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MULTIPLE SEPARATRIX CROSSING: CHAOS STRUCTURE

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Abstract

Numerical experiments on the structure of the chaotic component of motion under multiple crossing of the separatrix of a nonlinear resonance with time–varying amplitude are described with the main attention to the problem of ergodicity. The results clearly demonstrate nonergodicity of that motion due to the presence of a regular component of relatively small measure with a very complicated structure. A simple 2D–map per crossing has been constructed which qualitatively describes the main properties of both chaotic and regular components of the motion. An empirical relation for the correlation–affected diffusion rate has been found including a close vicinity of the chaos border where an evidence of the critical structure has been observed. Some unsolved problems and open questions are also discussed.

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1 Introduction

The present work continues the studies of chaotic motion under a slow separatrix crossing. This is a particular case of adiabatic processes which are very important in physics because of the adiabatic invariance, approximate though, that is of the conservation of action variables (J) under a slow parametric perturbation. The main problem here is the degree of accuracy or of violation of that invariance. Separatrix crossing produces the largest chaotic component in phase space whose size does not depend on the adiabatic parameter $\epsilon \to 0$ which, however, does affect the detailed structure of the motion as well as its time scale.

In our previous paper [1] the single separatrix crossing for a particular model was described in detail. Remarkably, a fairly simple relation for such a model in Ref.[2] we used turned out to be surprisingly accurate within the most part of the chaotic component.

In this paper we discribe the results of numerical experiments on multiple separatrix crossing. We focus on statistical properties of the motion, including the structure and measure of regular component disseminated into the chaotic 'sea' in a rather tricky way. The existence of regular component means nonergodicity of the motion, the question which has remained unclear for a long time until recently. To our knowledge, the nonergodicity of motion in a similar model was first predicted theoretically and estimated numerically in Ref.[3]. We have confirmed this result by different methods, and found many other characteristics of the motion structure. The present work, as well as the previous one [1], was stimulated by a very interesting study of the corresponding quantum adiabaticity [4]. We use the same classical model which is briefly described, for reader's convenience, in the next Section (for details see Ref.[1]).

The model is specified by the Hamiltonian:

$$H(x, p, t) \;=\; \frac{p^2}{2} \;+\; A_0 \, \sin{(\Omega t)} \cdot \cos{x} \; , \tag{2.1}$$

which describes a single nonlinear resonance in the pendulum approximation (see, e.g., Ref.[5, 6]) with a time–varying amplitude

$$A(t) \;=\; A_0 \, \sin{(\Omega t)} \; . \tag{2.2}$$

The dimensionless adiabaticity parameter is defined in the usual way as the ratio of perturbation/oscillation frequencies:

$$\epsilon \;=\; \frac{\Omega}{\sqrt{A_0}} \; , \tag{2.3}$$

where $\sqrt{A_0}$ is constant frequency of the small pendulum oscillation for the maximal amplitude.

Two branches of the instant, or 'frozen', separatrix at some $t = const$ are given by the relation

$$p_s(x'; t) \;=\; \pm 2 \sqrt{|A(t)|} \cdot \sin{\left(\frac{x'}{2} \right)} \; , \quad x' \;=\; \begin{cases} x, & A(t) > 0 \\ x - \pi, & A(t) < 0 \end{cases} \; . \tag{2.4}$$

Following previous studies of separatrix crossing we restrict ourselves below to this frozen approximation. As was shown in Ref.[1] the latter provides quite good accuracy of fairly simple theoretical relations.

In this approximation the action variable is defined in the standard way as

$$J \;=\; \frac{1}{2\pi} \oint p(x) \, dx \; , \tag{2.5}$$

where integral is taken over the whole period for x rotation (off the resonance) and over a half of that for x oscillation (inside the resonance). This difference is necessary to avoid the discontinuity of J at separatrix where the action is given by a simple expression

$$J \;=\; J_s(t) \;=\; \frac{4}{\pi} \sqrt{|A(t)|} \;\le\; J_{max} \;=\; \frac{4}{\pi} \sqrt{A_0} \; . \tag{2.6}$$

At $\Omega t \;=\; 0 \;\; (mod \, \pi)$ the action $J \;=\; |p|$, and the conjugated phase $\theta \;=\; x$. Notice that unlike p the action $J \geq 0$ is never negative.

transformation: $J/J_{max} \to J$. Then, the crossing region swept by separatrix is the unit interval, and J is simply related to the crossing time $t = t_{cr}$ by the expression

$$|A(t_{cr})| = J^2 , \quad 0 \le J \le 1 \tag{2.7}$$

while the adiabaticity parameter becomes $\epsilon = \Omega$.

Numerical integration of the motion equations for Hamiltonian (2.1) was performed in (x, p) variables using two algorithms. In most cases it was the so-called bilateral symplectic fourth-order Runge-Kutta algorithm as in Ref.[1]. However, in a few most long runs we applied a very simple and also symplectic first–order algorithm like in Ref.[2] which is actually the well known standard map [5] with the time–varying parameter:

$$\overline{\tilde{p}} = \tilde{p} + \tilde{A}_0 \cdot \sin\left(\tilde{\Omega}\tilde{t}\right) \cdot \sin x , \quad \overline{x} = x + \overline{\tilde{p}} , \tag{2.8}$$

where tilde marks the new set of quantities rescaled by the transformation

$$\tilde{A}_0 = \frac{1}{s^2} , \quad \tilde{t} = st , \quad \tilde{\Omega} = \frac{\Omega}{s} , \quad \tilde{p} = \frac{p}{s} . \tag{2.9}$$

Here s is the scaling parameter, and we remind that $A_0 = 1$. The primary goal of the rescaling was decreasing parameter $\tilde{A}_0$ which controls the computation accuracy. Usually, it was around $\tilde{A}_0 \approx 0.1$.

As is well known, the variation of J under adiabatic perturbation consists of the two qualitatively different parts: (i) the average action which is nearly constant between the crossings, up to an exponentially small correction, and which is of primary interest in our problem, and (ii) the rapid oscillation with the motion frequency. The ratio of the two time scales is $\sim \epsilon/\sqrt{|A(t)|} \ll 1$ which allows for efficient suppression of the second unimportant part of J variation by a simple averaging of $J(t)$ over a long time interval $\sim 1/\epsilon$ (see Ref.[1]).

3 Ergodicity

The ergodicity is the weakest statistical property in dynamical systems (see, e.g., Ref.[7]). Nevertheless, it is an important characteristic of the motion, necessary in the statistical theory (see, e.g., Ref.[8]).

The question of ergodicity of the motion under separatrix crossing remained open for a long time until recently. The upper bound for the measure (phase–space area) of a separate domain with regular motion ('stability islet') was estimated in Ref.[9] as $\mu_1 \lesssim \epsilon$.

first predicted theoretically and estimated numerically in Ref.[3]. The authors directly calculated the number and positions of stable trajectories for two different periods. Moreover, they were able to locate some of them in computation, and thus to measure their area in phase space which turned out to be surprisingly small.

Here, we make use of a different, statistical, approach. To this end, we first obtain from numerical experiments the steady–state distribution $f_s(J)$ in the action. In case of ergodic motion it should be constant. Examples of the distribution are shown in Fig.1 with the parameters listed in the Table below.

Table. Regular component under separatrix crossing

n	ϵ	$\mu_r \times 10^2$	$T \times N_{tr}$	N_b
1	0.1	0.68 ± 0.2	$2 \cdot 10^3 \times 1000$	200
2	0.05	0.75 ± 0.06	$4 \cdot 10^5 \times 200$	500
3	0.033	0.70 ± 0.2	$4 \cdot 10^5 \times 200$	200
4	0.033	0.81 ± 0.08	$4 \cdot 10^5 \times 150$	500
5	0.02	0.60 ± 0.05	$2 \cdot 10^6 \times 100$	200
6	0.01	0.75 ± 0.04	$4 \cdot 10^6 \times 100$	200

ϵ – parameter of adiabaticity
μ_r – total relative measure of regular component
T – number of separatrix crossings for each of N_{tr} trajectories
N_b – number of histogram bins in Fig.1
n – reference number for Fig.1

The striking feature of all the distributions is clear and rather specific inhomogeneity, reminiscent of a burst of 'icicles' hanging down from a nearly 'ergodic roof'. This directly demonstrates the generic nonergodic character of motion under separatrix crossing.

The histograms normalized in such a way that for ergodic motion the distribution $f_s(J) = 1$ while the sum over all the bins is also unity for any distribution. As a result the dips in the distribution ('icicles'), indicating the regular component, are compensated by an increase in the ergodic background. The latter is clearly seen in all distributions, especially for small J, and is a measure of the regular component. Namely, the relative measure (share) is given by the approximate relation

$$\mu_r \approx < f_s(J) - 1 > , \qquad J < J_1 , \tag{3.1}$$