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QUANTUM GROUP $\mathrm{U_q(D_\ell)}$ SINGULAR VECTORS IN POINCARÉ-BIRKHOFF-WITT BASIS

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Abstract

We give explicit expressions for the singular vectors of $U_q(D_\ell)$ in terms of the Poincaré-Birkhoff-Witt (PBW) basis. We relate these expressions with those in terms of the simple root vectors.

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1. Introduction

Singular vectors of Verma modules appeared in the representation theory of semisimple Lie algebras and groups, cf. [1], [2], [3], [4]. After that singular vectors have also made a great impact in physics and not only from the point of view of applications. In fact, the generalization of singular vectors to other symmetry objects was done primarily in the (mathematical) physics literature starting with the paper [5], where singular vectors of the Virasoro algebra were crucially used. More explicit examples in this case were given in [6], [7], [8], [9], [10]. Further, singular vectors were given for Kac-Moody algebras [11], [12], the conformal superalgebra $su(2,2/n)$ [13], the $N=1$ super-Virasoro algebras [14], [15], quantum groups [16], [17], [18], W-algebras [19], [20], [21], the $N=2$ super-Virasoro algebras [22], [23], [24], the $N=4$ super-Virasoro algebras [25], Kac-Moody superalgebras [26], [27].

In the present paper we consider singular vectors on Verma modules over the Drinfeld-Jimbo quantum groups [28], [29]. These are q-deformations $U_q(\mathcal{G})$ of the universal enveloping algebras $U(\mathcal{G})$ of simple Lie algebras $\mathcal{G}$ and are called quantum groups [28], quantum universal enveloping algebras [30], [31], or just *quantum algebras*. The present paper may be viewed as a natural continuation of the paper [17], where explicit formulae were given for the singular vectors of Verma modules over $U_q(\mathcal{G})$ for arbitrary $\mathcal{G}$ corresponding to a class of positive roots of $\mathcal{G}$, which were called straight roots, and some examples corresponding to arbitrary positive roots. Note that these results are complete only for $\mathcal{G} = A_\ell$ since in that case all positive roots are straight. The singular vectors were given only through the simple root vectors as in earlier work in the case $q=1$, cf. [32], [33]. (This basis turned out to be part of a more general basis introduced later in the context of quantum groups, though for other reasons, by Lusztig [34].) On the other hand, there were examples, both in the undeformed case [35] and the q-deformed case [36], [37], [38], when it was convenient to use singular vectors in the Poincaré-Birkhoff-Witt (PBW) basis. In principle, the paper [18] generalizes the results of [11], (from where PBW singular vectors may be extracted), to the quantum group case, however, the formulae are not so explicit as it is necessary for the applications.

Thus, the first result of the present paper gives explicit expressions for the singular vectors of $U_q(D_\ell)$ in terms of the PBW basis. The second result relates these expressions with those in terms of the simple root vectors. This we also do for the non-straight roots, which were omitted in [17]. The second result is not known also for $q=1$.

2. Preliminaries

Let $\mathcal{G}$ be a complex simple Lie algebra with Chevalley generators $X_i^\pm$, H_i , $i=1,\ldots,\ell=$ rank $\mathcal{G}$. Then the quantum algebra $U_q(\mathcal{G})$ is the q-deformation of the universal enveloping algebra $U(\mathcal{G})$ defined as the associative algebra over $\mathbb{C}$ with generators $X_i^\pm$, $K_i \equiv q_i^{H_i}$, $K_i^{-1} \equiv q_i^{-H_i}$, and with relations [29] :

$$[K_i\ ,\ K_j] \ =\ 0, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1\ , \quad K_i\ X_j^\pm\ K_i^{-1} \ =\ q_i^{\pm a_{ij}} X_j^\pm\ , \qquad (1a)$$

$$[X_i^+\ ,\ X_j^-] \ =\ \delta_{ij}\ \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}\ , \qquad (1b)$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}_{q_i} \left(X_i^\pm\right)^k\ X_j^\pm\ \left(X_i^\pm\right)^{n-k} \ =\ 0\ ,\ i\neq j\ , \qquad (1c)$$

where $q_i \equiv q^{(\alpha_i\,,\,\alpha_i)/2}$, $(a_{ij}) = (2(\alpha_i\,,\,\alpha_j)/(\alpha_i\,,\,\alpha_i))$ is the Cartan matrix of $\mathcal{G}$, $(\cdot\,,\,\cdot)$ is the scalar product of the roots normalized so that for the short roots α we have $(\alpha,\alpha)=2$, $n = 1 - a_{ij}$,

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}\,, \quad [m]_q! = [m]_q\,[m-1]_q\ldots[1]_q\,, \quad [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}} \qquad (1d)$$

Further we may omit the subscript q in $[m]_q$ if no confusion could arise.

The above definition is valid also when $\mathcal{G}$ is an affine Kac-Moody algebra [28].

We use the standard decompositions into direct sums of vector subspaces $\mathcal{G} = \mathcal{H} \oplus \bigoplus_{\beta\in\Delta} \mathcal{G}_\beta = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^-$, $\mathcal{G}^\pm = \bigoplus_{\beta\in\Delta^\pm} \mathcal{G}_\beta$, where $\mathcal{H}$ is the Cartan subalgebra spanned by the elements H_i , $\Delta = \Delta^+ \cup \Delta^-$ is the root system of $\mathcal{G}$, Δ^+, Δ^-, the sets of positive, negative, roots, respectively; Δ_S will denote the set of simple roots of Δ. We recall that H_j correspond to the simple roots α_j of $\mathcal{G}$, and if $\beta^\vee = \sum_j\, n_j\, \alpha_j^\vee$, $\beta^\vee \equiv 2\beta/(\beta,\beta)$, then β corresponds to $H_\beta = \sum_j\, n_j\, H_j$.

For the PBW basis of $U_q(\mathcal{G})$ besides $X_i^\pm$, $K_i^{\pm 1}$ we also need the Cartan–Weyl (CW) generators $X_\beta^\pm$ corresponding to the non-simple roots $\beta \in \Delta^+$. Naturally, we shall use a uniform notation, so that $X_{\alpha_i}^\pm \equiv X_i^\pm$. The CW generators $X_\beta^\pm$ are normalized so that [29], [16], [39] :

$$\begin{aligned} [X_\beta^+\,,\,X_\beta^-] &= \frac{K_\beta - K_\beta^{-1}}{q_\beta - q_\beta^{-1}}\,, \qquad q_\beta \equiv q^{(\beta,\beta)/2} \\ K_\beta &\equiv \prod_j K_j^{n_j(\beta,\beta)/(\alpha_j\,,\,\alpha_j)}\ (= q_\beta^{H_\beta}) \end{aligned} \qquad (2)$$

We shall not use the fact that the algebra $U_q(\mathcal{G})$ is a Hopf algebra and consequently we shall not introduce the corresponding structure.

The highest weight modules V over $U_q(\mathcal{G})$ are given by their highest weight $\Lambda \in \mathcal{H}^*$ and highest weight vector $v_0 \in V$ such that:

$$K_i\, v_0 = q_i^{\Lambda_i}\, v_0\,, \quad X_i^+\, v_0 = 0\,, \quad i = 1,\ldots,\ell\,, \quad \Lambda_i \equiv (\Lambda, \alpha_i^\vee) \qquad (3)$$

We start with the *Verma* modules V^Λ such that $V^\Lambda \cong U_q(\mathcal{G}^-)\otimes v_0$. We recall several facts from [16]. The Verma module V^Λ is reducible if there exists a root $\beta \in \Delta^+$ and $m \in I\!N$ such that

$$[(\Lambda + \rho\,,\,\beta^\vee) - m]_{q_\beta} = 0\,, \qquad (4)$$

holds, where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$. If q is not a root of unity then (4) is also a necessary condition for reducibility and then it may be rewritten as $2(\Lambda+\rho\,,\,\beta) = m(\beta,\beta)$. (In that case it is the generalization of the (necessary and sufficient) reducibility conditions for Verma modules over finite–dimensional $\mathcal{G}$ [1] and affine Lie algebras [40].) For uniformity we shall write the reducibility condition in the general form (4). If (4) holds then there exists a vector $v_s \in V^\Lambda$, called a *singular vector*, such that $v_s \notin \mathbb{C}\, v_0$, and:

$$K_i\, v_s = q_i^{\Lambda_i - m(\beta,\alpha_i^\vee)}\, v_s\,, \quad i = 1,\ldots,\ell\,, \qquad (5a)$$

$$X_i^+\, v_s = 0\,, \quad i = 1,\ldots,\ell\,, \qquad (5b)$$

The space $U_q(\mathcal{G}^-)v_s$ is a proper submodule of V^Λ isomorphic to the Verma module $V^{\Lambda-m\beta} = U_q(\mathcal{G}^-)\otimes v'_0$ where v'_0 is the highest weight vector of $V^{\Lambda-m\beta}$; the isomorphism being realized by $v_s \mapsto 1\otimes v'_0$. The singular vector is given by [32], [33], [16]:

$$v_s \ = \ v^{\beta,m} \ = \ \mathcal{P}^\beta_m \otimes v_0 \tag{6}$$

where $\mathcal{P}^\beta_m$ is a homogeneous polynomial of weight $m\beta$. The polynomial $\mathcal{P}^\beta_m$ is unique up to a non-zero multiplicative constant. The Verma module V^Λ contains a unique proper maximal submodule I^Λ. Among the HWM with highest weight Λ there is a unique irreducible one, denoted by L_Λ, i.e., $L_\Lambda = V^\Lambda/I^\Lambda$. If V^Λ is irreducible then $L_\Lambda = V^\Lambda$. Thus we discuss further L_Λ for which V^Λ is reducible. If V^Λ is reducible with respect to (w.r.t.) to every simple root (and thus w.r.t. to all positive roots), then L_Λ is a finite-dimensional highest weight module over $U_q(\mathcal{G})$ [41]. The representations of $U_q(\mathcal{G})$ are deformations of the representations of $U(\mathcal{G})$, and the latter are obtained from the former for $q\to 1$ [41].

In [17] the singular vectors were given only through the simple root vectors, namely:

$$v^{\beta,m} \ = \ \mathcal{P}^\beta_m(X^-_1,\ldots,X^-_\ell)\otimes v_0 \ , \tag{7}$$

so that $\mathcal{P}^\beta_m$ is a homogeneous polynomial in its variables of degrees mn_i, where $n_i \in \mathbb{Z}_+$ come from $\beta = \sum n_i\alpha_i$.

The aim of the present paper is to give expressions for the singular vectors in terms of the PBW basis and to relate these expressions with those of [17].

3. Singular vectors for the straight roots

3.1. Singular vectors in PBW basis

In this paper we consider $U_q(\mathcal{G})$ when the deformation parameter q is not a nontrivial root of unity. This generic case is very important for two reasons. First, for $q=1$ all formulae are valid also for the undeformed case and formulae for the relation with [17] are also new for $q=1$. Second, the formulae for the case when q is a root of unity use the formulae for generic q as important input as is explained in [17].

Let $\mathcal{G} = D_\ell$, $\ell\geq 4$. Let α_i , $i=1,\ldots,\ell$ be the simple roots, so that $(\alpha_i\ ,\ \alpha_j) = -1$ if either $|i-j|=1$, $i,j\neq\ell$ or $ij = \ell(\ell-2)$ and $(\alpha_i\ ,\ \alpha_j) = 2\delta_{ij}$ in other cases.

Then all positive roots are given as follows

$$\begin{aligned}
&\alpha_{ij} = \alpha_i+\alpha_{i+1}+\ldots+\alpha_j\,, \quad 1\leq i<j\leq \ell-2\,,\\
&\beta_j = \alpha_j+\alpha_{j+1}+\ldots+\alpha_{\ell-2}+\alpha_\ell\,, \quad 1\leq j\leq\ell-2\,,\\
&\tilde\beta_j = \alpha_j+\alpha_{j+1}+\ldots+\alpha_{\ell-2}+\alpha_{\ell-1}\,, \quad 1\leq j\leq\ell-2\,,\\
&\beta_0 = \alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell\,,\\
&\gamma_j = \alpha_j+\alpha_{j+1}+\ldots+\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell\,, \quad 1\leq j\leq \ell-3\,,\\
&\gamma_{ij} = \alpha_i+\alpha_{i+1}+\ldots+2(\alpha_j+\ldots+\alpha_{\ell-2})+\alpha_{\ell-1}+\alpha_\ell\,, \quad 1\leq i<j\leq\ell-2\,.
\end{aligned} \tag{8}$$

We recall that the roots α_{ij} , β_j , $\tilde\beta_j$, β_0 , are positive roots of various A_n subalgebras. Thus, we have to consider only the roots γ_j and γ_{ij} . We recall from [17] that γ_j are straight, while γ_{ij} are not straight.

In this section we deal with the straight roots γ_j . Now we recall that every root γ_j is the highest straight root of a $D_{\ell-j+1}$ subalgebra of D_ℓ . This means that it is enough to give the formula for the singular vector corresponding to the highest straight root γ_1 .

Further we shall need the explicit expressions for the non-simple-root Cartan-Weyl (CW) generators of $U_q(\mathcal{G})$. Let $X^\pm_{i,j}$, $Y^\pm_j$, $\tilde Y^\pm_j$, $Y^\pm_0$, $Z^\pm_j$ and $Z^\pm_{i,j}$ be the Cartan-Weyl generators corresponding respectively to the roots $\pm\alpha_{ij}$, $\pm\beta_j$, $\pm\tilde\beta_j$, $\pm\beta_0$, $\pm\gamma_j$ and $\pm\gamma_{ij}$.

The CW generators corresponding to the nonsimple roots are given as follows:

$$\begin{aligned}
X^\pm_{ij} &= \pm\, q^{\mp 1/2}\left(q^{1/2}\ X^\pm_i X^\pm_{i+1,j}\ -\ q^{-1/2}\ X^\pm_{i+1,j}X^\pm_i\right)\ = && (9a)\\
&= \pm\, q^{\mp 1/2}\left(q^{1/2}\ X^\pm_{i,j-1} X^\pm_{j}\ -\ q^{-1/2}\ X^\pm_{j}X^\pm_{i,j-1}\right)\ , \qquad 1\le i<j\le \ell-2 \\
Y^\pm_{i} &= \pm\, q^{\mp 1/2}\left(q^{1/2}\ X^\pm_\ell X^\pm_{i,\ell-2}\ -\ q^{-1/2}\ X^\pm_{i,\ell-2}X^\pm_\ell\right)\ = && (9b)\\
&= \pm\, q^{\mp 1/2}\left(q^{1/2}\ X^\pm_{i,j} Y^\pm_{j+1}\ -\ q^{-1/2}\ Y^\pm_{j+1}X^\pm_{i,j}\right)\ , \qquad 1\le i\le \ell-2 \\
\tilde Y^\pm_{i} &= \pm\, q^{\mp 1/2}\left(q^{1/2}\ X^\pm_{\ell-1} X^\pm_{i,\ell-2}\ -\ q^{-1/2}\ X^\pm_{i,\ell-2}X^\pm_{\ell-1}\right)\ = && (9c)\\
&= \pm\, q^{\mp 1/2}\left(q^{1/2}\ X^\pm_{i,j} \tilde Y^\pm_{j+1}\ -\ q^{-1/2}\tilde Y^\pm_{j+1}X^\pm_{i,j}\right)\ , \qquad 1\le j\le \ell-2 \\
Y^\pm_{0} &= \pm\, q^{\mp 1/2}\left(q^{1/2}\ X^\pm_{\ell-1} Y^\pm_{\ell-2}\ -\ q^{-1/2}\ Y^\pm_{\ell-2}X^\pm_{\ell-1}\right)\ = && (9d)\\
&= \pm\, q^{\mp 1/2}\left(q^{1/2}\ X^\pm_{\ell} \tilde Y^\pm_{\ell-2}\ -\ q^{-1/2}\tilde Y^\pm_{\ell-2}X^\pm_{\ell}\right) \\
Z^\pm_{i} &= \pm\, q^{\mp 1/2}\left(q^{1/2}\ X^\pm_{i,\ell-3} Y^\pm_{0}\ -\ q^{-1/2}\ Y^\pm_{0}X^\pm_{i,\ell-3}\right)\ = && (9e)\\
&= \pm\, q^{\mp 1/2}\left(q^{1/2}\ X^\pm_{\ell} \tilde Y^\pm_{i}\ -\ q^{-1/2}\tilde Y^\pm_{i}X^\pm_{\ell}\right), \\
&= \pm\, q^{\mp 1/2}\left(q^{1/2}\ X^\pm_{\ell-1} Y^\pm_{i}\ -\ q^{-1/2}\tilde Y^\pm_{i}X^\pm_{\ell-1}\right)\ , \qquad 1\le j\le \ell-3 \\
Z^\pm_{ij} &= \pm\, q^{\mp 1/2}\left(q^{1/2}\ Z^\pm_{i} X^\pm_{j,\ell-2}\ -\ q^{-1/2}\ X^\pm_{j,\ell-2}Z^\pm_{i}\right), \qquad 1\le i<j\le \ell-2 && (9f)
\end{aligned}$$

Now the PBW basis of $U_q(\mathcal{G}^-)$ is given by the following monomials:

$$\begin{aligned}
&(X^-_{\ell-2})^{a_{\ell-2}}(X^-_{\ell-3,\ell-2})^{t_{\ell-3,\ell-2}}\ldots\ (X^-_{1,\ell-2})^{t_{1,\ell-2}}(\tilde Y^-_{\ell-2})^{\tilde t_{\ell-2}}(Y^-_{\ell-2})^{t_{\ell-2}}\times\\
&\times (Z^-_{\ell-3,\ell-2})^{s_{\ell-3,\ell-2}}(X^-_{\ell-4,\ell-2})^{s_{\ell-4,\ell-2}}\ldots\ (Z^-_{1,\ell-2})^{s_{1,\ell-2}}(\tilde Y^-_{\ell-3})^{\tilde t_{\ell-3}}(Y^-_{\ell-3})^{t_{\ell-3}}\times\\
&\times (Z^-_{\ell-4,\ell-3})^{s_{\ell-4,\ell-3}}\ldots\ (Z^-_{1,\ell-3})^{s_{1,\ell-3}}\ldots\ (\tilde Y^-_{1})^{\tilde t_{1}}(Y^-_{1})^{t_{1}}(Y^-_{0})^{t}\times\\
&\times (Z^-_{\ell-3})^{s_{\ell-3}}\ldots\ (Z^-_{1})^{s_{1}}(X^-_{\ell})^{a_{\ell}}(X^-_{\ell-1})^{a_{\ell-1}}(X^-_{\ell-3})^{a_{\ell-3}}\times\\
&\times (X^-_{\ell-4,\ell-3})^{t_{\ell-4,\ell-3}}\ldots\ (X^-_{1,\ell-3})^{t_{1,\ell-3}}(X^-_{\ell-4})^{a_{\ell-4}}\times\\
&\times\ldots\ (X^-_{2})^{a_{2}}(X^-_{12})^{t_{12}}(X^-_{1})^{a_{1}}
\end{aligned}\qquad (10)$$

These monomials are in the so-called normal order [39]. Namely, we put the simple root vectors X^-_j in the order $X^-_{\ell-2}$, X^-_ℓ , $X^-_{\ell-1}$, $X^-_{\ell-3}$, ... , X^-_2 , X^-_1. Then we put a root vector E^-_α corresponding to the nonsimple root α between the root vectors E^-_β and E^-_γ if $\alpha = \beta+\gamma,\, \alpha,\beta,\gamma\in\Delta^+$. This order is not complete but this is not a problem, since when two roots are not ordered this means that the corresponding root vectors commute, e.g., $[X^-_i\ ,\ X^-_{i-k,i+k}]\ =0$, and $[Y^-_i\ ,\ \tilde{Y}^-_i]\ = 0, \quad 1\le i\le \ell-2$.

Let us have condition (4) fulfilled for γ_1, but not for any of its subroots $\gamma_i\ ,\ i>1$:

$$[(\Lambda+\rho,{\gamma_1}^\vee)-m]_q \ =\ 0\ ,\quad m\in I\!\!N\ , \tag{11a}$$
$$[(\Lambda+\rho,\gamma_i^\vee)-m']_q\ \neq 0\ ,\quad \forall m'\in I\!\!N\ . \tag{11b}$$

(The necessity of the condition ($11b$) was explained in [17].) Let us denote the singular vector corresponding to ($11a$) by

$$\begin{aligned} v^{\gamma_1,m}_s = \sum_T D^{\gamma_1,m}_T\ & (X^-_{\ell-2})^{a_{\ell-2}} (X^-_{\ell-3,\ell-2})^{t_{\ell-3,\ell-2}} \dots\ (X^-_{1,\ell-2})^{t_{1,\ell-2}} (\tilde{Y}^-_{\ell-2})^{\tilde{t}_{\ell-2}} (Y^-_{\ell-2})^{t_{\ell-2}} \times \\ & \times (Z^-_{\ell-3,\ell-2})^{s_{\ell-3,\ell-2}} (X^-_{\ell-4,\ell-2})^{s_{\ell-4,\ell-2}} \dots\ (Z^-_{1,\ell-2})^{s_{1,\ell-2}} (\tilde{Y}^-_{\ell-3})^{\tilde{t}_{\ell-3}} (Y^-_{\ell-3})^{t_{\ell-3}} \times \\ & \times (Z^-_{\ell-4,\ell-3})^{s_{\ell-4,\ell-3}} \dots\ (Z^-_{1,\ell-3})^{s_{1,\ell-3}} \dots\ (\tilde{Y}^-_1)^{\tilde{t}_1} (Y^-_1)^{t_1} (Y^-_0)^{t} \times \\ & \times (Z^-_{\ell-3})^{s_{\ell-3}} \dots\ (Z^-_1)^{s_1} (X^-_\ell)^{a_\ell} (X^-_{\ell-1})^{a_{\ell-1}} (X^-_{\ell-3})^{a_{\ell-3}} \times \\ & \times (X^-_{\ell-4,\ell-3})^{t_{\ell-4,\ell-3}} \dots\ (X^-_{1,\ell-3})^{t_{1,\ell-3}} (X^-_{\ell-4})^{a_{\ell-4}} \times \\ & \times \dots\ (X^-_2)^{a_2} (X^-_{12})^{t_{12}} (X^-_1)^{a_1} \otimes v_0\ , \end{aligned} \tag{12}$$

where T denotes the set of summation variables $a_i, t_{ij}, s_{ij}, \tilde{t}_i, t_i, s_i, t$, all of which are nonnegative integers. Next we impose the conditions (5) with $\beta\to\gamma_1$. (Inequalities ($11b$) mean that no other conditions need to be imposed.) Conditions ($5a$) restrict the linear combination to terms of weight $m\gamma_1$. In our parametrization these are the following ℓ conditions:

$$\begin{aligned} a_p \ &=\ m - \sum_{i=1}^p \big((t_i+\tilde{t}_i+s_i) + \sum_{j=p}^{\ell-2} t_{ij} + \sum_{j=p+1}^{\ell-2} s_{ij} + 2\sum_{1\le i<j\le p} s_{ij}\big), \quad 1\le p\le \ell-3 \\ a_\ell &= m - \big(t + \sum_{i=1}^{\ell-2} t_i + \sum_{i=1}^{\ell-3} s_i + \sum_{1\le i<j\le \ell-2} s_{ij}\big) \\ a_{\ell-1} &= m - \big(t + \sum_{i=1}^{\ell-2} \tilde{t}_i + \sum_{i=1}^{\ell-3} s_i + \sum_{1\le i<j\le \ell-2} s_{ij}\big) \\ a_{\ell-2} &= m - \big(t + \sum_{i=1}^{\ell-3} (t_i+\tilde{t}_i) + \sum_{i=1}^{\ell-3} (s_i + t_{i,\ell-2}) + 2\sum_{1\le i<j\le \ell-2} s_{ij}\big) \end{aligned} \tag{13}$$

This eliminates the summation in a_i in (12) and also restricts further the summation $t_{ij}, s_{ij}, \tilde{t}_i, t_i, s_i, t$ so that the a_i in (13) would be all nonnegative.

Furthermore, conditions $(5b)$ fix the coefficients $D_T^{\gamma_1,m}$ completely and we have:

$$
\begin{aligned}
D_T^{\gamma_1,m} \; &= D^\ell \; (-1)^{\sum\limits_{i\leq j} s_{ij}} \frac{\prod\limits_{p=2}^{\ell-3} \frac{[\tilde{a}_p]!}{[a_p]!}}{[t]! \prod\limits_{j=2}^{\ell-2} [s_{1j}]![s_{j-1}]! \prod\limits_{j=1}^{\ell-2} [t_j]![\tilde{t}_j]! \prod\limits_{1\leq i<j\leq \ell-2} [t_{ij}]!} \times \\
&\times \frac{q^{\mathbf{A}} q^{(\Lambda+\rho, a_\ell \alpha_\ell + a_{\ell-1}\alpha_{\ell-1})}}{[m - 2t - \sum\limits_{i=1}^{\ell-2}(t_i + \tilde{t}_i) - 2 \sum\limits_{1\leq i<j\leq \ell-2} s_{ij} - 2\sum\limits_{i=1}^{\ell-3} s_i - \sum\limits_{i=1}^{\ell-3} t_{i,\ell-2}]!} \times \qquad (14) \\
&\times \prod_{j=1}^{\ell-3} q^{a_j(\Lambda+\rho)(H^j)} \frac{,_q(\Lambda^j + j - a_j + t_{j-1,j})}{,_q(\Lambda^j + j + 1)} \times \\
&\times \frac{,_q(\Lambda_{\ell-1} + 1 - a_{\ell-1}), \; _q(\Lambda_\ell + 1 - a_\ell)}{,_q(\Lambda_{\ell-1} + 2), \; _q(\Lambda_\ell + 2)}, \qquad D^\ell \neq 0 \\
&\Lambda^r := (\Lambda, \beta^r), \text{ with } \beta^r := \alpha_1 + \ldots + \alpha_r
\end{aligned}
$$

where

$$
\tilde{a}_p \;=\; m - \sum_{i=1}^{p} \big((t_i + \tilde{t}_i + s_i) + \sum_{j=p+1}^{\ell-2} (t_{ij} + s_{ij}) + 2 \sum_{1\leq i<j\leq p} s_{ij} \big), \quad 1 \leq p \leq \ell - 3
$$

and the factor $\mathbf{A}$ is given by:

$$
\begin{aligned}
\mathbf{A} \;=\; & \sum_{1\le i<j\le \ell-2} \Big\{ t_{ij} \sum_{p=0}^{\ell-4} t^{p+j-1} + s_{ij} \sum_{p=0}^{\ell-4} s^{p+j-1} \Big\} + \sum_{1\le i<j\le \ell-2} s_{ij}^2 + \sum_{i=1}^{\ell-3} s_i^2 + \\
& + \sum_{1\le i<j\le \ell-2} t_{ij}^2 - \Big((\ell-2) \sum_{1\le i<j\le \ell-2} t_{ij} \;+\; (\ell+1) \sum_{1\le i<j\le \ell-2} s_{ij} \;+ \ell \sum_{i=1}^{\ell-3} s_i \Big) m \\
& + \sum_{1\le i<j\le \ell-2} t_{ij} \sum_{i=1}^{\ell-3} (t_i + d_i) \;+\; \sum_{i=1}^{\ell-4} (t_i + d_i) \sum_{j=1}^{\ell-3} (t_j + d_j) \\
& + \sum_{p=1}^{\ell-2} \Big\{ t_p \big(\sum_{j=1}^{p} t_j + \sum_{1\le i<j\le \ell-2} s_{ij} + \sum_{i=1}^{\ell-3} s_i - (\ell-p) m \big) \\
& + \tilde{t}_p \big(\sum_{j=1}^{p} \tilde{t}_j + \sum_{1\le i<j\le \ell-2} s_{ij} + \sum_{i=1}^{\ell-3} s_i - (\ell-p) m \big) \Big\} \\
& + t \big(t + t_{\ell-2} + \tilde{t}_{\ell-2} + \sum_{i=1}^{\ell-3} s_i + \sum_{1\le i<j\le \ell-3} s_{ij} - 3m \big) \\
& + \sum_{1\le i<j\le \ell-3} s_i s_j \;+\; (\ell-2) \sum_{1\le i<j\le \ell-2} s_{ij} \sum_{k=1}^{\ell-3} s_k \\
& + \sum_{1\le i<j\le \ell-2} t_{ij} \big(\sum_{1\le i<j\le \ell-2} s_{ij} + \sum_{i=1}^{\ell-3} s_i \big) \\
& + \sum_{1\le i<j\le \ell-2} (j-i) t_{ij} + \sum_{\substack{1\le i\le \ell-4 \\ i<j}} (\ell-j+3) s_{ij} + 4 s_{\ell-3,\ell-2} \\
& + \sum_{i=1}^{\ell-3} (\ell-i) s_i + \sum_{i=1}^{\ell-2} (\ell-i-1)(t_i + \tilde{t}_i)
\end{aligned}
\qquad (15)
$$

where $t^b \;:= \sum\limits_{k=j+1}^{\ell-3} t_{bk}$.

Finally, we explicate how to obtain the singular vectors for the roots $\gamma_i\,,\ i>1$ from the above formulae. For this one has to replace $\ell \to \ell - i + 1$, and then to shift the enumeration of the roots, namely, to replace $1, ..., \ell - i + 1$ by $i, ..., \ell$.

3.2. Relation between the two expressions for the singular vectors

Here we would like to present the relation between the expressions for the singular vectors in the PBW basis given in (14) and in the simple root vectors basis given in [17]. The latter formula is (cf. f-la (16):

$$
\begin{aligned}
v^{\gamma_1,m} &= \sum_{k_1=0}^{m} \cdots \sum_{k_{\ell-1}=0}^{m} d_{k_1\ldots k_{\ell-1}} (X_1^-)^{m-k_1} \cdots (X_{\ell-3}^-)^{m-k_{\ell-3}}\ (X_{\ell-1}^-)^{m-k_{\ell-1}} \times \\
&\times (X_\ell^-)^{m-k_{\ell-2}}\ (X_{\ell-2}^-)^m (X_\ell^-)^{k_{\ell-2}} (X_{\ell-1}^-)^{k_{\ell-1}} \times \\
&\times (X_{\ell-3}^-)^{k_{\ell-3}} \ldots\ (X_1^-)^{k_1} \otimes v_0 \quad ,
\end{aligned}
\tag{16a}
$$

$$
\begin{aligned}
d_{k_1\ldots k_{\ell-1}} &= d\ (-1)^{k_1+\cdots+k_{\ell-1}} \binom{m}{k_1}_q \cdots \binom{m}{k_{\ell-1}}_q \times \\
&\times \frac{[(\Lambda+\rho,\beta^1)]_q}{[(\Lambda+\rho,\beta^1)-k_1]_q} \cdots \frac{[(\Lambda+\rho,\beta^{\ell-3})]_q}{[(\Lambda+\rho,\beta^{\ell-3})-k_{\ell-3}]_q} \times \\
&\times \frac{[(\Lambda+\rho,\alpha_\ell)]_q}{[(\Lambda+\rho,\alpha_\ell)-k_{\ell-2}]_q} \frac{[(\Lambda+\rho,\alpha_{\ell-1})]_q}{[(\Lambda+\rho,\alpha_{\ell-1})-k_{\ell-1}]_q} \\
&= d\ (-1)^{k_1+\cdots+k_{\ell-1}} \binom{m}{k_1}_q \cdots \binom{m}{k_{\ell-1}}_q \times \\
&\times \frac{[\Lambda^1+1]_q}{[\Lambda^1-k_1]_q} \cdots \frac{[\Lambda^{\ell-3}+\ell-3]_q}{[\Lambda^{\ell-3}-k_{\ell-3}]_q} \times \\
&\times \frac{[\Lambda_\ell+1]_q}{[\Lambda_\ell-k_{\ell-2}]_q} \frac{[\Lambda_{\ell-1}+1]_q}{[\Lambda_{\ell-1}-k_{\ell-1}]_q} \quad , \quad d \neq 0
\end{aligned}
\tag{16b}
$$

where $\Lambda_s = (\Lambda,\alpha_s)$,

The $D-$coefficients are given in terms of the $d-$coefficients by the following formula:

$$
\begin{aligned}
D_T^{\gamma_1,m} &= \frac{\prod_{p=2}^{\ell-3} \frac{[\tilde{a}_p]!}{[a_p]!}}{[t]! \prod_{j=2}^{\ell-2} [s_{1j}]![s_{j-1}]! \prod_{j=1}^{\ell-2} [t_j]![\tilde{t}_j]! \prod_{1\le i<j\le \ell-2} [t_{ij}]!} \times \\
&\times \frac{(-1)^{\sum_{i=1}^{\ell} a_i}\ q^{\mathbf{A}}}{[m-2t-\sum_{i=1}^{\ell-2}(t_i+\tilde{t}_i) - 2\sum_{1\le i<j\le \ell-2} s_{ij} - 2\sum_{i=1}^{\ell-3} s_i - \sum_{i=1}^{\ell-3} t_{i,\ell-2}]!} \times \\
&\times \sum_{k_1,k_2,\ldots,k_{\ell-1}} d_{k_1,k_2,\ldots,k_{\ell-1}} \prod_{p=1}^{\ell-3} \frac{[m-k_p]!q^{k_p(a_p-t_{p-1,p})}}{[a_p-t_{p-1,r}-k_p]!} \times \\
&\times \frac{[m-k_{\ell-1}]!}{[a_{\ell-1}-k_{\ell-1}]!} \frac{[m-k_{\ell-2}]!}{[a_\ell-k_{\ell-2}]!} q^{(k_{\ell-1}a_{\ell-1}+k_{\ell-2}a_\ell)}
\end{aligned}
\tag{17}
$$

where $0 \le k_p \le a_p,\quad 0 \le p \le \ell-3,\quad k_{\ell-1} \le a_{\ell-1}$ and $k_{\ell-2} \le a_\ell\,.$

4. Singular vectors for the nonstraight roots

4.1. Singular vectors in the PBW basis

The nonstraight roots of D_ℓ are given in (8). We shall also write them as:

$$
\begin{aligned}
\gamma_{rp} \ &= \ \sum_{j=r}^{\ell} n_j\,\alpha_j\ , \qquad 1 \le r < p \le \ell-2 \\
n_j \ &= \ \begin{cases} 1 & \text{for } \ r \le j < p \\ 2 & \text{for } \ p \le j \le \ell-2 \\ 1 & \text{for } \ j = \ell-1, \ell \end{cases}
\end{aligned}
\tag{18}
$$

Like in the case of straight roots we could use the fact that every root γ_{rp} can be treated as the root γ_{1p} of a $D_{\ell-r+1}$ subalgebra of D_ℓ. This means that it would be enough to give the formula for the singular vector corresponding to the roots γ_{1p}. However, we shall not do this for these roots, since anyway it is not reduced to single root.

Let us have condition (4) fulfilled for γ_{rp}, but not for any of its subroots. The singular vectors corresponding to these roots are given by:

$$
\begin{aligned}
v_s^{\gamma_{rp},m} \ =\ \sum_T\ & D_T^{\gamma_{rp},m}\ (X^-_{\ell-2})^{2m-b_{\ell-2}} (X^-_{\ell-3,\ell-2})^{t_{\ell-3,\ell-2}} \ldots\ (X^-_{r,\ell-2})^{t_{r,\ell-2}} (\tilde{Y}^-_{\ell-2})^{\tilde{t}_{\ell-2}} (Y^-_{\ell-2})^{t_{\ell-2}} \times \\
& \times (Z^-_{\ell-3,\ell-2})^{s_{\ell-3,\ell-2}} (X^-_{\ell-4,\ell-2})^{s_{\ell-4,\ell-2}} \ldots\ (Z^-_{r,\ell-2})^{s_{r,\ell-2}} (\tilde{Y}^-_{\ell-3})^{\tilde{t}_{\ell-3}} (Y^-_{\ell-3})^{t_{\ell-3}} \times \\
& \times (Z^-_{\ell-4,\ell-3})^{s_{\ell-4,\ell-3}} \ldots\ (Z^-_{r,\ell-3})^{s_{r,\ell-3}} \ldots\ (\tilde{Y}^-_r)^{\tilde{t}_r} (Y^-_r)^{t_r} (Y^-_0)^{t} \times \\
& \times (Z^-_{\ell-3})^{s_{\ell-3}} \ldots\ (Z^-_r)^{s_r} (X^-_\ell)^{m-b_\ell} (X^-_{\ell-1})^{m-b_{\ell-1}} (X^-_{\ell-3})^{mn_{\ell-3}-b_{\ell-3}} \times \\
& \times (X^-_{\ell-4,\ell-3})^{t_{\ell-4,\ell-3}} \ldots\ (X^-_{r,\ell-3})^{t_{r,\ell-3}} (X^-_{\ell-4})^{mn_{\ell-4}-b_{\ell-4}} \times \\
& \times \ldots\ (X^-_{r+1})^{mn_{r+1}-b_{r+1}} (X^-_{r,r+1})^{t_{r,r+1}} (X^-_r)^{m-b_r} \otimes v_0
\end{aligned}
\tag{19}
$$

where the coefficients $D_T^{\gamma_{rp},m}$ are given by:

$$
\begin{aligned}
D_T^{\gamma_{rp},m} \ &= D^{ns}\ (-1)^{\sum_{r\le j} s_{ij}}\ \frac{\prod_{s=r+1}^{\ell-3} \frac{[mn_s-\tilde{b}_s]!}{[mn_s-b_s]!}}{[t]!\ \prod_{j=r+1}^{\ell-2}[s_{rj}]![s_{j-1}]!\ \prod_{j=r}^{\ell-2}[t_j]![\tilde{t}_j]!\ \prod_{r\le i<j\le \ell-2}[t_{ij}]!} \times \\
& \times\ \frac{q^{\mathbf{A}^{ns}} q^{(\Lambda+\rho, b_\ell\alpha_\ell + b_{\ell-1}\alpha_{\ell-1})}}{[2m-2t-\sum_{i=r}^{\ell-2}(t_i+\tilde{t}_i) - 2\sum_{r\le i<j\le\ell-2} s_{ij} - 2\sum_{i=r}^{\ell-3} s_i - \sum_{i=r}^{\ell-3} t_{i,\ell-2}]!} \times \\
& \times \prod_{j=r}^{\ell-3} q^{(mn_j - b_j\Lambda'^j)} \frac{,_q(\Lambda'^j - mn_j + b_j + t_{j-1,j})}{,_q(\Lambda^j+1)} \times \\
& \times \frac{,_q(\Lambda_{\ell-1}+1-m+b_{\ell-1}),\ _q(\Lambda_\ell+1-m+b_\ell)}{,_q(\Lambda_{\ell-1}+2),\ _q(\Lambda_\ell+2)} \\
& \Lambda'^j := \sum_{i=r}^{j} n_i(\Lambda_i+1)\,, \qquad D^{ns} \neq 0
\end{aligned}
\tag{20}
$$

where we have set for $r \leq p \leq \ell-3$:

$$\begin{aligned}
\tilde{b}_p &= \sum_{i=r}^{p} \Big((t_i+\tilde{t}_i+s_i)+\sum_{j=p+1}^{\ell-2}(t_{ij}+s_{ij})+2\sum_{r\leq i<j\leq p} s_{ij}\Big),\\
b_p &= \sum_{i=r}^{p} \Big((t_i+\tilde{t}_i+s_i)+\sum_{j=p}^{\ell-2} t_{ij}+\sum_{j=p+1}^{\ell-2} s_{ij}+2\sum_{1\leq i<j\leq p} s_{ij}\Big),\\
b_\ell &= t+\sum_{i=r}^{\ell-2} t_i+\sum_{i=r}^{\ell-3} s_i+\sum_{r\leq i\leq<j\leq \ell-2} s_{ij},\\
b_{\ell-1} &= t+\sum_{i=r}^{\ell-2} \tilde{t}_i+\sum_{i=r}^{\ell-3} s_i+\sum_{r\leq i<j\leq \ell-2} s_{ij},\\
b_{\ell-2} &= t+\sum_{i=r}^{\ell-3}(t_i+\tilde{t}_i)+\sum_{i=r}^{\ell-3}(s_i+t_{i,\ell-2})+2\sum_{r\leq i<j\leq \ell-2} s_{ij}. \qquad (21)
\end{aligned}$$

4.2. Singular vectors in the simple roots basis

The singular vectors corresponding to the nonstraight roots, γ_{rp}, $1 \leq r < p \leq \ell-2$, in the simple root basis are given by

$$\begin{aligned}
v^{\gamma_{rp},m} &= \sum_{k_r=0}^{m}\sum_{k_{r+1}=0}^{mn_{r+1}}\cdots\sum_{k_{\ell-1}=0}^{m} d_{k_1\ldots k_{\ell-1}}(X_r^-)^{m-k_r}(X_{r+1}^-)^{mn_{r+1}-k_{r+1}}\cdots\times\\
&\times (X_{\ell-3}^-)^{2m-k_{\ell-3}}\ (X_{\ell-1}^-)^{m-k_{\ell-1}}(X_\ell^-)^{m-k_{\ell-2}}\ (X_{\ell-2}^-)^{2m}(X_\ell^-)^{k_{\ell-2}}\times\\
&\times (X_{\ell-1}^-)^{k_{\ell-1}}(X_{\ell-3}^-)^{k_{\ell-3}}\ldots\ (X_r^-)^{k_r}\otimes v_0 \qquad (22)
\end{aligned}$$

Now the coefficients d are given by the following expression

$$\begin{aligned}
d_{k_r,\ldots,k_{\ell-1}} &= (-1)^{k_r+\ldots+k_{\ell-1}}\times \sum_{\substack{mn_r-b_r\leq k_r\\ \vdots\\ mn_{\ell-3}-b_{\ell-3}\leq k_{\ell-3}}}\ \sum_{\substack{m-b_{\ell-1}\leq k_{\ell-1}\\ m-b_\ell\leq k_{\ell-2}}} D^{ns}\times\\
&\times \prod_{j=r}^{\ell-3}\frac{q^{(mn_j-b_j)(1-k_j)-k_j}[mn_j-b_j]!}{[mn_j-k_j]![mn_j-\tilde{b}_j]![k_j-mn_j+b_j]!}\times\\
&\times \frac{q^{(m-b_\ell)(1-k_{\ell-2})-k_{\ell-2}}}{[m-b_\ell]![k_{\ell-2}-m-b_\ell]![m-k_{\ell-2}]!}\,\frac{q^{(m-b_{\ell-1})(1-k_{\ell-1})-k_{\ell-1}}}{[m-b_{\ell-1}]![k_{\ell-1}-m-b_{\ell-1}]![m-k_{\ell-1}]!}\times\\
&\times [2m-2t-\sum_{i=r}^{\ell-2}(t_i+\tilde{t}_i)-2\sum_{r\leq i<j\leq\ell-2} s_{ij}-2\sum_{i=r}^{\ell-3}s_i-\sum_{i=r}^{\ell-3}t_{i,\ell-2}]!\times\\
&\times [t]!\prod_{j=r+1}^{\ell-2}[s_{rj}]![s_{j-1}]!\prod_{j=r}^{\ell-2}[t_j]![\tilde{t}_j]!\prod_{r\leq i<j\leq\ell-2}[t_{ij}]!q^{-A^{ns}} \qquad (23)
\end{aligned}$$

more explicitly the $d-$coefficients are given by

$$d_{k_1\ldots k_{\ell-1}} = d^{ns}\ (-1)^{k_r+\cdots+k_{\ell-1}} \binom{mn_r}{k_r}_q \cdots \binom{mn_{\ell-1}}{k_{\ell-1}}_q \times$$

$$
\begin{aligned}
&\times \frac{[(\Lambda+\rho,\beta^{r,r})]_q}{[(\Lambda+\rho,\beta^{r,r})-k_r]_q}\cdots\frac{[(\Lambda+\rho,\beta^{r,\ell-3})]_q}{[(\Lambda+\rho,\beta^{r,\ell-3})-k_{\ell-3}]_q}\times\\
&\times \frac{[(\Lambda+\rho,\alpha_\ell)]_q}{[(\Lambda+\rho,\alpha_\ell)-k_{\ell-2}]_q}\frac{[(\Lambda+\rho,\alpha_{\ell-1})]_q}{[(\Lambda+\rho,\alpha_{\ell-1})-k_{\ell-1}]_q}\\
=\ & d^{ns}\ (-1)^{k_r+\cdots+k_{\ell-1}}\ \binom{mn_r}{k_r}_q\cdots\binom{mn_{\ell-1}}{k_{\ell-1}}_q\times\\
&\times \frac{[\Lambda'^r+n^r]_q}{[\Lambda'^r+n^r-k_1]_q}\cdots\frac{[\Lambda'^{\ell-3}+n^{\ell-3}]_q}{[\Lambda'^{\ell-3}+n^{\ell-3}-k_{\ell-3}]_q}\times\\
&\times \frac{[\Lambda_\ell+1]_q}{[\Lambda_\ell+1-k_{\ell-2}]_q}\frac{[\Lambda_{\ell-1}+1]_q}{[\Lambda_{\ell-1}+1-k_{\ell-1}]_q}\,,\qquad d^{ns}\neq 0\\
&\beta^{r,j}:=\sum_{i=r}^{j}n_i\alpha_i\,,\quad \Lambda'^j=(\Lambda,\beta^{r,j})\,,\quad n^j:=\sum_{i=r}^{j}n_i\ . \qquad (24)
\end{aligned}
$$

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