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EMITTANCE GROWTH OF THE LHC BEAM DUE TO THE EFFECT OF HEAD-ON BEAM-BEAM INTERACTION AND GROUND MOTION

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Emittance Growth of the LHC Beam due to the Effect of Head-on Beam-Beam Interaction and Ground Motion

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Abstract

The influence of ground motion on the LHC beam is estimated by applying the existing theories of particle diffusion due to a weak-strong beam-beam collision with random offset at the interaction point. Noise at odd harmonics of the betatron frequency contributes significantly to particle diffusion. Extrapolating the characteristics of the random offset from the ground motion spectrum at the LHC site shows a fast fall-off with frequency and the amplitude is very small even at the first harmonic. We find that the head-on beam-beam force in the weak-strong approximation and ground motion by themselves do not induce significant diffusion over the lifetime of the beam.

1 DYNAMICAL SYSTEM

Random quadrupole oscillations create distortions of the beam orbit and random beam-beam offsets at the interaction point. It is known that ground vibration at frequencies higher than $f_m = 11$ Hz will cause uncorrelated quadrupole motion in the LHC ring [1]. Our aim is to estimate the effect of ground motion and the head-on beam-beam collision, which is the dominant nonlinearity at high energy, on the beam emittance.

We consider the dynamics of a test particle whose motion is followed over N turns, assuming linear betatron motion and a weak-strong beam-beam collision at one interaction point. At this interaction point, the particle experiences a deflection caused by the field of a counter rotating Gaussian beam. Our system of normalized variables in two transverse degrees of freedom is $(x,y) = (X/\sigma_x, Y/\sigma_y)$, $(v_x, v_y) = (\beta_x X'/\sigma_x, \beta_y Y'/\sigma_y)$, where (X,Y) is the position of the particle, (σ_x, σ_y) are the nominal rms sizes and (β_x, β_y) the nominal betatron function. We assume that the coupling between the transverse planes is negligible so that the linear map from one interaction point to the next is

$$\begin{pmatrix} x(n+1) \\ v_x(n+1) \end{pmatrix} = R(2\pi Q_x) \begin{pmatrix} x(n) \\ v_x(n) + \Delta v_x(n) \end{pmatrix} \quad (1)$$

$$R(2\pi Q_x) = \begin{pmatrix} \cos(2\pi Q_x) & \sin(2\pi Q_x) \\ -\sin(2\pi Q_x) & \cos(2\pi Q_x) \end{pmatrix} \quad (2)$$

A similar map is applied to the vertical plane. The beam-beam kick $\Delta v_x(n)$ at turn n depends on the distance from the test particle with position (x, y) to the centroid of the counter rotating beam. The beam oscillates randomly due to the ground motion. The position (in units of σ) of the centroid of the counter rotating beam at turn n is represented by the random variable $(\eta_x(n), \eta_y(n))$ whose spectral characteristics have to simulate those of the orbit offset spectrum. The kick due to the beam-beam interaction is

$$\Delta v_x(n) = \frac{2C\beta_x^*(x-\eta_x(n))}{\sigma_x^2(x-\eta_x(n))^2 + (y-\eta_y(n))^2} \times \quad (3)$$
$$[1 - \exp(-1/2[(x-\eta_x(n))^2 + (y-\eta_y(n))^2])]$$

with $C = N_p r_p/\gamma_p$, N_p the number of protons per bunch in the opposing beam, r_p the classical radius of the proton, γ_p the relativistic kinematic factor of the protons, (β_x^*, β_y^*) the beta functions at the interaction point. We shall use for our study the LHC beam parameters: $\sigma_{x,y} = 0.0159$ mm, $\gamma_p = E_p/E_0$, $E_0 = 0.93827$ GeV, $E_p = 7000$ GeV, $N_p = 1.05 \times 10^{11}$, $r_p = 1.5347 \times 10^{-15}$ mm $\beta_x^* = \beta_y^* = 500$ mm, , which correspond to a beam-beam parameter of $\xi = 0.0034$.

The spectral density of the ground motion measured at 10 Hz in the LEP tunnel is $S_{gm} = 5 \times 10^{-15}\text{mm}^2/\text{Hz}$ while the logarithmic slope with frequency at low frequencies is about -2.5. Assuming that this fall-off rate continues at high frequencies, we can then expect the spectral density of the ground motion in the vicinity of the betatron tune to be about $S_{gm} = 10^{-20}\text{mm}^2/\text{Hz}$ [1]. The effect of plane ground waves on the closed orbit of LHC has been studied for the collision configuration of the LHC lattice Version 4.3 [2], using MAD for computation of the closed orbit. The mean square response for LHC in the vicinity of the betatron frequency is $R^2 = 10$. Therefore we can estimate the spectral density of the orbit offset in the vicinity of the betatron tune to be about $S_o(Q_{beta}) = R^2 \times S_{gm}(Q_{beta}) = 10^{-19}\text{mm}^2/\text{Hz}$.

We have seen that the orbit offset spectrum decays very rapidly with frequency, having a logarithmic slope of -2.5. We can model these fluctuations by an Ornstein-Uhlenbeck (OU) process whose spectrum has a $1/f^2$ dependence. If $\eta(t)$ is a stochastic variable of zero mean following an OU process, then its correlation function decays exponentially with time. The stochastic equation describing a discrete OU process η_n is

$$\eta_{n+1} = (1 - \frac{1}{\tau_c})\eta_n + \sqrt{\frac{2}{\tau_c}}|\eta|\xi_{n+1} \quad (4)$$

where n is the number of turns, $|\eta|$ the amplitude of the fluctuations in units of σ, τ_c is the correlation time measured in number of turns, and ξ is a Gaussian white noise process of zero mean and unit variance. For stationary processes, the spectral density is the Fourier transform of the correlation function. The spectral density of this process is

$$S_{OU}(Q) = \frac{T_{rev}}{2\pi} \frac{(|\eta|\sigma)^2 \sinh\theta}{(1 - 1/(2\tau_c))[\cosh\theta - \cos(2\pi Q)]} \quad (5)$$

where $\theta = -\ln(1 - 1/\tau_c)$ and T_{rev} is the revolution period. The fall in noise power with increasing frequency is characterized by the correlation time τ_c. For instance for a correlation time $\tau_c = 100$ (in units of turns), $S_{OU}(0.28)/S_{OU}(0) \approx 4 \times 10^{-5}$ which is similar to the expected ratio in the LHC offset spectrum. A discrete OU process with oscillation amplitude of $|\eta| = 10^{-4}$ (in units of σ) and correlation time $\tau_c = 100$ (turns) has a spectral density at $Q_\beta = 0.28$ of about $10^{-19}\mathrm{mm}^2/\mathrm{Hz}$ which is the expected spectral density of the orbit offset in the vicinity of the betatron tune.

2 ANALYTICAL EVALUATION OF THE DIFFUSION COEFFICIENT

The diffusion coefficient has been studied analytically, in the case of tunes far from resonances, using action angle variables ($x = \frac{\sqrt{2J_x\beta^*}}{\sigma}\cos\psi_x$, $v_x = -\frac{\sqrt{2J_x\beta^*}}{\sigma}\sin\psi_x$) [3]. The 1D Hamiltonian is $H = Q_x J_x + U(x)\delta_p(\theta)$ where θ is the azimuthal variable. $U(x)$ is the beam-beam potential that can be expressed as a Fourier series $U(x) = C\sum_{k=0}^{\infty} U_k(a)\cos(2k\psi_x)$, $U_k = \int_0^a \frac{1}{w}[\delta_{0k} - (2-\delta_{0k})(-1)^k e^{-w} I_k]dw$, with $a = \frac{\beta^* J_x}{2\sigma^2}$ and I_k the modified Bessel functions. The one turn map in action angle variables to first order in the beam-beam parameter reads $\Delta\psi_x = 2\pi Q_x + \frac{\partial U}{\partial J_x}$, $\Delta J_x = -\frac{\partial U}{\partial \psi_x}$ For small closed orbit offsets η, we can expand the potential in a Taylor series $U(J_x, \psi_x) = U(x) + U'(x)\eta + O(\eta^2)$, $f(J_x, \psi_x) \equiv U'(x) = \frac{\partial J_x}{\partial x}\frac{\partial U_x}{\partial J_x} + \frac{\partial \psi_x}{\partial x}\frac{\partial U_x}{\partial \psi_x} = C\sum_{k=0}^{\infty} G_k(J_x)\cos((2k+1)\psi_x)$ where G_k are the Fourier coefficients of the beam-beam force given by $G_k(a) = \frac{\sqrt{a}}{\sigma}(U'_{k+1} + U'_k) + \frac{((k+1)U_{k+1} - kU_k)}{\sqrt{a}\sigma}$.

We wish to calculate the change in action due to the fluctuating offset alone, given that we know that in the absence of fluctuation the change in action is negligible. To first order in η the change at turn m is $\Delta J_x(m) = -\frac{\partial}{\partial\psi_x} f(J_x(m), \psi_x(m))\eta(m)$. If $J(0)$ is the initial value of the action of a particle and $J(N)$ the particle action at turn N, the total change at turn N is obtained by summing over all previous turns. The diffusion coefficient is defined as $D_{\mathrm{off}}(J) = \lim_{N\to\infty} < (\Delta J_x(N))^2 > /N$ where the average is over many noise realizations. Extracting the dominant terms, and introducing the correlation function of the offset $K(n) < \eta(l)\eta(n+l) >= \sigma^2 K(n)$ one gets the diffusion coefficient due to collisions at a single IP

$$D(J_x) = \frac{\pi C^2\sigma^2}{4T_{rev}}\sum_{k=0}^{\infty}(2k+1)^2 G_k^2(a) S_{off}[(2k+1)Q_x] \quad (6)$$

In the expression for the diffusion coefficient, $S_{off}[(2k+1)Q_x]$ is the spectral density of the fluctuating offsets at odd harmonics of the betatron frequency.

When the noise is described by a OU process, the expression can be simplified to

$$D(J_x) = \frac{(C\sigma|\eta|)^2}{8 - 8/(2\tau_c)}\sum_{k=0}^{k=\infty}\frac{\sinh\theta(2k+1)^2 G_k^2(a)}{\cosh\theta - \cos[2\pi(2k+1)Q_x]} \quad (7)$$

(for two uncorrelated IPs use $\sqrt{2} \times D(J_x)$).

2.1 Check of the 1D diffusion coefficient

The parameters of the random process are set to $\tau_c = 100$ and $|\eta| = 0.01$. Notice that the amplitude of the random offset $|\eta|$ is 10^2 times stronger than the one expected at the site of the LHC ring (for a realistic parameter we expect the diffusion coefficient $D_{\mathrm{off}}(J)$ to be 10^4 times smaller). In Fig. 1, we compare this analytical expression with the diffusion rate obtained from tracking. We follow the dynamics of a set of 50 initial conditions with action J_0 and random angle ψ (distributed with a random uniform distribution in $[0, 2\pi]$), subject to the one dimensional version of the maps (2) and (3) and the OU process of Eq. (4). We evaluate the diffusion coefficient $D_{\mathrm{off}}(J) = \lim_{N\to\infty} < (\Delta J_x(N))^2 > /N$ in the limit of $N = 10^7$ turns.

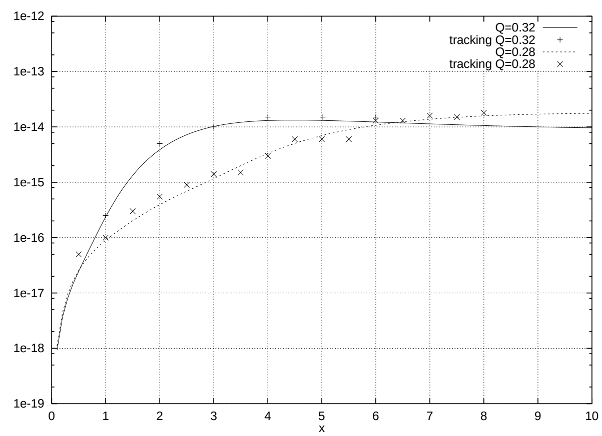

Figure 1: Diffusion coefficient in units of $[(\mathrm{mm\text{-}mrad})^2/\mathrm{turn}]$ as a function of the particle amplitude in units of σ. The lines are the diffusion coefficient as evaluated from the analytical expression, $+$ are the tracking results for $Q = 0.32$ and $\times$ for $Q = 0.28$. Theory and simulation are in perfect agreement.

3 TWO-DIMENSIONAL CASE

This approach has been extended to a two dimensional model of the beam, see [5, 6]. The corresponding Fokker-Planck equation reads

$$\begin{aligned}\frac{\partial\rho(J_x, J_y)}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial J_x}\left(DJ_x(J_x, J_y)\frac{\partial\rho(J_x, J_y)}{\partial J_x}\right) \quad (8)\\ &+ \frac{1}{2}\frac{\partial}{\partial J_y}\left(DJ_y(J_x, J_y)\frac{\partial\rho(J_x, J_y)}{\partial J_y}\right).\end{aligned}$$

The analytical expression for the diffusion coefficients $DJ_x(J_x, J_y)$ and $DJ_y(J_x, J_y)$ can be found in [6]. See in Fig. 2 these coefficients evaluated for $Q_x = 0.31$, $Q_y = 0.32$ abd $\eta = 0.01$.

Integrating the Fokker-Planck Eq. (8) with an initial Gaussian distribution, absorbing boundaries at the action

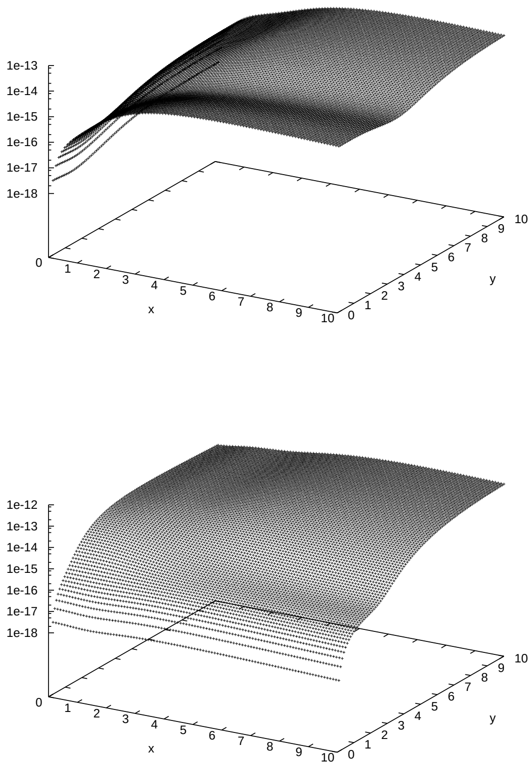

Figure 2: Vertical axis: 2D diffusion coefficient in [(mm-mrad)^2/turn], x and y are, respectively, the horizontal and vertical particle amplitudes in units of the rms beam size. Left: $DJ_{x,}$ off(J_x, J_y), right: $DJ_{y,}$ off(J_x, J_y).

corresponding to 10σ and reflecting boundaries at $J = 0$ and assuming the same parameters for both planes (this is a pessimistic approximation since the ground motion will have mainly an effect on the vertical plane) we evaluate the relative increment of the mean action in each plane as a function of time, see Fig. 3. The emittance doubling time is about 11 hours for the horizontal plane and 5 hours for the vertical plane. For realistic offset amplitudes of $|\eta_x| = |\eta_y| = 10^{-4}$ (in units of σ), we expect an emittance doubling time of 11×10^4 hours horizontally and 5×10^4 hours vertically.

4 CONCLUSIONS

We have estimated the influence of ground motion on the LHC beam applying ory of particle diffusion induced by the beam-beam head-on collision with random offset at the interaction point. We have found that the analytical expression of the one dimensional diffusion coefficient is in perfect agreement with the results of tracking. These calculations have been extended to a two-dimensional model. In these calculations we have used an OU spectrum for the noise. However the theory developed can also be applied to

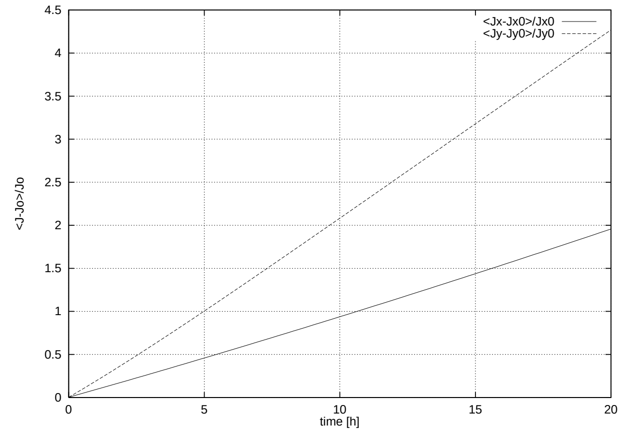

Figure 3: Relative increment of the average action in each plane ($< J_x - J_{x,0} > / J_{x,0}$, $< J_x - J_{x,0} > / J_{x,0}$) as a function of time. Evaluated using a two-dimensional model with $Q_x = 0.31$, $Q_y = 0.32$ (with $\eta = 0.01$, closed orbit spectral density 10^4 times greater than expected).

a measured noise spectrum by direct use of the expressions which require a knowledge of the correlation functions.

We have integrated the Fokker-Planck equation in a one- and two-dimensional case predicting for the LHC beam an emittance doubling time of about 5×10^4 hours for $Q = 0.32$. In order to keep the emittance doubling time larger than 1 day the spectral density of the offset fluctuations in the neighbourhood of the betatron tune should be below 10^{-16}mm^2/Hz which is three orders of magnitude below the expected density. We conclude that, under the weak-strong approximation and considering only head-on collisions, the ground motion alone has a negligible influence on the emittance of the beam. Several factors not included in this calculation may increase the emittance growth rate beyond the above estimates: the numerous long-range interactions, machine non-linearities and other effects which drive the nearby third order resonances have not been included.

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