The shape function in field theory

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The shape function describes (non-perturbative) Fermi motion effects in semi-inclusive heavy flavour decay. Its renormalization properties are substantially dependent on the kind of ultraviolet regulator used. For example, the double logarithm that appears at one loop is larger by a factor 2 in dimensional regularization than in lattice regularization. We show that factorization of long-distance effects inside the shape function is achieved with any regulator considered.

1. Introduction

In general, the shape function [1] is introduced to describe the decay of a hadron H_Q containing a heavy flavour Q into an inclusive hadronic state X with a large energy and a small invariant mass, plus non-QCD partons, i.e.

$$H_Q \to X + \mathrm{non-QCD\ \ partons} \quad (m_X \ll E_X)\,. \tag{1}$$

Specifically, the selected kinematics is

$$\frac{m_X^2}{E_X} \sim O(\Lambda_{QCD}), \qquad E_X \gg \Lambda_{QCD}. \tag{2}$$

In a more formal language, the conditions (2) correspond to the limit

$$E_X \to \infty, \quad m_X \to \infty, \quad \text{with } \frac{m_X^2}{E_X} \to \text{const.} \tag{3}$$

The limit (3) implies the infinite mass limit of the heavy flavour,

$$m_Q \to \infty, \tag{4}$$

as $m_Q \geq E_X$. The heavy quark can then be treated in the Heavy Quark Effective Theory (HQET) [2]. The divergence of m_X - even though it is slower than the one of E_X - implies that the final hadronic state can be replaced by a partonic one and that perturbation theory (PT) can be applied; however, the decay has also a non-perturbative component - to be factorized in the shape function - which we introduce with the following model.

We identify a hard subprocess in (1) consisting in the fragmentation of the heavy flavour Q,

$$Q \to \widehat{X} + \mathrm{non-QCD\ \ partons}. \tag{5}$$

$\widehat{X}$ differs from X in that it does not contain the light valence quark(s) of H_Q. The momentum of Q can be written as [2] [1]

$$p_Q = m_H v + k', \tag{6}$$

where m_H is the mass and

$$v^\mu = (1;0,0,0) \tag{7}$$

is the velocity of H_Q, which we take at rest without any loss of generality. The momentum of the light degrees of freedom in H_Q is $-k'$. The point $k'=0$ is the elastic one where all the initial light partons have soft momenta. It is natural to assume that [2,4]

$$k'_+ \sim k'_- \sim k'_\perp \sim O(\Lambda_{QCD})\,. \tag{8}$$

The distribution of the k'-momenta is clearly non-perturbative and it is related to the well-known Fermi motion of the heavy quark inside the hadron [5]. Let us denote by q^μ the momentum carried away by the non-QCD partons, taken along the $+z$ axis; the final hadronic system flies along the minus direction. $\widehat{X}$ has a mass

$$m_{\widehat{X}}^2 = (Q+k')^2 = m_X^2 + 2E_X\, k'_+ + \frac{m_X^2}{2E_X} k'_- + k'^2, \tag{9}$$

where Q^μ is the momentum available to the final partons [2],

$$Q \;\equiv\; m_H v - q = (m_H - q_0; 0, 0, -q_3)$$

[1]We use m_H instead of m_Q to define the momentum (6); this amounts to a shift in the range of k'_+ [3].
[2]The light-cone components are defined as $a_\pm \equiv a_0 \pm a_3$.

$$\begin{aligned} &= \left(E_X;0,0,-\sqrt{E_X^2-m_X^2}\right) \\ &\cong E_X\left(v_- + \frac{m_X^2}{4E_X^2}v_+\right), \end{aligned} \qquad (10)$$

with $v_\pm = (1;0,0,\pm 1)$. The sizes of the terms in the third member of eq. (9) are

$$\begin{aligned} 2E_X\, k'_+ &\sim O\left(E_X \Lambda_{QCD}\right), \\ \frac{m_X^2}{2E_X} k'_- &\sim k'^2 \sim O\left(\Lambda_{QCD}^2\right). \end{aligned} \qquad (11)$$

We now perform two different approximations:

1. We linearize the problem by dropping the k'^2 term in eq. (9), i.e. we describe the final hadronic system with the *HQET* or, in geometrical language, with a Wilson line *off* the light cone;

2. We drop the term $m_X^2/(2E_X)\,k'_-$ in the last member of eq.(9). Q^μ is replaced with a vector lying exactly on the light cone, in the minus direction. Neglecting these mass (virtuality) effects, the final hadronic system is described by the LEET [6] or, in geometrical language, by a Wilson line *on* the light cone.

Therefore, we have

$$m_{\widehat{X}}^2 \simeq m_X^2 + 2E_X\, k'_+. \qquad (12)$$

Equation (12) is the main result of this section. Let us comment on it. $m_{\widehat{X}}^2$ depends on a single light-cone component, k'_+ in our reference frame. Because of eqs. (2) and (8), the two terms on the r.h.s. of eq.(12) are of the same order: thus $m_{\widehat{X}}^2$ is affected in a substantial way by the distribution of the k'_+ momenta and cannot be considered constant.

The rate of the elementary process (5) contains perturbative corrections of double-logarithmic kind:

$$\alpha_S^n \left(\frac{\log^k[1-x]}{1-x}\right)_+ \qquad (0 \le k \le 2n-1), \qquad (13)$$

where [3]

$$x \equiv 1 - \frac{m_{\widehat{X}}^2}{E_X^2} \qquad (E_{\widehat{X}} \cong E_X). \qquad (14)$$

Since m_X^2 and $m_{\widehat{X}}^2$ are of the same order of magnitude, the limit (3) implies that the threshold region (also called large-x region) is approached:

$$x \to 1^-. \qquad (15)$$

The perturbative corrections (13) — enhanced in the threshold region — are large and radically modify the tree-level distribution [7]. The physical distribution for (1) is obtained by convoluting the perturbative corrections of the form (13) with the primordial k'_+-distribution: this is the way non-perturbative effects enter the game. We conclude that the process has a substantial non-perturbative component related to the k'_+-distribution.

Let us also present another way to establish the non-perturbative component in the decay (1), which is a critical analysis of dynamics using only perturbation theory. We neglect all the binding effects (confinement, Fermi motion, etc.) and consider an isolated on-shell heavy quark Q, i.e. we take $k' = 0$ [4]. Q decays into a massless quark q plus non-QCD partons:

$$Q \to q + \mathrm{non-QCD\ \ partons}. \qquad (16)$$

We now consider the emission of a soft gluon, with momentum components of the order of the QCD scale:

$$k_+ \sim k_- \sim k_\perp \sim O\left(\Lambda_{QCD}\right). \qquad (17)$$

The invariant mass of the final hadronic state is

$$m_X^2 = (p_q + k)^2 \simeq 2E_X\, k_+ \sim O\left(E_X\, \Lambda_{QCD}\right). \qquad (18)$$

An invariant mass of the order of (2), i.e. rather large, is generated by the emission of a very soft gluon. A kinematical amplification by a factor E_X has occurred as the kinematics goes from

[3] The plus-distribution $P(x)_+$ is defined on test functions $f(x)$ in the unit interval $x \in [0,1]$ as $\int_0^1 P(x)\,[f(x) - f(1)]\,dx$.

[4] The physical process and the subprocess coincide in this case.

time-like to light-like with the fragmentation of the heavy flavour. Higher-order perturbative corrections replace the bare coupling with the running coupling evaluated at the transverse gluon momentum [8],

$$\alpha_S \to \alpha_S\left(k_\perp\right). \tag{19}$$

For a soft gluon with the momentum (17), the coupling is evaluated close to the Landau pole, indicating the presence of non-perturbative effects. We conclude again that the decay (1) has the non-perturbative component identified before.

Let us observe that the linearization introduced in 1. and the light-cone limit introduced in 2. are not valid approximations for a hard collinear gluon; by this we mean a gluon with momentum components of order

$$k_- \sim O\left(E_X\right), \quad k_+ \sim O\left(\Lambda_{QCD}\right)$$

and, in order to have $k^2 \sim 0$:

$$k_\perp \sim O\left(\sqrt{E_X\,\Lambda_{QCD}}\right). \tag{20}$$

For such a gluon, all the terms in the last member of eq. (9) are of the same order. Its contribution can be considered a short-distance effect, since the transverse momentum (20) is substantially larger than that of a soft gluon (17) and the related coupling constant is in the perturbative region.

According to the above discussion, it is natural to identify the non-perturbative component in (1) as the following matrix element:

$$f\left(k_+\right) \equiv \langle H_Q\left(v\right)|h_v^\dagger \delta\left(k_+ - iD_+\right) h_v|H_Q\left(v\right)\rangle, \tag{21}$$

where h_v is a field in the HQET with velocity v. This function gives the probability that the heavy quark in the hadron has a plus virtuality equal to k_+, independently from the other components. In the following section we will critically analyse how the above matrix element factorizes the non-perturbative effects. We already found that the hard collinear region (20) cannot be described by the shape function but can consistently be incorporated into a coefficient function [5] [7].

[5]The coefficient function is also called jet factor, collinear factor, matching constant, hard factor and short-distance cross-section.

2. The shape function in various regularizations

In order to avoid distributions and to deal only with ordinary functions, it is convenient to consider the light-cone function [9]

$$F\left(k_+\right) \equiv \langle H_Q|h_v^\dagger \frac{1}{k_+ - iD_+ + i\epsilon} h_v|H_Q\rangle, \tag{22}$$

from which the shape function is obtained by taking the imaginary part,

$$f\left(k_+\right) = -\frac{1}{\pi}\mathrm{Im}\;F\left(k_+\right). \tag{23}$$

We want to study if the light-cone function factorizes the non-perturbative effects by performing a perturbative computation. This involves 3 steps:

1. To replace the hadronic light-cone function with a partonic light-cone function [6],

   $$F\left(k_+\right) = \langle Q|h_v^\dagger \frac{1}{k_+ - iD_+ + i\epsilon} h_v|Q\rangle, \tag{24}$$

   and to perform a perturbative computation. We take the heavy quark Q with the momentum (6), i.e off-shell by k';

2. To perform the same perturbative computation — with the same external states — in the original high-energy theory, i.e. full QCD;

3. To compare the results to see if the difference is a short-distance effect or not.

All this procedure has a meaning if we accept the following assumption: the long-distance effects of *perturbative* kind — i.e. the leading infrared logarithms — are able to trace the long-distance effects of *non-perturbative* nature. Therefore, if two matrix elements have the *same perturbative* long-distance contributions, they manifest also the *same non-perturbative* long-distance effects if computed with a non-perturbative technique, such as lattice QCD [7].

[6]The same symbol $F\left(k_+\right)$ is used for the two different matrix elements, as this should not cause confusion.

[7]An exception to this rule seems to be the observed difference of the fragmentation functions of the heavy flavours, as extracted from electron and proton collisions [10].

At the tree level, we have the expected result

$$F(k_+)^{tree} = \frac{1}{k'_+ - k_+ + i\epsilon}, \tag{25}$$

i.e.

$$f(k_+)^{tree} = \delta\left(k'_+ - k_+\right). \tag{26}$$

The shape function is a spike when its argument matches the plus virtuality of the external state, $k_+ = k'_+$, independently of the other components, k'_- and $k'_\perp$. We now consider the double logarithm that appears at one-loop.

Let us first discuss the case of a simple regularization cutting the space momenta:

$$|\overrightarrow{l}| < \Lambda_S, \qquad -\infty < l_0 < +\infty. \tag{27}$$

Since infrared logarithms are associated to quasi-real configurations, for which $l_0 \sim |\overrightarrow{l}|$, we expect this regularization to give the same double logarithm as lattice regularization (after continuation from Euclidean to Minkowski space). On the lattice, all the momentum components are cut off,

$$|l_\mu| < \frac{\pi}{a}, \tag{28}$$

where a is the lattice spacing. The result of the computation done with the regularization (27) is [9]

$$F(k_+)^{\Lambda_S} = \frac{1}{k+i0}\,\left(-\frac{a}{2}\right)\,\log^2\left(\frac{\Lambda_S}{-k-i0}\right), \tag{29}$$

where we defined the usual combination

$$k \equiv k'_+ - k_+. \tag{30}$$

and $a \equiv \alpha_S C_F/\pi$.

The result of the computation done with Dimensional Regularization (DR) is [11,9,12] [8]

$$\begin{aligned} F_B(k_+) &= \frac{1}{k+i0}\left(-\frac{a}{2}\right)\Gamma(1+\epsilon) \\ &\quad \frac{\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{\epsilon^2}\left(\frac{\mu}{-k-i0}\right)^{2\epsilon} \\ &= \frac{1}{k+i0}\,a\left[-\frac{1}{2\epsilon^2} - \frac{1}{\epsilon}\log\left(\frac{\mu}{-k-i0}\right)\right. \\ &\quad \left. - \log^2\left(\frac{\mu}{-k-i0}\right)\right], \end{aligned} \tag{31}$$

where $\epsilon \equiv 2 - D/2$ (with D the space-time dimension) and μ is the regularization scale [9]. We find a double pole of UV nature because infrared singularities (soft and collinear) are completely regulated by the light-quark leg being off-shell. The first problem we encounter is how to renormalize the above expression. The point is that, to obtain a finite result, one has to subtract not only the double pole, whose coefficient is just a number, but also the simple pole, which has a $\log\mu/k$ coefficient [14,12]. It seems that the subtraction of the simple pole cannot be performed with a local counter-term, as standard textbooks on renormalization teach one should do. However, if we blindly subtract all the poles, we obtain

$$F(k_+)^{\text{"}ren\text{"}} = \frac{1}{k+i0}\,(-a)\,\log^2\left(\frac{\mu}{-k-i0}\right). \tag{32}$$

We see that the coefficient of the double logarithm is 2 times larger than in eq. (29).

Even though we are not able to justify the renormalization procedure, we believe that the above result is correct. A hint in favour of this conjecture is obtained by repeating the same computation in the bare theory at finite ultraviolet (UV) cut-off, using a regularization similar to DR, i.e. cutting only the transverse momenta and not the longitudinal ones [10]:

$$|\overrightarrow{l}_\perp| < \Lambda_\perp, \qquad -\infty < l_+,\, l_- < \infty. \tag{33}$$

The unbounded integration over the longitudinal components does not give rise to UV divergences and we obtain the result [9]

$$F(k_+)^{\Lambda_\perp} = \frac{1}{k+i0}\,(-a)\,\log^2\left(\frac{\Lambda_\perp}{-k-i0}\right). \tag{34}$$

The transposition of symbols should be clear: the renormalization point μ is replaced by the UV cut-off $\Lambda_\perp$. As anticipated, there is agreement between the two results (32) and (34).

The origin of the factor 2 in the difference of the double logarithms is easily seen by considering

[8] This result is also in [13], but we disagree with the renormalization procedure [9].

[9] Actually, in the last member of (31), μ^2 should be replaced by $\mu^2 4\pi \exp[-\gamma_E]$ (γ_E is the Euler constant), even though this rescaling does not affect DLA results.

[10] The loop measure in DR is indeed regulated as $\int d^D l = 1/2 \int dl_+ dl_- d^{D-2} l_\perp$.

the integrals

$$I_{\Lambda_S,\Lambda_\perp} \equiv \int_0^\infty \frac{d\epsilon}{\epsilon}\int_0^{\epsilon^2}\frac{dl_\perp^2}{l_\perp^2}\theta\left(l_\perp^2-\epsilon k_+\right) \theta\left(\Lambda_S-\epsilon\right),\,\theta\left(\Lambda_\perp^2-l_\perp^2\right).$$

A straightforward computation indeed gives:

$$I_{\Lambda_S}=\frac{1}{2}\log^2\frac{\Lambda_S}{k_+},\quad I_{\Lambda_\perp}=\log^2\frac{\Lambda_\perp}{k_+}.$$

Let us analyse the domains of the 2 integrals in the plane $(\epsilon, l_\perp)$. For I_{Λ_S}, the integration region is a triangle-like domain D_S limited by the curves

$$l_\perp=\sqrt{k_+\epsilon},\quad l_\perp=\epsilon\quad\text{and}\quad\epsilon=\Lambda_S\qquad(D_S)\,.$$

The energies and the transverse momenta in D_S are in the ranges (see fig.1)

$$k_+<\epsilon<\Lambda_S,\qquad k_+<l_\perp<\Lambda_S\qquad(D_S).$$

Transverse momenta do become as small as $k_+\sim O\left(\Lambda_{QCD}\right)$, entering the non-perturbative region. For $I_{\Lambda_\perp}$, the integration region $D_\perp$ has again a triangle-like shape and it is limited by the curves

$$l_\perp=\sqrt{k_+\epsilon},\quad l_\perp=\epsilon\quad\text{and}\quad l_\perp=\Lambda_\perp\qquad(D_\perp)\,.$$

Assuming $\Lambda_\perp=\Lambda_S$, we see that $D_\perp$ contains D_S plus another region ΔD (see fig. 2),

$$D_\perp=D_S\cup\Delta D.$$

In the "new" region ΔD

$$\sqrt{k_+\Lambda_\perp}<l_\perp<\Lambda_\perp,\qquad\Lambda_\perp<\epsilon<\frac{\Lambda_\perp^2}{k_+}\qquad(\Delta D)\,.$$

The integration over ΔD gives a double logarithmic contribution equal to the one coming from the integration over D_S: that is the origin of the factor 2 in the difference between I_{Λ_S} and $I_{\Lambda_\perp}$. The contribution coming from ΔD is a short-distance contribution because it always involves rather large transverse momenta,

$$l_\perp\geq\sqrt{k_+\Lambda_\perp}\gg\Lambda_{QCD}\qquad(\Lambda_\perp\gg\Lambda_{QCD}).\quad(35)$$

Let us observe also that in ΔD the energy ϵ of the gluon can become much larger than the UV cutoff, $\epsilon\gg\Lambda_\perp$. One usually relates the cutoff to the hard scale E_X in the full QCD process, which, on the contrary, sets the upper bound for the gluon energies: $\epsilon\leq E_X$. It is clear that we are dealing with highly unphysical regularization scheme effects.

We now compare with the QCD rate, from which a full QCD light-cone function can be defined [9], given by

$$F(k_+)^{QCD}=\frac{1}{k+i0}\left(-\frac{a}{2}\right)\log^2\left(\frac{E_X}{-k-i0}\right).\quad(36)$$

It is clear that $\Lambda_\perp$ or μ is replaced by E_X. It is immediate to see that the double logarithms match for the Λ_S-regularization, while they do not match for the $\Lambda_\perp$-regularization. In the latter case the "spurious" contribution from ΔD can be subtracted by means of the coefficient function, which includes also "true" hard collinear effects, finite terms, etc. While in the Λ_S case the coefficient function contains at most a single logarithm of k_+, in the $\Lambda_\perp$ case it contains also a double logarithm of k_+.

3. Conclusions

We tried to describe in a way as transparent as possible the problem of factorization of non-perturbative effects in the decay (1) by means of the shape function. The starting point of our analysis is the observation that the shape function exhibits a different double logarithm at one loop, depending on the kind of UV regularization adopted [9]. In a regularization such as DR — cutting only transverse loop momenta — the double logarithm has a coefficient 2 times larger than the one derived with a lattice-like regularization. We have explicitly shown that the difference between the double logarithms is related to the integration over a region of large transverse momenta, so it is a short-distance effect. The QCD rate has a double logarithm equal to the one derived in lattice-like regularization. This implies that the coefficient function in lattice-like regularization contains at most a single logarithm of k_+ (the infrared scale of the problem), while in DR it contains also a double logarithm of k_+. In both regularizations, however, non-perturbative effects are factorized inside the shape function.

In general, we observe that there are UV regularization effects similar to those found for the shape function in other operators incorporating Wilson lines on the light cone [15].

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Figure captions

Fig.1 (above): Integration region for the soft gluon in the plane $(\epsilon, l_\perp)$ with the Λ_S regularization. We have taken $k_+ = 0.3$ GeV and $\Lambda_S = 1$ GeV.

Fig.2 (below): Integration region for the soft gluon in the plane $(\epsilon, l_\perp)$ with the $\Lambda_\perp$ regularization. We have taken $k_+ = 0.3$ GeV and $\Lambda_\perp = 1$ GeV. The domain D_S is to the left of the dashed line, ΔD is to the right.