) The first type is described by a superfield  $W^{\{i_1 \dots i_n\}}(x, \theta)$ ,  $1 \leq n \leq N$ , which is antisymmetric and traceless in the external  $U\text{Sp}(2N)$  indices (for even  $n$  one can impose a reality condition). It satisfies the constraint (see <sup>27</sup> and <sup>42</sup>)

$$D_{\alpha}^{(k} W^{\{i_1 i_2 \dots i_n\}} = 0 \quad \Rightarrow \quad \mathcal{D}(2; 0, 0, 0; 0, \dots, 0, a_n = 1, 0, \dots, 0) \quad (2.11)$$

The components of this superfield are massless fields. In the case  $N = n = 1$  this is the on-shell  $(1, 0)$  hypermultiplet and for  $N = n = 2$  it is the on-shell  $(2, 0)$  tensor multiplet <sup>27,28</sup>.

(ii) The second type is described by a (real) superfield without external indices,  $w(x, \theta)$  obeying the constraint

$$D_{[\alpha}^{(i} D_{\beta]}^{j)} w = 0 \quad \Rightarrow \quad \mathcal{D}(2; 0, 0, 0; 0, \dots, 0) . \quad (2.12)$$

(iii) Finally, there exists an infinite series of multiplets described by superfields with  $n$  totally symmetrized external Lorentz spinor indices,  $w_{(\alpha_1 \dots \alpha_n)}(x, \theta)$  (they can be made real in the case of even  $n$ ) obeying the constraint

$$D_{[\beta}^i w_{(\alpha_1 \dots \alpha_n)} = 0 \quad \Rightarrow \quad \mathcal{D}(2 + n/2; n, 0, 0; 0, \dots, 0) . \quad (2.13)$$

As shown in ref. <sup>16</sup>, the six-dimensional massless conformal fields only carry reps  $(J_1, 0)$  of the little group  $SU(2) \times SU(2)$  of a light-like particle momentum. This result is related to the analysis of conformal fields in  $d$  dimensions <sup>43,44</sup>. This fact implies that massless superconformal multiplets are classified by a single  $SU(2)$  and  $U\text{Sp}(2N)$  R-symmetry and are therefore identical to massless super-Poincaré multiplets in five dimensions. Some physical implication of the above circumstance have recently been discussed in ref. <sup>45</sup> where it was suggested that certain strongly coupled  $d = 5$  theories effectively become six-dimensional.

### 2.3. Harmonic superspace

The massless multiplets (i), (ii) admit an alternative formulation in harmonic superspace (see <sup>46,47,29</sup> for  $N = 1, 2$ ). The advantage of this formulation is that the constraints (2.11) become conditions for G-analyticity. We introduce harmonic variables describing the coset  $U\text{Sp}(2N)/[U(1)]^N$ :

$$u \in U\text{Sp}(2N) : \quad u_i^I u_j^i = \delta_j^I, \quad u_i^I \Omega^{ij} u_j^J = \Omega^{IJ}, \quad u_i^I = (u_i^i)^* . \quad (2.14)$$

Here the indices  $i, j$  belong to the fundamental representation of  $U\text{Sp}(2N)$  and  $I, J$  are labels corresponding to the  $[U(1)]^N$  projections. The harmonic derivatives

$$D^{IJ} = \Omega^{K(I} u_i^{J)} \frac{\partial}{\partial u_i^K} \quad (2.15)$$

form the algebra of  $U\text{Sp}(2N)_R$  realized on the indices  $I, J$  of the harmonics.

Let us now project the defining constraint (2.11) with the harmonics  $u_k^K u_{i_1}^1 \dots u_{i_n}^n$ ,  $K = 1, \dots, n$ :

$$D_\alpha^1 W^{12\dots n} = D_\alpha^2 W^{12\dots n} = \dots = D_\alpha^n W^{12\dots n} = 0 \quad (2.16)$$

where  $D_\alpha^K = D_\alpha^i u_i^K$  and  $W^{12\dots n} = W^{\{i_1\dots i_n\}} u_{i_1}^1 \dots u_{i_n}^n$ . Indeed, the constraint (2.11) now takes the form of a G-analyticity condition. In the appropriate basis in superspace the solution to (2.16) is a short superfield depending on part of the odd coordinates:

$$W^{12\dots n}(x_A, \theta^1, \theta^2, \dots, \theta^{2N-n}, u). \quad (2.17)$$

In addition to (2.16), the projected superfield  $W^{12\dots n}$  automatically satisfies the  $\text{USp}(2N)$  harmonic irreducibility conditions

$$D^{K \ 2N-K} W^{12} = 0, \quad K = 1, \dots, N \quad (2.18)$$

(only the simple roots of  $\text{USp}(2N)$  are shown). The equivalence between the two forms of the constraint follows from the obvious properties of the harmonic products  $u_{[k}^K u_{j]}^K = 0$  and  $\Omega^{ij} u_i^K u_j^L = 0$  for  $1 \leq K < L \leq n$ . The harmonic constraints (2.18) make the superfield ultrashort.

Finally, in case (ii), projecting the constraint (2.12) with  $u_i^I u_j^I$  where  $I = 1, \dots, N$  (no summation), we obtain the condition

$$D_\alpha^I D_\beta^I w = 0. \quad (2.19)$$

It implies that the superfield  $w$  is *linear* in each projection  $\theta^{\alpha I}$ .

#### 2.4. Series of UIR's of $\text{OSp}(8^*/2N)$ and shortening

It is now clear that we can realize the BPS series of UIR's (2.10) as products of the different G-analytic superfields (supersingletons) (2.16).<sup>a</sup> BPS shortening is obtained by setting the first  $p - 1$   $\text{USp}(2N)$  Dynkin labels to zero:

$$\frac{p}{2N} \text{BPS} : W^{[0, \dots, 0, a_p, \dots, a_N]}(\theta^1, \theta^2, \dots, \theta^{2N-p}) = (W^{1\dots p})^{a_p} \dots (W^{1\dots N})^{a_N} \quad (2.20)$$

(note that even if  $a_1 \neq 0$  we still have  $1/2N$  shortening).

We remark that our harmonic coset  $\text{USp}(2N)/[\text{U}(1)]^N$  is effectively reduced to

$$\frac{\text{USp}(2N)}{\text{U}(p) \times [\text{U}(1)]^{N-p}} \quad (2.21)$$

in the case of  $p/2N$  BPS shortening (just as it happened in four dimensions). Such a smaller harmonic space was used in Ref. <sup>29</sup> to formulate the  $(2, 0)$  tensor multiplet.

A study of the most general UIR's of  $\text{OSp}(8^*/2N)$  (similar to the one of Ref. <sup>48</sup> for the case of  $\text{SU}(2, 2/N)$ ) is presented in Ref. <sup>30</sup>. We can construct these UIR's by multiplying the three types of supersingletons above:

$$w_{\alpha_1 \dots \alpha_{m_1}} w_{\beta_1 \dots \beta_{m_2}} w_{\gamma_1 \dots \gamma_{m_3}} w^k W^{[a_1, \dots, a_N]} \quad (2.22)$$

<sup>a</sup>As a bonus, we also prove the unitarity of these series, since they are obtained by multiplying massless unitary multiplets.

where  $m_1 \geq m_2 \geq m_3$  and the spinor indices are arranged so that they form an  $SU^*(4)$  UIR with Young tableau  $(m_1, m_2, m_3)$  or Dynkin labels  $J_1 = m_1 - m_2, J_2 = m_2 - m_3, J_3 = m_3$ . Thus we obtain four distinct series:

$$\begin{aligned}
 \text{A)} \quad & \ell \geq 6 + \frac{1}{2}(J_1 + 2J_2 + 3J_3) + 2 \sum_{k=1}^N a_k ; \\
 \text{B)} \quad & J_3 = 0, \quad \ell \geq 4 + \frac{1}{2}(J_1 + 2J_2) + 2 \sum_{k=1}^N a_k ; \\
 \text{C)} \quad & J_2 = J_3 = 0, \quad \ell \geq 2 + \frac{1}{2}J_1 + 2 \sum_{k=1}^N a_k ; \\
 \text{D)} \quad & J_1 = J_2 = J_3 = 0, \quad \ell = 2 \sum_{k=1}^N a_k .
 \end{aligned} \tag{2.23}$$

The superconformal bound is saturated when  $k = 0$  in (2.22). Note that the values of the conformal dimension we can obtain are “quantized” since the factor  $w^k$  has  $\ell = 2k$  and  $k$  must be a non-negative integer to ensure unitarity. With this restriction eq. (2.23) reproduces the results of Ref. <sup>30</sup>. However, we cannot comment on the existence of a “window” of dimensions  $2 + \frac{1}{2}J_1 + 2 \sum_{k=1}^N a_k \leq \ell \leq 4 + \frac{1}{2}J_1 + 2 \sum_{k=1}^N a_k$  conjectured in Ref. <sup>30</sup>.<sup>b</sup>

In the generic case the multiplet (2.22) is “long”, but for certain special values of the dimension some shortening can take place <sup>30</sup>.

### 3. The three-dimensional case

In this section we carry out the analysis of the  $d = 3$   $N = 8$  superconformal algebra  $OSp(8/4, \mathbb{R})$  in a way similar to the above. Some of the results have already been presented in <sup>15</sup>. As in the previous cases, our results could easily be extended to  $OSp(N/4, \mathbb{R})$  superalgebras with arbitrary  $N$ . The  $N = 2$  and  $N = 3$  cases were considered in Ref. <sup>50</sup>.

#### 3.1. The conformal superalgebra $OSp(8/4, \mathbb{R})$ and Grassmann analyticity

The standard realization of the conformal superalgebra  $OSp(8/4, \mathbb{R})$  makes use of the superspace

$$\mathbb{R}^{3|16} = \frac{OSp(8/4, \mathbb{R})}{\{K, S, M, D, T\}} = (x^\mu, \theta^{\alpha i}) . \tag{3.1}$$

In order to study G-analyticity we need to decompose the generators  $Q_\alpha^i$  under  $[U(1)]^4 \subset SO(8)$ . Besides the vector representation  $8_v$  of  $SO(8)$  we are also going to use the spinor ones,  $8_s$  and  $8_c$ . In this context we find it convenient to introduce the four subgroups  $U(1)$  by successive reductions:  $SO(8) \rightarrow SO(2) \times SO(6) \sim$

<sup>b</sup>In a recent paper <sup>49</sup> the UIR’s of the six-dimensional conformal algebra  $SO(2, 6)$  have been classified. Note that the superconformal bound in case A (with all  $a_i = 0$ ) is stronger than the purely conformal unitarity bounds found in <sup>49</sup>.

$U(1) \times SU(4) \rightarrow [SO(2)]^2 \times SO(4) \sim [U(1)]^2 \times SU(2) \times SU(2) \rightarrow [SO(2)]^4 \sim [U(1)]^4$ . Denoting the four  $U(1)$  charges by  $\pm$ ,  $(\pm)$ ,  $[\pm]$  and  $\{\pm\}$ , we decompose the three 8-dimensional representations as follows:

$$8_v : Q^i \rightarrow Q^{\pm\pm}, Q^{(\pm\pm)}, Q^{[\pm]\{\pm\}}, \quad (3.2)$$

$$8_s : \phi^a \rightarrow \phi^{+(+)\{\pm\}}, \phi^{-(-)\{\pm\}}, \phi^{+(-)\{\pm\}}, \phi^{-(+)\{\pm\}} \quad (3.3)$$

$$8_c : \sigma^{\dot{a}} \rightarrow \sigma^{+(+)\{\pm\}}, \sigma^{-(-)\{\pm\}}, \sigma^{+(-)\{\pm\}}, \sigma^{-(+)\{\pm\}} \quad (3.4)$$

Let us denote a quasi primary superconformal field of the  $OSp(8/4, \mathbb{R})$  algebra by the quantum numbers of its HWS:

$$\mathcal{D}(\ell; J; a_1, a_2, a_3, a_4) \quad (3.5)$$

where  $\ell$  is the conformal dimension,  $J$  is the Lorentz spin and  $a_i$  are the Dynkin labels (see, e.g., <sup>51</sup>) of the  $SO(8)$  R symmetry.

In order to build G-analytic superspaces we have to add one or more projections of  $Q_\alpha^i$  to the coset denominator. In choosing the subset of projections we have to make sure that: i) they anticommute among themselves; ii) the subset is closed under the action of the raising operators of  $SO(8)$ . Then we have to examine the consistency of the vanishing of the chosen projections with the conformal superalgebra. Thus we find the following sequence of G-analytic superspaces corresponding to BPS states:

$$\frac{1}{8} \text{ BPS} : \begin{cases} q_\alpha^{++}\Phi = 0 \rightarrow \\ \Phi(\theta^{++}, \theta^{(\pm\pm)}, \theta^{[\pm]\{\pm\}}) \\ \mathcal{D}(a_1 + a_2 + \frac{1}{2}(a_3 + a_4); 0; a_1, a_2, a_3, a_4) \end{cases} \quad (3.6)$$

$$\frac{1}{4} \text{ BPS} : \begin{cases} q_\alpha^{++}\Phi = q_\alpha^{(++)}\Phi = 0 \rightarrow \\ \Phi(\theta^{++}, \theta^{(++)}, \theta^{[\pm]\{\pm\}}) \\ \mathcal{D}(a_2 + \frac{1}{2}(a_3 + a_4); 0; 0, a_2, a_3, a_4) \end{cases} \quad (3.7)$$

$$\frac{3}{8} \text{ BPS} : \begin{cases} q_\alpha^{++}\Phi = q_\alpha^{(++)}\Phi = q_\alpha^{[+]\{+\}}\Phi = 0 \rightarrow \\ \Phi(\theta^{++}, \theta^{(++)}, \theta^{[+]\{\pm\}}, \theta^{[-]\{+\}}) \\ \mathcal{D}(\frac{1}{2}(a_3 + a_4); 0; 0, 0, a_3, a_4) \end{cases} \quad (3.8)$$

$$\frac{1}{2} \text{ BPS (type I)} : \begin{cases} q_\alpha^{++}\Phi = q_\alpha^{(++)}\Phi = q_\alpha^{[+]\{\pm\}}\Phi = 0 \rightarrow \\ \Phi(\theta^{++}, \theta^{(++)}, \theta^{[+]\{\pm\}}) \\ \mathcal{D}(\frac{1}{2}a_3; 0; 0, 0, a_3, 0) \end{cases} \quad (3.9)$$

$$\frac{1}{2} \text{ BPS (type II)} : \begin{cases} q_\alpha^{++}\Phi = q_\alpha^{(++)}\Phi = q_\alpha^{[\pm]\{+\}}\Phi = 0 \rightarrow \\ \Phi(\theta^{++}, \theta^{(++)}, \theta^{[\pm]\{+\}}) \\ \mathcal{D}(\frac{1}{2}a_4; 0; 0, 0, 0, a_4) \end{cases} \quad (3.10)$$

Note the existence of two types of  $1/2$  BPS states due to the two possible subsets of projections of  $q^i$  closed under the raising operators of  $SO(8)$ .

### 3.2. *Supersingletons and harmonic superspace*

The supersingletons are the simplest  $OSp(8/4, \mathbb{R})$  representations of the type (3.9) or (3.10) and correspond to  $\mathcal{D}(1/2; 0; 0, 0, 1, 0)$  or  $\mathcal{D}(1/2; 0; 0, 0, 0, 1)$ . The



existence of two distinct types of  $d = 3$   $N = 8$  supersingletons has first been noted in Ref. <sup>52</sup>. Each of them is just a collection of eight Dirac supermultiplets <sup>26</sup> made out of “Di” and “Rac” singletons <sup>25</sup>.

In order to realize the supersingletons in superspace we note that the HWS in the two supermultiplets above has spin 0 and the Dynkin labels of the  $8_s$  or  $8_c$  of  $SO(8)$ , correspondingly. Therefore we take a scalar superfield  $\Phi_a(x^\mu, \theta_i^\alpha)$  (or  $\Sigma_{\dot{a}}(x^\mu, \theta_i^\alpha)$ ) carrying an external  $8_s$  index  $a$  (or an  $8_c$  index  $\dot{a}$ ). These superfields are subject to the following on-shell constraints <sup>c</sup>:

$$\text{type I: } D_\alpha^i \Phi_a = \frac{1}{8} \gamma_{ab}^i \tilde{\gamma}_{bc}^j D_\alpha^j \Phi_c ; \quad (3.11)$$

$$\text{type II: } D_\alpha^i \Sigma_{\dot{a}} = \frac{1}{8} \tilde{\gamma}_{\dot{a}b}^i \gamma_{bc}^j D_\alpha^j \Sigma_{\dot{c}} . \quad (3.12)$$

The two multiplets consist of a massless scalar in the  $8_s$  ( $8_c$ ) and spinor in the  $8_c$  ( $8_s$ ).

The harmonic superspace description of these supersingletons can be realized by taking the harmonic coset <sup>d</sup>

$$\frac{SO(8)}{[SO(2)]^4} \sim \frac{Spin(8)}{[U(1)]^4} . \quad (3.13)$$

Since  $SO(8) \sim Spin(8)$  has three inequivalent fundamental representations,  $8_s, 8_c, 8_v$ , following <sup>57</sup> we introduce three sets of harmonic variables:

$$u_a^A, w_{\dot{a}}^{\dot{A}}, v_i^I \quad (3.14)$$

where  $A, \dot{A}$  and  $I$  denote the decompositions of an  $8_s, 8_c$  and  $8_v$  index, correspondingly, into sets of four  $U(1)$  charges (see (3.2)-(3.4)). Each of the  $8 \times 8$  real matrices (3.14) belongs to the corresponding representation of  $SO(8) \sim Spin(8)$ . This implies that they are orthogonal matrices (this is a peculiarity of  $SO(8)$  due to triality):

$$u_a^A u_a^B = \delta^{AB}, \quad w_{\dot{a}}^{\dot{A}} w_{\dot{a}}^{\dot{B}} = \delta^{\dot{A}\dot{B}}, \quad v_i^I v_i^J = \delta^{IJ} . \quad (3.15)$$

Further, we introduce harmonic derivatives (the covariant derivatives on the coset (3.13)):

$$D^{IJ} = u_a^A (\gamma^{IJ})^{AB} \frac{\partial}{\partial u_a^B} + w_{\dot{a}}^{\dot{A}} (\gamma^{IJ})^{\dot{A}\dot{B}} \frac{\partial}{\partial w_{\dot{a}}^{\dot{B}}} + v_i^I \frac{\partial}{\partial v_i^J} . \quad (3.16)$$

They respect the algebraic relations (3.15) among the harmonic variables and form the algebra of  $SO(8)$  realized on the indices  $A, \dot{A}, I$  of the harmonics.

<sup>c</sup>See also <sup>29</sup> for the description of a supersingleton related to ours by  $SO(8)$  triality. Superfield representations of other  $OSp(N/4)$  superalgebras have been considered in <sup>53,54</sup>.

<sup>d</sup>A formulation of the above multiplet in harmonic superspace has been proposed in Ref. <sup>29</sup> (see also <sup>55</sup> and <sup>56</sup> for a general discussion of three-dimensional harmonic superspaces). The harmonic coset used in <sup>29</sup> is  $Spin(8)/U(4)$ . Although the supersingleton itself does indeed live in this smaller coset (see Section ), its residual symmetry  $U(4)$  would not allow us to multiply different realizations of the supersingleton. For this reason we prefer from the very beginning to use the coset (3.13) with a minimal residual symmetry.

We now use the harmonic variables for projecting the supersingleton defining constraints (3.11), (3.12). It is easy to show that the projections  $\Phi^{+(+)[+]}$  and  $\Sigma^{+(+){+}}$  satisfy the following G-analyticity constraints:

$$D^{++}\Phi^{+(+)[+]} = D^{(++)}\Phi^{+(+)[+]} = D^{[+]{\pm}}\Phi^{+(+)[+]} = 0, \quad (3.17)$$

$$D^{++}\Sigma^{+(+){+}} = D^{(++)}\Sigma^{+(+){+}} = D^{[+]{\pm}}\Sigma^{+(+){+}} = 0 \quad (3.18)$$

where  $D_\alpha^I = v_i^I D_\alpha^i$ ,  $\Phi^A = u_a^A \Phi_a$  and  $\Sigma^{\dot{A}} = w_{\dot{a}}^{\dot{A}} \Sigma_{\dot{a}}$ . This is the superspace realization of the 1/2 BPS shortening conditions (3.9), (3.10). In the appropriate basis in superspace  $\Phi^{+(+)[+]}$  and  $\Sigma^{+(+){+}}$  depend on different halves of the odd variables as well as on the harmonic variables:

$$\text{type I:} \quad \Phi^{+(+)[+]}(x_A, \theta^{++}, \theta^{(++)}, \theta^{[+]{\pm}}, u, w), \quad (3.19)$$

$$\text{type II:} \quad \Sigma^{+(+){+}}(x_A, \theta^{++}, \theta^{(++)}, \theta^{[\pm]{+}}, u, w). \quad (3.20)$$

In addition to the G-analyticity constraints (3.17), (3.18), the on-shell superfields  $\Phi^{+(+)[+]}$ ,  $\Sigma^{+(+){+}}$  are subject to the SO(8) irreducibility harmonic conditions obtained by replacing the SO(8) raising operators by the corresponding harmonic derivatives. The combination of the latter with eq. (3.17) is equivalent to the original constraint (3.11).

### 3.3. $OSp(8/4, \mathbb{R})$ supersingleton composites

One way to obtain short multiplets of  $OSp(8/4, \mathbb{R})$  is to multiply different analytic superfields describing the type I supersingleton. The point is that above we chose a particular projection of, e.g., the defining constraint (3.11) which lead to the analytic superfield  $\Phi^{+(+)[+]}$ . In fact, we could have done this in a variety of ways, each time obtaining superfields depending on different halves of the total number of odd variables. Leaving out the  $8_v$  lowest weight  $\theta^{--}$ , we can have four distinct but equivalent analytic descriptions of the type I supersingleton:

$$\begin{aligned} & \Phi^{+(+)[+]}(\theta^{++}, \theta^{(++)}, \theta^{[+]{+}}, \theta^{[+]{-}}), \\ & \Phi^{+(+)[-]}(\theta^{++}, \theta^{(++)}, \theta^{[-]{+}}, \theta^{[-]{-}}), \\ & \Phi^{+(-){+}}(\theta^{++}, \theta^{(--)}, \theta^{[+]{+}}, \theta^{[-]{+}}), \\ & \Phi^{+(-){-}}(\theta^{++}, \theta^{(--)}, \theta^{[+]{-}}, \theta^{[-]{-}}). \end{aligned} \quad (3.21)$$

Then we can multiply them in the following way:

$$(\Phi^{+(+)[+]})^{p+q+r+s} (\Phi^{+(+)[-]})^{q+r+s} (\Phi^{+(-){+}})^{r+s} (\Phi^{+(-){-}})^s \quad (3.22)$$

thus obtaining three series of  $OSp(8/4, \mathbb{R})$  UIR's exhibiting 1/8, 1/4 or 1/2 BPS shortening:

$$\begin{aligned} \frac{1}{8} \text{ BPS:} & \quad \mathcal{D}(a_1 + a_2 + \frac{1}{2}(a_3 + a_4), 0; a_1, a_2, a_3, a_4), \quad a_1 - a_4 = 2s \geq 0; \\ \frac{1}{4} \text{ BPS:} & \quad \mathcal{D}(a_2 + \frac{1}{2}a_3, 0; 0, a_2, a_3, 0); \\ \frac{1}{2} \text{ BPS:} & \quad \mathcal{D}(\frac{1}{2}a_3, 0; 0, 0, a_3, 0) \end{aligned} \quad (3.23)$$

where

$$a_1 = r + 2s, \quad a_2 = q, \quad a_3 = p, \quad a_4 = r. \quad (3.24)$$

We see that multiplying only one type of supersingletons cannot reproduce the general result of Section for all possible short multiplets. Most notably, in (3.23) there is no 3/8 series. The latter can be obtained by mixing the two types of supersingletons:

$$[\Phi^{+(+)[+]}(\theta^{++}, \theta^{(++)}, \theta^{[+]\{\pm\}})]^{a_3} [\Sigma^{+(+)\{+\}}(\theta^{++}, \theta^{(++)}, \theta^{[\pm]\{+\}})]^{a_4} \quad (3.25)$$

(or the same with  $\Phi$  and  $\Sigma$  exchanged). Counting the charges and the dimension, we find exact matching with the series (3.8):

$$\frac{3}{8} \text{ BPS: } \mathcal{D}\left(\frac{1}{2}(a_3 + a_4); 0; 0, 0, a_3, a_4\right). \quad (3.26)$$

Further, mixing two realizations of type I and one of type II supersingletons, we can construct the 1/4 series

$$[\Phi^{+(+)[+]}]^{a_2+a_3} [\Phi^{+(+)[-}]^{a_2} [\Sigma^{+(+)\{+\}}]^{a_4} \quad (3.27)$$

which corresponds to (3.7):

$$\frac{1}{4} \text{ BPS: } \mathcal{D}\left(a_2 + \frac{1}{2}(a_3 + a_4); 0; 0, a_2, a_3, a_4\right). \quad (3.28)$$

Finally, the full 1/8 series (3.6) (i.e., without the restriction  $a_1 - a_4 = 2s \geq 0$  in (3.23)) can be obtained in a variety of ways.

### 3.4. BPS states of $OSp(8/4, \mathbb{R})$

Here we give a summary of all possible  $OSp(8/4, \mathbb{R})$  BPS multiplets. Denoting the UIR's by

$$\mathcal{D}(\ell; J; a_1, a_2, a_3, a_4) \quad (3.29)$$

where  $\ell$  is the conformal dimension,  $J$  is the spin and  $a_1, a_2, a_3, a_4$  are the  $SO(8)$  Dynkin labels, we find four BPS conditions:

#### 3.4.1.

$$\frac{1}{8} \text{ BPS: } \quad q_\alpha^{++} = 0. \quad (3.30)$$

The corresponding UIR's are:

$$\mathcal{D}\left(a_1 + a_2 + \frac{1}{2}(a_3 + a_4); 0; a_1, a_2, a_3, a_4\right) \quad (3.31)$$

and the harmonic coset is

$$\frac{\text{Spin}(8)}{[\text{U}(1)]^4}. \quad (3.32)$$

If  $a_2 = a_3 = a_4 = 0$  this coset becomes  $\text{Spin}(8)/\text{U}(4)$ .

3.4.2.

$$\frac{1}{4} \text{ BPS} : \quad q_\alpha^{++} = q_\alpha^{(++)} = 0 . \quad (3.33)$$

The corresponding UIR's are:

$$\mathcal{D}(a_2 + \frac{1}{2}(a_3 + a_4); 0; 0, a_2, a_3, a_4) \quad (3.34)$$

and the harmonic coset is

$$\frac{\text{Spin}(8)}{[\text{U}(1)]^2 \times \text{U}(2)} . \quad (3.35)$$

If  $a_3 = a_4 = 0$  this coset becomes  $\text{Spin}(8)/\text{U}(1) \times [\text{SU}(2)]^3$ .

3.4.3.

$$\frac{3}{8} \text{ BPS} : \quad q_\alpha^{++} = q_\alpha^{(++)} = q_\alpha^{[+]\{+\}} = 0 . \quad (3.36)$$

The corresponding UIR's are:

$$\mathcal{D}(\frac{1}{2}(a_3 + a_4); 0; 0, 0, a_3, a_4) \quad (3.37)$$

and the harmonic coset is

$$\frac{\text{Spin}(8)}{\text{U}(1) \times \text{U}(3)} . \quad (3.38)$$

3.4.4.

$$\frac{1}{2} \text{ BPS (type I)} : \quad q_\alpha^{++} = q_\alpha^{(++)} = q_\alpha^{[+]\{+\}} = q_\alpha^{[+]\{\pm\}} = 0 ; \quad (3.39)$$

$$\frac{1}{2} \text{ BPS (type II)} : \quad q_\alpha^{++} = q_\alpha^{(++)} = q_\alpha^{[+]\{+\}} = q_\alpha^{[\pm]\{+\}} = 0 . \quad (3.40)$$

The corresponding UIR's are:

$$\frac{1}{2} \text{ BPS (type I)} : \quad \mathcal{D}(\frac{1}{2}a_3; 0; 0, 0, a_3, 0) ; \quad (3.41)$$

$$\frac{1}{2} \text{ BPS (type II)} : \quad \mathcal{D}(\frac{1}{2}a_4; 0; 0, 0, 0, a_4) . \quad (3.42)$$

and the harmonic coset is

$$\frac{\text{Spin}(8)}{\text{U}(4)} . \quad (3.43)$$

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## References

1. J. M. Maldacena, *Adv. Theor. Math. Phys.* 2 (1998) 231; *Int. J. Theor. Phys.* 38 (1999) 1113.
2. S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Phys. Lett.* B428 (1998) 105, hep-th/9802109.
3. E. Witten, *Adv. Theor. Math. Phys.* 2 (1998) 253, hep-th/9802150.
4. J. Wess and B. Zumino, *Phys. Lett.* B49 (1974) 52; J. Iliopoulos and B. Zumino, *Nucl. Phys.* B76 (1974) 310; S. Ferrara, J. Iliopoulos and B. Zumino, *Nucl. Phys.* B77 (1974) 413.
5. I. Bars and M. Günaydin, *Commun. Math. Phys.* 91 (1983) 31; M. Günaydin and S. J. Hyun, *J. Math. Phys.* 29 (1988) 2367; M. Günaydin and C. Saclioğlu, *Commun. Math. Phys.*, 87 (1982) 159; *Phys. Lett.* B108 (1982) 180.
6. A. Salam and J. Strathdee, *Nucl. Phys.* B76 (1974) 477.
7. S. Ferrara, J. Wess and B. Zumino, *Phys. Lett.* B51 (1974) 239.
8. A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, *Class. Quant. Grav.* 1 (1984) 469.
9. O. Aharony, Y. Oz and Z. Yin, *Phys. Lett.* B430 (1998) 87; hep-th/9803051.
10. S. Minwalla, *JHEP* 10 (1998) 002; hep-th/9803053.
11. K. G. Leigh and M. Rozsli, *Phys. Lett.* B431 (1998) 311; hep-th/9803068.
12. E. Halyo, *JHEP* 04 (1998) 011; hep-th/9803077.
13. P. Claus, R. Kallosh and A. Van Proeyen, *Nucl. Phys.* B518 (1998) 117.
14. O. Aharony, S.S. Gubser, J. Maldacena, Hiroshi Ooguri and Yaron Oz, *Phys. Rept.* 323 (2000) 183, hep-th/9905111.
15. S. Ferrara and E. Sokatchev, *Conformal primaries of  $OSp(8/4, \mathbb{R})$  and BPS states in  $AdS_4$* , hep-th/0003051.
16. S. Ferrara and E. Sokatchev, *Representations of  $(1,0)$  and  $(2,0)$  superconformal algebras in six dimensions: massless and short superfields*, hep-th/0001178.
17. S. Ferrara and E. Sokatchev, *Superconformal interpretation of BPS states in  $AdS$  geometries*, hep-th/0005151.
18. A. Galperin, E. Ivanov and V.I. Ogievetsky *JETP Lett.* 33 (1981) 168.
19. A.A. Rosly, *Constraints in supersymmetric Yang-Mills theory as integrability conditions*, in "Theor. Group Methods in Physics" Editors M. Markov, V. Man'ko, Nauka, Moskva, (1983) 263; *Class. Quantum. Grav.* 2 (1985) 693.
20. E. Ivanov, S. Kalitsyn, Nguyen Ai Viet and V. Ogievetsky, *J. Phys.* A 18 (1985) 3433.
21. A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, *Class. Quant. Grav.* 2 (1985) 155; R. Kallosh, *Pis'ma ZHETF*, 41 (1985) 172; A.A. Rosly and A.S. Schwarz, *Commun. Math. Phys.* 105 (1986) 645.
22. A. Galperin, E. Ivanov and V. Ogievetsky, *Sov. J. Nucl. Phys.* 46 (1987) 543.
23. I. Bandos, *Theor. Math. Phys.* 76 (1988) 783.
24. G.G. Hartwell and P.S. Howe, *Int. J. Mod. Phys.* A10 (1995) 3901, hep-th/9412147; *Class. Quant. Grav.* 12 (1995) 1823.
25. M. Flato and C. Fronsdal, *Lett. Math. Phys.* 2 (1978) 421; *Phys. Lett.* 97B (1980)

- 236; *J. Math. Phys.* 22 (1981) 1100; *Phys. Lett.* B172 (1986) 412.
26. C. Fronsdal, *Phys. Rev.* D26 (1982) 1988.
27. P.S. Howe, G. Sierra and P.K. Townsend, *Nucl. Phys. B* 221 (1983) 331.
28. E. Bergshoeff, E. Sezgin and A. Van Proeyen, *Nucl. Phys.* B264 (1986) 653.
29. P.S. Howe, in *Supersymmetries and Quantum Symmetries*, eds. J. Wess and E.A. Ivanov, Lecture Notes in Physics, Springer (1999), p. 68-78.
30. S. Minwalla, *Adv.Theor.Math.Phys.* 2 (1998) 781.
31. M. Günaydin, P. van Nieuwenhuizen and N. Warner, *Nucl. Phys.* B255 (1985) 63; M. Pernici, K. Pilch and P. Van Nieuwenhuizen, *Phys. Lett.* B143 (1984) 103.
32. E. D'Hoker and B. Pioline, *Near-Extremal Correlators and Generalized Consistent Truncation for  $AdS_{4/7} \times S^{4/7}$* , hep-th/0006103.
33. R. Corrado, B. Florea and R. McNees, *Correlation Functions of Operators and Wilson Surfaces in the  $d=6$ ,  $(0,2)$  Theory in the Large  $N$  Limit*, hep-th/9902153; F. Bastianelli, S. Frolov and A.A. Tseytlin, *Three point correlators of Stress Tensors in Maximally Supersymmetric Conformal Theories in  $D=3$  and  $D=6$* , hep-th/9911135; F. Bastianelli and R. Zucchini, *Three point functions for a Class of Chiral Operators in Maximally Supersymmetric CFT at Large  $N$* , hep-th/9909179; F. Bastianelli and R. Zucchini, *Phys. Lett.* B467 (1999) 61;
34. S.W. Hawking, C.J. Hunter and M.M. Taylor-Robinson, *Phys. Rev.* D59 (1999) 064005.
35. A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, hep-th/9902170.
36. K. Behrndt, M. Cvetič and W. Sabra, hep-th/9810227.
37. M.J. Duff and J.T. Liu, hep-th/9901149.
38. S. Ferrara and J. Maldacena, *Class. Quant. Grav.* 15 (1998) 749; S. Ferrara and M. Günaydin, *Int. J. Mod. Phys.* A13 (1998) 2075.
39. L. Andrianopoli, S. Ferrara, E. Sokatchev and B. Zupnik, *Shortening of primary operators in  $N$ -extended  $SCFT_4$  and harmonic-superspace analyticity*, hep-th/9912007.
40. S. Ferrara and E. Sokatchev, *Short representations of  $SU(2,2/N)$  and harmonic superspace analyticity*, hep-th/9912168.
41. M. Günaydin and S. Takemae, *Unitary supermultiplets of  $OSp(8/4)$  and the  $AdS_7/CFT_6$  duality*, hep-th/9910110.
42. J.-H. Park, *Nucl. Phys.* B539 (1999) 599.
43. W. Siegel, *Int. J. Mod. Phys.* A4 (1989) 2015.
44. E. Angelopoulos and M. Laoues, *Rev. Math. Phys.* 10 (1998) 271.
45. C. Hull, *Strongly coupled gravity and duality*, hep-th/0004195.
46. P.S. Howe, K. Stelle, P. West, *Class. Quantum Grav.* 2 (1985) 815-821.
47. B.M. Zupnik, *Sov. J. Nucl. Phys.* 44 (1986) 512; Proceedings of the 1985 Yurmala conference "Theoretical-group methods in physics", v. 1, p.52-59, Nauka, Moscow, 1986.
48. V. K. Dobrev and V. B. Petkova, *Phys. Lett.* B162 (1985) 127; *Fortschr. Phys.* 35 (1987) 7, 537.
49. S. Ferrara and C. Fronsdal, *Conformal fields in higher dimensions*, hep-th/0006009.
50. D. Fabbri, P. Fré, L. Gualtieri and P. Termonia, *Class. Quant. Grav.* 17 (2000) 55; P. Fré, L. Gualtieri and P. Termonia, *Phys. Lett.* B471 (1999) 27.
51. L. Frappat, A. Sciarrino and P. Sorba, *Dictionary on Lie algebras and superalgebras*, Academic Press (2000).
52. M. Günaydin, B. Nilsson, G. Sierra and P. Townsend, *Phys. Lett.* B176 (1986) 45.
53. E.A. Ivanov and A. Sorin, *J. Phys. A* 13 (1980) 1159.
54. D. Fabbri, P. Fré, L. Gualtieri, P. Termonia, *Class. Quant. Grav.* 17 (2000) 55.
55. B.M. Zupnik and D.V. Khetselius, *Sov. J. Nucl. Phys.* 47 (1988) 730.
56. P.S. Howe and M.I. Leeming, *Class. Quant. Grav.* 11 (1994) 2843.
57. A. Galperin, P.S. Howe and K.S. Stelle, *Nucl. Phys.* B368 (1992) 248.