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HIGHER DIMENSIONAL CLASS GROUPS OF GROUP-RINGS AND ORDERS IN ALGEBRAS OVER NUMBER FIELDS

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Introduction

For a Dedekind domain R with quotient field F, the notion of class groups of R-orders Λ is a natural generalisation of the notion of class groups of rings of integers in number fields as well as class groups of group-rings RG where G is a finite group. The class groups of group-rings, apart from their intimate connections with representation theory and number theory also house some topological invariants (e.g. Swan-Wall invariants) where G is usually the fundamental group of some spaces.

Now, if R is a Dedekind domain with quotient field F and Λ any R-order in a semi-simple F-algebra Σ, the higher class groups $C\ell_n(\Lambda)$ $n \geq 0$, is defined as

$$C\ell_n(\Lambda) := \mathrm{Ker}(SK_n(\Lambda) \to \underset{\mathbf{p}}{\oplus}\ SK_n(\hat{\Lambda}_{\mathbf{p}})) \tag{I}$$

where $\mathbf{p}$ runs through all the prime ideals of R and coincides with the usual class group $C\ell(\Lambda)$ at zero-dimensional level. Our attention in this paper is focussed on $C\ell_n(\Lambda)$ for R-orders Λ when R is the ring of integers in a number field, and we assume in the ensuing discussion that our R-orders are of this form.

The groups $C\ell_1(\Lambda), C\ell_1(RG)$ which are intimately connected with Whitehead groups and Whitehead torsion have been extensively studied by R. Oliver (see [19]). It is classical that $C\ell_0(\Lambda)$, $C\ell_1(\Lambda)$ are finite groups. However, it follows from some more recent results of this author that $C\ell_n(\Lambda)$ is finite for all $n \geq 1$ (see 2.2 or [17], [18]). If , is a maximal R-order, it follows from [8] that $C\ell_n(,) = 0$ for all $n \geq 1$.

We obtain in 2.3 the basic expression involving $C\ell_n(\Lambda)$ $n \geq 0$ that we want to analyse in this paper, namely

$$0 \longrightarrow \frac{K_{n+1}(\Sigma)}{\mathrm{Im}(K_{n+1}(\Lambda))} \longrightarrow \underset{\mathbf{p}}{\oplus}\ \frac{K_{n+1}(\hat{\Sigma}_{\mathbf{p}})}{\mathrm{Im}(K_{n+1}(\hat{\Lambda}_{\mathbf{p}}))} \longrightarrow C\ell_n(\Lambda) \longrightarrow 0 \tag{II}$$

and use this to show in 2.6 that if Λ, Λ' are two R-orders in semi-simple algebras Σ, Σ', then a surjective map $\varphi : \Sigma \to \Sigma'$ such that $\varphi(\Lambda) \subset \Lambda'$ induces a surjection $C\ell_n(\Lambda) \to C\ell_n(\Lambda')$ and we then deduce some consequences of this fact (see (2.7)). We show in 2.4 that there exists a finite set $\mathcal{P}$ of prime ideals $\mathbf{p}$ of R such that for $\mathbf{p} \notin \mathcal{P}$, $\hat{\Lambda}_{\mathbf{p}}$ is a maximal $\hat{R}_{\mathbf{p}}$-order in a semi-simple $\hat{F}_{\mathbf{p}}$-algebra $\hat{\Sigma}_{\mathbf{p}}$ which is unramified over its centre, in which case, $SK_n(\hat{\Lambda}_{\mathbf{p}}) = 0$ for all $n \geq 1$. We denote by $\check{\mathcal{P}}$ the set of rational primes lying below the primes $\mathbf{p} \in \mathcal{P}$.

Our analysis of the exact sequence (II) above at first concentrates on $\underset{\mathbf{p}}{\oplus} \frac{K_{n+1}(\hat{\Sigma}_{\mathbf{p}})}{\mathrm{Im}(K_{n+1}(\hat{\Lambda}_{\mathbf{p}}))}$ which maps on $C\ell_n(\Lambda)$ for all $n \geq 0$. We show that for all $n \geq 1$, the group $\underset{\mathbf{p} \notin \mathcal{P}}{\oplus} \frac{K_{n+1}(\hat{\Sigma}_{\mathbf{p}})}{\mathrm{Im}\ K_{n+1}(\hat{\Lambda}_{\mathbf{p}})}$ has no p-torsion for any rational prime $p \notin \check{\mathcal{P}}$ and deduce that for all $n \geq 1$, p-torsion can occur in $C\ell_n(\Lambda)$ only for rational primes $p \in \check{\mathcal{P}}$.

In [9], the authors considered only odd dimensional class groups and showed by a different method that p-torsion could occur in $C\ell_{2n-1}(\Lambda)$ only for primes lying in $\check{\mathcal{P}}$. Our result in 2.9 shows that this holds also for even dimensional class groups.

In 2.1.2, we obtain further analysis of $\underset{\mathbf{p}\in\mathcal{P}}{\oplus} \frac{K_n(\hat{\Sigma}_\mathbf{p})}{\mathrm{Im}(K_n(\hat{\Lambda}_\mathbf{p}))}$ which maps onto $C\ell_n(\Lambda)$ to prepare the way for our applications to finite p-groups in Section 3. In Section 3, we at first show that (I) and (II) above have particularly simple forms for even dimensional class groups i.e., we have exact sequences

$$0 \longrightarrow \frac{K_{2r+1}(,\ )}{\mathrm{Im}\ K_{2r+1}(\mathbb{Z}G)} \longrightarrow \frac{K_{2r+1}(\hat{,}\ _p)}{\mathrm{Im}\ K_{2r+1}(\hat{\mathbb{Z}}_pG)} \longrightarrow C\ell_{2r}(\mathbb{Z}G) \longrightarrow 0 \qquad \text{(III)}$$

where , is a maximal order containing $\mathbb{Z}G$, and

$$0 \longrightarrow C\ell_{2r}(\mathbb{Z}G) \longrightarrow SK_{2r}(\mathbb{Z}G) \longrightarrow SK_{2r}(\hat{\mathbb{Z}}_pG) \longrightarrow 0 \qquad \text{(IV)}$$

where G is a finite p-group (see 3.1 (i)). We show also that $\frac{K_{2r+1}(\Gamma)}{\mathrm{Im}\ K_{2r+1}(\mathbb{Z}G)}$ and $\frac{K_{2r+1}(\hat{\Gamma}_p)}{\mathrm{Im}\ K_{2r+1}(\hat{\mathbb{Z}}_pG)}$ are finite groups.

For odd-dimensional class groups, we show that for all $r \geq 1$, $C\ell_{2r-1}(\mathbb{Z}G)$ is a finite p-group (3.1 (vi)) if G is a finite p-group and in the process show that $\frac{K_{2r}(\Gamma)}{\mathrm{Im}\ K_{2r}(\mathbb{Z}G)}$ is a *finite* p-group and that $\frac{K_{2r}(\hat{\Gamma}_p)}{\mathrm{Im}\ K_{2r}(\hat{\mathbb{Z}}_pG)}$ is a p-torsion group (see 3.1 (iii) and (iv)).

In Section 4, we indicate how to use induction techniques of "Mackey functors" to obtain further results on $C\ell_n(RG)$ $n \geq 0$, G a finite group, R a Dedekind domain. First we show that for all $n \geq 0$, the $C\ell_n(R-)$ are Mackey functors that are modules over the Green functors $G_0(R-)$ and that they are hyper-elementary computable. Furthermore, we exploit the properties of $C\ell_n(R-)$ $(n \geq 0)$ as p-local Mackey functors to obtain a decomposition of $C\ell_n(\mathbb{Z}G)_{(p)}$, G any finite group, in terms of certain twisted group rings of p-groups (see 4.11).

Notes on Notation For any ring A, we write $K_n(A)$ for the Quillen K-groups $\pi_{n+1}(BQ\mathbf{P}(A)) = \pi_n(BGL(A)^+)$ where $\mathbf{P}(A)$ is the category of finitely generated projective A-modules. If A is Noetherian, we write $G_n(A)$ for $\pi_{r+1}(BQ\mathbf{M}(A))$ where $\mathbf{M}(A)$ is the category of finitely generated A-modules. If R is the ring of integers in a number field F, and Λ is an R-order in a semi-simple F-algebra Σ, we write $SK_n(\Lambda) = \mathrm{Ker}(K_n(\Lambda) \to K_n(\Sigma)), SK_n(\hat{\Lambda}_\mathbf{p}) = \mathrm{Ker}(K_n(\hat{\Lambda}_\mathbf{p}) \to K_n(\hat{\Sigma}_\mathbf{p}))$ where $\hat{\Lambda}_\mathbf{p} = \hat{R}_\mathbf{p} \otimes_R \Lambda, \hat{\Sigma}_\mathbf{p} = \hat{F}_\mathbf{p} \otimes_F \Sigma$ are the completions of Λ and Σ respectively at a prime $\mathbf{p}$ of R. We shall write $\mathcal{P}(\Lambda)$ or just $\mathcal{P}$ for the finite set of prime ideals $\mathbf{p}$ of R such that $\hat{\Lambda}_\mathbf{p}$ is not a maximal order in $\hat{\Sigma}_\mathbf{p}$ and $\check{\mathcal{P}}(\Lambda)$ or $\check{\mathcal{P}}$ for the finite set of rational primes lying below the prime ideals $\mathbf{p} \in \mathcal{P} := \mathcal{P}(\Lambda)$ (see 2.5).

1 Definitions of class groups and higher class group of orders and group-rings

In this section, we give the classical definition of class group of orders, which coincides with the zero dimensional form of higher class groups and record some of the known properties of class groups.

Definition 1.1 *Let R be a Dedekind domain with quotient field F. An R-order Λ in a finite dimensional semi-simple F-algebra Σ is a subring of Σ such that (i) R is contained in the centre of Λ, (ii) Λ is a finitely generated R-module and (iii) $F \otimes_R \Lambda = \Sigma$. For example, if G is any finite group, RG is an R-order in FG. A maximal R-order , in Σ is an order that is not contained in any other R-order in Σ. Note that every R-order is contained in at least one maximal order and every semi-simple F-algebra Σ contains at least one maximal order.*

Definition 1.2 *Let R, F, Σ, Λ be as in 1.1. A left-Λ-lattice is a left Λ-module which is also an R-lattice (i.e. finitely generated and projective as an R-module).*

A Λ-ideal in Σ is a left Λ-lattice $M \subset \Sigma$ such that $FM \subset \Sigma$.

Two left Λ-lattices M, N are said to be in the same genus if $M_p \simeq N_p$ for each prime ideal $\mathbf{p}$ of R. A left Λ-ideal is said to be locally free if $M_{\mathbf{p}} \simeq \Lambda_{\mathbf{p}}$ for all $\mathbf{p} \in \mathrm{Spec}(R)$. We write $M \vee N$ if M and N are in the same genus.

Let $\mathcal{P}(\Lambda) := \{\mathbf{p} \in \mathrm{Spec}(R) | \hat{\Lambda}_{\mathbf{p}}$ is not a maximal $\hat{R}_{\mathbf{p}}$-order in $\hat{\Sigma}_{\mathbf{p}}\}$. Then $\mathcal{P}(\Lambda)$ is a finite set and $\mathcal{P}(\Lambda) = \emptyset$ iff Λ is a maximal order. Note that the genus of a Λ-lattice M is determined by isomorphism classes of modules $\{M_{\mathbf{p}} | \mathbf{p} \in \mathcal{P}(\Lambda)\}$ see [4].

Theorem 1.3 [4] *Let L, M, N be lattices in the same genus. Then $M \oplus N \simeq L \oplus L'$ for some lattice L' in the same genus. Hence, if M, M' are locally free Λ-ideals in Σ, then $M \oplus M' = \Lambda \oplus M''$ for some locally free ideal M''.*

Definition 1.4 *Let R, F, Σ be as in 1.1. The idèle group of Σ, denoted $J(\Sigma)$ is defined by $J(\Sigma) := \{(\alpha_{\mathbf{p}}) \in \Pi(\hat{\Sigma}_{\mathbf{p}})^* | \alpha_{\mathbf{p}} \in \hat{\Lambda}^*_{\mathbf{p}}$ almost everywhere$\}$. For $\alpha = (\alpha_{\mathbf{p}}) \in J(\Sigma)$, define*

$$\Lambda\alpha := \Sigma \cap \left\{ \bigcap_{\mathbf{p}} \hat{\Lambda}_{\mathbf{p}} \alpha_{\mathbf{p}} \right\} = \bigcap_{\mathbf{p}} \left\{ \Sigma \cap \hat{\Lambda}_{\mathbf{p}} \alpha_{\mathbf{p}} \right\}$$

The group of principal idèles, denoted $u(\Sigma)$ is defined by $u(\Sigma) = \{\alpha = (\alpha_{\mathbf{p}}) | \alpha_{\mathbf{p}} = x \in \Sigma^*$ for all $\mathbf{p} \in \mathrm{Spec}(R)\}$. The group of unit idèles is defined by

$$U(\Lambda) = \prod_{\mathbf{p}} (\Lambda_{\mathbf{p}})^* \subseteq J(\Sigma)$$

Remarks 1.5 (i) $J(\Sigma)$ is independent of the choice of the R-order Λ in Σ since if Λ' is another R-order, then $\Lambda_{\mathbf{p}} = \Lambda'_{\mathbf{p}}$ a.e.

(ii) $\Lambda\alpha$ is isomorphic to a left ideal of Λ and $\Lambda\alpha$ is in the same genus as Λ. Call $\Lambda\alpha$ a locally free (rank 1) Λ-lattice or a locally free fractional Λ-ideal in Σ. Note that any $M \in g(\Lambda)$ can be written in the form $M = \Lambda\alpha$ for some $\alpha \in J(\Sigma)$ (see [4]).

(iii) If $\Sigma = F$ and $\Lambda = R$, we also have $J(F), u(F)$ and $U(R)$ as defined above.

(iv) For $\alpha, \beta \in J(\Sigma), \Lambda\alpha \oplus \Lambda\beta \cong \Lambda \oplus \Lambda\alpha\beta$ (see [4]).

Definition 1.6 *Let F, Σ, R, Λ be as in 1.1. Two left Λ-modules M, N are said to be stably isomorphic if $M \oplus \Lambda^{(k)} \simeq N \oplus \Lambda^{(k)}$ for some positive integer k. If F is a number field, then $M \oplus \Lambda^{(k)} \simeq N \oplus \Lambda^{(k)}$ iff $M \oplus \Lambda \simeq N \oplus \Lambda$. We write $[M]$ for the stable isomorphism class of M.*

Theorem 1.7 [4] *The stable isomorphism classes of locally free ideals form an Abelian group $C\ell(\Lambda)$ called the locally free class group of Λ where addition is given by $[M] + [M'] = [M'']$ whenever $M \oplus M' \simeq \Lambda \oplus M''$. The zero element is (Λ) and inverses exist since $(\Lambda\alpha) \oplus (\Lambda\alpha^{-1}) \simeq \Lambda \oplus \Lambda$ for any $\alpha \in J(\Sigma)$.*

Theorem 1.8 [4] *Let R, F, Λ, Σ be as in 1.1. If F is an algebriac number field, then $C\ell(\Lambda)$ is a finite group.*

Remarks 1.9 Let R, F, Λ, Σ be as in 1.1.

(i) If $\Lambda = R$, then $C\ell(\Lambda)$ is the ideal class group of R.

(ii) If , is a maximal R-order in Σ, then very left-ideal in Σ is locally free. So, $C\ell(,)$ is the group of stable isomorphism classes of all left , -ideals in Σ.

(iii) Define a map $J(\Sigma) \to C\ell(\Lambda); \alpha \to [\Lambda\alpha]$. Then one can show that this map is surjective and that the kernel is $J_0(\Sigma)\Sigma^* U(\Lambda)$ where $J_0(\Sigma)$ is the kernel of the reduced norm acting on $J(\Sigma)$. So $J(\Sigma)/(J_0(\Sigma)\Sigma^* U(\Lambda)) \simeq C\ell(\Lambda)$ (see [4]).

(iv) If G is a finite group such that no proper divisor of $|G|$ is a unit in R, then $C\ell(RG) \simeq SK_0(RG)$. Hence $C\ell(\mathbb{Z}G) \simeq SK_0(\mathbb{Z}G)$ for every finite group G (see [4]).

For computations of $C\ell(RG)$ for various R and G see [4].

Definition 1.10 *Let R be a Dedekind domain with quotient field F, Λ any R-order in a semi-simple F-algebra Σ. For $n \geq 0$, let $SK_n(\Lambda) = \mathrm{Ker}(K_n(\Lambda) \to K_n(\Sigma))$ and for any prime ideal $\mathbf{p}$ of R, let $SK_n(\hat{\Lambda}_{\mathbf{p}}) = \mathrm{Ker}(K_n(\hat{\Lambda}_{\mathbf{p}}) \to K_n(\hat{\Sigma}_{\mathbf{p}}))$. We now define $C\ell_n(\Lambda) := \mathrm{Ker}(SK_n(\Lambda) \xrightarrow{\gamma} \bigoplus_{\mathbf{p}} SK_n(\hat{\Lambda}_{\mathbf{p}}))$.*

Theorem 1.11 [4] $C\ell(\Lambda) = C\ell_0(\Lambda) \cong \mathrm{Ker}(SK_0(\Lambda) \to \bigoplus_{\mathbf{p}} SK_0(\hat{\Lambda}_{\mathbf{p}}))$.

2 Higher dimensional class groups of orders and group-rings

2.1 Quite a lot of work has been done, notably by R. Oliver, on $C\ell_1(\Lambda)$ and $C\ell_1(\mathbb{Z}G)$ where G is a finite group in connection with his intensive study of $SK_1(\Lambda)$ and $SK_1(\mathbb{Z}G)$, $SK_1(\hat{\mathbb{Z}}_p G)$ etc. (see [19]). We note in particular the following properties of $C\ell_1(\Lambda)$, where R is the ring of integers in a number field and Λ any R-order in a semi-simple F-algebra.

(i) $C\ell_1(\Lambda)$ is finite;

(ii) If G is any Abelian group, $C\ell_1(RG) = SK_1(RG)$

(iii) $C\ell_1(\mathbb{Z}G) \neq 0$ if G is a non-Abelian p-group

(iv) $C\ell_1(\mathbb{Z}G) = 0$ if G is a Dihedral or quaternion 2-group.

For further information on computations of $C\ell_1(\mathbb{Z}G)$, (see [19]).

We now endeavour to obtain information on $C\ell_n(\Lambda)$ for all $n \geq 1$.

We first show that for all $n \geq 1$, $C\ell_n(\Lambda)$ is a finite group. This follows from some earlier results of the author. We state this result formally.

Theorem 2.2 *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ. Then, $C\ell_n(\Lambda)$ is a finite group for all $n \geq 1$.*

Sketch of Proof It suffices to show that for all $n \geq 1$, $SK_n(\Lambda)$ is a finite group. Now in [16], it was shown that $K_n(\Lambda)$ is a finitely generated Abelian group. So $SK_n(\Lambda)$ is also finitely generated. Also it was shown in [17], that $SK_n(\Lambda)$ is torsion. Hence $SK_n(\Lambda)$ is finite. Also, see [18], 1.7, for a more direct proof that $SK_n(\Lambda)$ is finite.

We next present a fundamental sequence involving $C\ell_n(\Lambda)$ in the following

Theorem 2.3 *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ. If $\mathbf{p}$ is any prime=maximal ideal of R, write $\hat{\Lambda}_{\mathbf{p}} = \hat{R}_{\mathbf{p}} \otimes_R \Lambda$, $\hat{\Sigma}_{\mathbf{p}} = \hat{F}_{\mathbf{p}} \otimes \Sigma$ where $\hat{R}_{\mathbf{p}}$, $\hat{F}_{\mathbf{p}}$ are completions of R, F respectively at $\mathbf{p}$. Then we have the following exact sequence:*

$$0 \to K_{n+1}(\Sigma)/\mathrm{Im}(K_{n+1}(\Lambda)) \to \bigoplus_{\mathbf{p}\in\max(R)} (K_{n+1}(\hat{\Sigma}_{\mathbf{p}})/\mathrm{Im}(K_{n+1}(\hat{\Lambda}_{\mathbf{p}})) \to C\ell_n(\Lambda) \to 0 \qquad \text{(I)}$$

Proof. Consider the following commutative diagram of localisation sequences of Quillen where $S = R - 0$; $\hat{S}_{\mathbf{p}} = \hat{R}_{\mathbf{p}} - 0$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{K_{n+1}(\Sigma)}{\mathrm{Im}(K_{n+1}(\Lambda))} & \longrightarrow & K_n(\mathbf{H}_S(\Lambda)) & \longrightarrow & SK_n(\Lambda) & \longrightarrow & 0 \\ & & \downarrow \alpha_n & & \downarrow \rho & & \downarrow \gamma_n & & \\ 0 & \longrightarrow & \bigoplus_{\mathbf{p}} \frac{K_{n+1}(\hat{\Sigma}_{\mathbf{p}})}{\mathrm{Im}(K_{n+1}(\hat{\Lambda}_{\mathbf{p}})} & \longrightarrow & \bigoplus_{\mathbf{p}} K_n(\mathbf{H}_{\hat{S}_{\mathbf{p}}}(\hat{\Lambda}_{\mathbf{p}}) & \longrightarrow & \bigoplus_{\mathbf{p}} SK_n(\hat{\Lambda}_{\mathbf{p}}) & \longrightarrow & 0 \end{array} \qquad \text{(II)}$$

where $K_n(H_S(\Lambda) \overset{\rho}{\simeq} \bigoplus_{\mathbf{p}} K_n(H_{\hat{S}_{\mathbf{p}}}(\hat{\Lambda}_{\mathbf{p}}))$ is an isomorphism. By applying the Snake lemma to diagram (II), we have Ker $\gamma_n \simeq$ Coker α_n and Coker $\gamma_n = 0$ and Ker $\alpha_n = 0$. Since, by definition, $C\ell_n(\Lambda) = \mathrm{Ker}(SK_n(\Lambda) \to \bigoplus_{\mathbf{p}} SK_n(\hat{\Lambda}_{\mathbf{p}}))$, we have the required exact sequence

Lemma 2.4 In the exact sequence

$$0 \to C\ell_n(\Lambda) \to SK_n(\Lambda) \to \bigoplus_{\mathbf{p}} SK_n(\hat{\Lambda}_{\mathbf{p}}) \to 0\ ,$$

$SK_n(\hat{\Lambda}_{\mathbf{p}}) = 0$ for almost all $\mathbf{p}$, i.e. $\bigoplus_{\mathbf{p}} SK_n(\hat{\Lambda}_{\mathbf{p}})$ is a finite direct sum.

Proof. It is well known that for almost all $\mathbf{p}$, $\hat{\Lambda}_{\mathbf{p}}$ is a maximal order in a split semi-simple algebra $\hat{\Sigma}_{\mathbf{p}}$. Now, when $\hat{\Lambda}_{\mathbf{p}}$ is a maximal order in $\hat{\Sigma}_{\mathbf{p}}$, we have by [12], 1.1, that $SK_{2n-1}(\hat{\Lambda}_{\mathbf{p}}) = 0$ iff $\hat{\Sigma}_{\mathbf{p}}$ splits. Moreover, $SK_{2n}(\hat{\Lambda}_{\mathbf{p}}) = 0$ for all $n \geq 1$ by [10], 1.3(b). So, for almost all $\mathbf{p}$, $SK_n(\hat{\Lambda}_{\mathbf{p}}) = 0$ for all $n \geq 1$.

Remarks 2.5 (i) In view of 2.4, there exists a finite set $\mathcal{P}(\Lambda)$ of prime ideals $\mathbf{p}$ of R such that for $\mathbf{p} \notin \mathcal{P}(\Lambda)$, $\hat{\Lambda}_{\mathbf{p}}$ is maximal and $\hat{\Sigma}_{\mathbf{p}}$ splits in which case $SK_n(\hat{\Lambda}_{\mathbf{p}}) = 0$ for all $n \geq 1$. We shall often write $\mathcal{P}$ for $\mathcal{P}(\Lambda)$ when the context is clear, as well as $\check{\mathcal{P}} = \check{\mathcal{P}}(\Lambda)$ for the set of rational primes lying below the prime ideals in $\mathcal{P} = \mathcal{P}(\Lambda)$.

(ii) If $\Lambda = RG$ where G is a finite group, then the prime ideals $\mathbf{p} \in \mathcal{P}$ lies above the prime divisors of $|G|$. In particular if $R = \mathbb{Z}$, then $\check{\mathcal{P}}$ consists of the prime divisors of $|G|$.

(iii) If , is a maximal order containing Λ such that p does not divide [, $: \Lambda] :=$ the index of Λ in , , then $p \notin \check{\mathcal{P}}$ (see [19]).

Theorem 2.6 *Let R be the ring of integers in a number field F, Λ, Λ' R-orders in semi-simple F-algebras Σ, Σ' respectively. Suppose that $\varphi : \Sigma \to \Sigma'$ is a surjection of algebras such that $\varphi(\Lambda) \subset \Lambda'$. Then φ induces a surjection*

$$C\ell_n(\Lambda) \to C\ell_n(\Lambda') \text{ for all } n \geq 1$$

Proof. Consider the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{K_{n+1}(\Sigma)}{\mathrm{Im}(K_{n+1}(\Lambda))} & \longrightarrow & \bigoplus\limits_{\mathbf{p}} \frac{K_{n+1}(\hat{\Sigma}_{\mathbf{p}})}{\mathrm{Im}K_{n+1}(\hat{\Lambda}_{\mathbf{p}})} & \longrightarrow & C\ell_n(\Lambda) & \longrightarrow & 0 \\
 & & \Big\downarrow \varphi_\Sigma & & \Big\downarrow \varphi_{\hat{\Sigma}} & & \Big\downarrow \varphi_\Lambda & & \\
0 & \longrightarrow & \bigoplus\limits_{\mathbf{p}} \frac{K_{n+1}(\Sigma')}{\mathrm{Im}(K_{n+1}(\Lambda')} & \longrightarrow & \bigoplus\limits_{\mathbf{p}} \frac{K_{n+1}(\hat{\Sigma}'_{\mathbf{p}})}{\mathrm{Im}K_{n+1}(\hat{\Lambda}'_{\mathbf{p}})} & \longrightarrow & C\ell_n(\Lambda') & \longrightarrow & 0
\end{array}$$

By the Snake lemma, we have

$$0 \to \text{Ker } \varphi_\Sigma \to \text{Ker } \varphi_{\hat{\Sigma}} \to \text{Ker } \varphi_\Lambda \to \text{Coker } \varphi_\Sigma \to \text{Coker } \varphi_{\hat{\Sigma}} \to \text{Coker } \varphi_\Lambda \to 0$$

Now, since $\varphi : \Sigma \to \Sigma'$ is a projection onto a direct summand, then φ_Σ is onto (that is Coker $\varphi_\Sigma = 0$) and so $\varphi_{\hat{\Sigma}}$ is also onto i.e., Coker $\varphi_{\hat{\Sigma}} = 0$. Hence coker $\varphi_\Lambda = 0$ i.e., φ_Λ is onto as required.

Corollary 2.7 (i) Let R be the ring of integers in a number field F. If $\Lambda \subseteq \Lambda'$ are R-orders in a semi-simple F-algebra Σ, then the induced maps $C\ell_n(\Lambda) \to C\ell_n(\Lambda')$ are surjective for all $n \geq 1$

(ii) If $G \to G'$ is an epimorphism of finite groups, then the induced maps $C\ell_n(RG) \to C\ell_n(RG')$ are surjective for all $n \geq 1$.

Proof. Follows from 2.6 by considering $\Sigma = \Sigma'$.

Remarks 2.8 Since from 2.3, the group $C\ell_n(\Lambda)$ is the homomorphic image of $\underset{\mathbf{p}}{\oplus} \frac{K_{n+1}(\hat{\Sigma}_{\mathbf{p}})}{\text{Im } K_{n+1}(\hat{\Lambda}_{\mathbf{p}})}$, we analyse the latter group as much as possible. First we prove the following.

Theorem 2.9 *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ, $\mathcal{P}(\Lambda), \check{\mathcal{P}}(\Lambda)$ as in 2.5. Then, for all $n \geq 1$, we have*

(i) $C\ell_{2n}(\Lambda)$ is the homomorphic image of $\underset{\mathbf{p}\in\mathcal{P}}{\oplus} K_{2n+1}(\hat{\Sigma}_{\mathbf{p}})/\text{Im}(K_{2n+1}(\hat{\Lambda}_{\mathbf{p}}))$.

(ii) $C\ell_{2n-1}(\Lambda)$ is the homomorphic image of $\underset{\mathbf{p}\notin\mathcal{P}}{\oplus} G_{2n-1}(\hat{\Lambda}_{\mathbf{p}}/\mathbf{p}\hat{\Lambda}_{\mathbf{p}}) \oplus (\underset{\mathbf{p}\in\mathcal{P}}{\oplus} (K_{2n}(\hat{\Sigma}_{\mathbf{p}})/\text{Im}(K_{2n}(\hat{\Lambda}_{\mathbf{p}}))$.
Hence <u>for all</u> $n \geq 1$, p-torsion can occur in $C\ell_n(\Lambda)$ only for rational primes p lying in $\check{\mathcal{P}}(\Lambda)$.

Remarks 2.10 In [9], the authors considered only odd-dimensional class groups and showed by a different method that p-torsion could occur in $C\ell_{2n-1}(\Lambda)$ only for primes p lying in $\check{\mathcal{P}}(\Lambda)$. It follows from 2.9 above that this is also the case for even-dimensional class groups.

Proof of 2.9 Let $\mathcal{P} = \mathcal{P}(\Lambda)$ be as defined in 2.5. We first show that for all $n \geq 1$,

$$\underset{\mathbf{p}}{\oplus} \frac{K_{n+1}(\hat{\Sigma}_{\mathbf{p}})}{\text{Im } K_{n+1}(\hat{\Lambda}_{\mathbf{p}})} \cong \left(\underset{\mathbf{p}\notin\mathcal{P}}{\oplus} G_n(\hat{\Lambda}_{\mathbf{p}}/\mathbf{p}\hat{\Lambda}_{\mathbf{p}})\right) \oplus \left(\underset{\mathbf{p}\in\mathcal{P}}{\oplus} \frac{K_{n+1}(\hat{\Sigma}_{\mathbf{p}})}{\text{Im } (K_{n+1}(\hat{\Lambda}_{\mathbf{p}})}\right) \qquad \text{(I)}$$

Now, for $\mathbf{p} \notin \mathcal{P}$, $\hat{\Lambda}_{\mathbf{p}}$ is a maximal order (a regular ring) and so, $K_n(\hat{\Lambda}_{\mathbf{p}}) \simeq G_n(\hat{\Lambda}_{\mathbf{p}})$. So, for each $\mathbf{p} \notin \mathcal{P}$, we have Quillen's localisation sequence

$$\ldots K_{n+1}(\hat{\Lambda}_{\mathbf{p}}) \longrightarrow K_{n+1}(\hat{\Sigma}_{\mathbf{p}}) \longrightarrow K_n(\mathcal{M}_{\hat{S}_{\mathbf{p}}}(\hat{\Lambda}_{\mathbf{p}})) \longrightarrow K_n(\hat{\Lambda}_{\mathbf{p}}) \longrightarrow K_n(\hat{\Sigma}_{\mathbf{p}}) \longrightarrow \ldots$$

where $\hat{S}_{\mathbf{p}} = \hat{R}_{\mathbf{p}} - 0$ and $\mathcal{M}_{\hat{S}_{\mathbf{p}}}(\hat{\Lambda}_{\mathbf{p}})$ is the category of finitely generated $\hat{S}_{\mathbf{p}}$-torsion $\hat{\Lambda}_{\mathbf{p}}$-modules. Now $K_n(\mathcal{M}_{\hat{S}_{\mathbf{p}}}(\hat{\Lambda}_{\mathbf{p}})) \simeq G_n(\hat{\Lambda}_{\mathbf{p}}/\mathbf{p}\hat{\Lambda}_{\mathbf{p}})$ by Devissage. This proves (I) above.

Now, $G_n(\hat{\Lambda}_{\mathbf{p}}/\mathbf{p}\hat{\Lambda}_{\mathbf{p}}) \simeq K_n((\hat{\Lambda}_{\mathbf{p}}/\mathbf{p}\hat{\Lambda}_{\mathbf{p}})/\text{rad}(\hat{\Lambda}_{\mathbf{p}}/\mathbf{p}\hat{\Lambda}_{\mathbf{p}})) \simeq K_n(\hat{\Lambda}_{\mathbf{p}}/\text{rad}(\hat{\Lambda}_{\mathbf{p}}))$ where $\hat{\Lambda}_{\mathbf{p}}/\text{rad}\hat{\Lambda}_{\mathbf{p}}$, as a finite semi-simple ring is a product of matrix algebras over finite fields and so, when n is even, i.e. $n = 2r, r \geq 1$, we have $G_{2r}(\hat{\Lambda}_{\mathbf{p}}/\mathbf{p}\hat{\Lambda}_{\mathbf{p}}) \simeq K_{2r}(\hat{\Lambda}_{\mathbf{p}}/\text{rad}\hat{\Lambda}_{\mathbf{p}}) = 0$.

If n is odd, i.e. $n = 2r-1, r \geq 1$, then $|K_{2r-1}(\hat{\Lambda}_{\mathbf{p}}/\text{rad}\hat{\Lambda}_{\mathbf{p}})| \equiv 1(p)$ by Quillen's results on K-theory of finite fields where p is a rational prime lying below $\mathbf{p}$.

So, we have shown that in (I), for all $n \geq 1$, $\underset{\mathbf{p}\notin\mathcal{P}}{\oplus} \frac{K_{n+1}(\hat{\Sigma}_{\mathbf{p}})}{\text{Im } K_{n+1}(\hat{\Lambda}_{\mathbf{p}})} \simeq \underset{\mathbf{p}\notin\mathcal{P}}{\oplus} (G_n(\hat{\Lambda}_{\mathbf{p}}/\mathbf{p}\hat{\Lambda}_{\mathbf{p}})$ has no p-torsion for $p \notin \check{\mathcal{P}}$ and hence that for all $n \geq 1$, p-torison can occur in $C\ell_n(\Lambda)$ only for primes p lying in $\check{\mathcal{P}}$.

Remarks 2.11 We now obtain a further analysis of $\underset{\mathbf{p}\in\mathcal{P}}{\oplus} \frac{K_n(\hat{\Sigma}_{\mathbf{p}})}{\text{Im}(K_n(\hat{\Lambda}_{\mathbf{p}})}$. So let $\hat{\Gamma}_{\mathbf{p}}$ be a maximal order containing $\hat{\Lambda}_{\mathbf{p}}$, $\mathbf{p} \in \mathcal{P}$. Then the inclusion $\hat{\Lambda}_{\mathbf{p}} \to \hat{\Gamma}_{\mathbf{p}}$ induces homomorphisms $K_n(\hat{\Lambda}_{\mathbf{p}}) \to K_n(\hat{\Gamma}_{\mathbf{p}})$. We now have the following

Lemma 2.12 (i) For $\mathbf{p} \in \mathcal{P}$ and all $r \geq 1$ we have an exact sequence

$$0 \longrightarrow \frac{K_{2r}(\hat{\Gamma}_{\mathbf{p}})}{\text{Im } K_{2r}(\hat{\Lambda}_{\mathbf{p}})} \longrightarrow \frac{K_{2r}(\hat{\hat{\Sigma}}_{\mathbf{p}})}{\text{Im } K_{2r}(\hat{\Lambda}_{\mathbf{p}})} \longrightarrow G_{2r-1}(\hat{\Gamma}_{\mathbf{p}}/\mathbf{p}\hat{\Gamma}_{\mathbf{p}}) \longrightarrow 0$$

where $|G_{2r-1}(\hat{\Gamma}_{\mathbf{p}}/\mathbf{p}\hat{\Gamma}_{\mathbf{p}})| \equiv 1$ (p) for some rational prime lying below $\mathbf{p}$

(ii)

$$\frac{K_{2r+1}(\hat{\Sigma}_{\mathbf{p}})}{\text{Im } K_{2r+1}(\hat{\Lambda}_{\mathbf{p}})} \simeq \left(\frac{K_{2r+1}(\hat{\Gamma}_{\mathbf{p}})}{\text{Im } K_{2r+1}(\hat{\Lambda}_{\mathbf{p}})} \right) \Big/ H_{\mathbf{p}} \ \text{ for any } \ \mathbf{p} \in \mathcal{P}, \quad r \geq 1,$$

where $H_{\mathbf{p}}$ is a subgroup of $\frac{K_{2r+1}(\hat{\Gamma}_{\mathbf{p}})}{\text{Im} K_{2r+1}(\hat{\Lambda}_{\mathbf{p}})}$ of order $\equiv 1 \pmod p$ for some rational prime p lying below $\mathbf{p}$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} K_n(\hat{\Lambda}_{\mathbf{p}}) & \xrightarrow{\beta} & K_n(\hat{\Gamma}_{\mathbf{p}}) \\ & \searrow^{\gamma} \qquad\qquad {}^{\alpha}\swarrow & \\ & K_n(\hat{\Sigma}_{\mathbf{p}}) & \end{array} \tag{I}$$

Then we have an exact sequence

$$0 \longrightarrow \text{Ker } \beta \longrightarrow SK_n(\hat{\Lambda}_{\mathbf{p}}) \longrightarrow SK_n(\hat{\Gamma}_{\mathbf{p}}) \longrightarrow \frac{K_n(\hat{\Lambda}_{\mathbf{p}})}{\text{Im}(K_n(\hat{\Lambda}_{\mathbf{p}})}$$

$$\longrightarrow K_n(\hat{\Sigma}_{\mathbf{p}})/\text{Im}K_n(\hat{\Lambda}_{\mathbf{p}}) \longrightarrow \frac{K_n(\hat{\Sigma}_{\mathbf{p}})}{\text{Im } K_n(\hat{\Gamma}_{\mathbf{p}})} \longrightarrow 0$$

Now, since $SK_{2r}(\hat{\Gamma}_{\mathbf{p}}) = 0$ for all $\mathbf{p}$ (see [10] or [13]), we have an exact sequence

$$0 \longrightarrow \frac{K_{2r}(\hat{\Gamma}_{\mathbf{p}})}{\text{Im } K_{2r}(\hat{\Lambda}_{\mathbf{p}})} \longrightarrow \frac{K_{2r}(\hat{\Sigma}_{\mathbf{p}})}{\text{Im } K_{2r}(\hat{\Gamma}_{\mathbf{p}})} \xrightarrow{\nu} \frac{K_{2r}(\hat{\Sigma}_{\mathbf{p}})}{\text{Im } K_{2r}(\hat{\Gamma}_{\mathbf{p}})} \longrightarrow 0$$

Now, consider the localisation sequence

$$0 \longrightarrow K_{2r}(\hat{\Gamma}_{\mathbf{p}}) \longrightarrow K_{2r}(\hat{\Sigma}_{\mathbf{p}}) \xrightarrow{\delta} G_{2r-1}(\hat{\Gamma}_{\mathbf{p}}/\mathbf{p}\hat{\Gamma}_{\mathbf{p}})$$

$$\xrightarrow{\rho} SK_{2r-1}(\hat{\Gamma}_{\mathbf{p}}) \longrightarrow 0\ .$$

By a similar argument to that given in the proof of 2.9, we have that

$$G_{2r-1}(\hat{\Gamma}_{\mathbf{p}}/\mathbf{p}\hat{\Gamma}_{\mathbf{p}}) \simeq K_{2r-1}(\hat{\Gamma}_{\mathbf{p}}/\text{rad } \hat{\Gamma}_{\mathbf{p}})$$

has order relatively prime to the rational prime lying below $\mathbf{p}$. Hence (i) is proved.

(ii) From diagram (I), we obtain an exact sequence

$$\ldots \longrightarrow SK_n(\hat{\Gamma}_{\mathbf{p}}) \xrightarrow{\delta} \frac{K_n(\hat{\Gamma}_{\mathbf{p}})}{\text{Im } (K_n(\hat{\Lambda}_{\mathbf{p}}))} \longrightarrow \frac{K_n(\hat{\Sigma}_{\mathbf{p}})}{\text{Im } (K_n(\hat{\Lambda}_{\mathbf{p}}))} \longrightarrow \frac{K_n(\hat{\Sigma}_{\mathbf{p}})}{\text{Im}(K_n(\hat{\Gamma}_{\mathbf{p}}))} \longrightarrow 0 \tag{II}$$

If $n = 2r+1$, we have an exact sequence

$$\ldots K_{2r+1}(\hat{\Gamma}_{\mathbf{p}}/\mathrm{rad}\hat{\Gamma}_{\mathbf{p}}) \longrightarrow K_{2r+1}(\hat{\Gamma}_{\mathbf{p}}) \longrightarrow K_{2r+1}(\hat{\Sigma}_{\mathbf{p}}) \longrightarrow 0 \qquad \text{(II}'\text{)}$$

Hence

$$\frac{K_{2r+1}(\hat{\Sigma}_{\mathbf{p}})}{\mathrm{Im}\ K_{2r+1}(\hat{\Gamma}_{\mathbf{p}})} = 0\ . \qquad \text{(III)}$$

Next, we show that $|SK_{2r+1}(\hat{\Gamma}_{\mathbf{p}})| \equiv 1 \pmod{p}$ for a rational prime lying below $\mathbf{p}$. To do this, first note that

$$\hat{\Gamma}_{\mathbf{p}} \simeq \prod_{i=1}^{m} M_{r_i}(\Gamma_i) \text{ and } \hat{\Sigma}_{\mathbf{p}} = \prod_{i=1}^{m} M_{n_i}(D_i)$$

where Γ_i is a maximal $\hat{R}_{\mathbf{p}}$-order is a division algebra D_i over $\hat{F}_{\mathbf{p}}$. Then $K_n(\hat{\Gamma}_{\mathbf{p}}) \simeq \prod_{i=1}^{m} K_n(\Gamma_i)$ and $K_n(\hat{\Sigma}_{\mathbf{p}}) = \prod_{i=1}^{m} K_n(D_i)$. So it suffices to show that for each i, $SK_{2r+1}(\Gamma_i)| \equiv 1(p)$. Put $\bar{\Gamma}_i = \Gamma_i/\mathrm{rad}\,\Gamma_i$, $\bar{\hat{R}}_{\mathbf{p}} = \hat{R}_{\mathbf{p}}/\mathrm{rad}\hat{R}_{\mathbf{p}}$. Let $(\bar{\Gamma}_i : \bar{\hat{R}}_{\mathbf{p}}) = t$. Also $|SK_{2r+1}(\bar{\Gamma}_i)| = \frac{|K_{2r+1}(\bar{\Gamma}_i)|}{|K_{2r+1}(\bar{\hat{R}}_{\mathbf{p}})|}$ (see [8]).

If $|\bar{\hat{R}}_{\mathbf{p}}| = p^{\ell}$ for some integer $\ell \geq 1$. Then $|K_{2r+1}(\bar{\hat{R}}_{\mathbf{p}})| = p^{\ell(r+1)} - 1$ and $|K_{2r+1}(\bar{\Gamma}_i)| = p^{\ell(r+1)t} - 1$. Hence $SK_{2r+1}(\Gamma_i) \equiv 1(p)$ as required.

Now putting $H_{\mathbf{p}} = \mathrm{Im}\ (\delta)$ in the sequence (II) above, we have an exact sequence

$$0 \to H_{\mathbf{p}} \longrightarrow \frac{K_{2r+1}(\hat{\Gamma}_{\mathbf{p}})}{\mathrm{Im}\ K_{2r+1}(\hat{\Lambda}_{\mathbf{p}})} \longrightarrow \frac{K_{2r+1}(\hat{\Sigma}_{\mathbf{p}})}{\mathrm{Im}\ K_{2r+1}(\hat{\Lambda}_{\mathbf{p}})} \longrightarrow 0\ .$$

Hence the result (ii).

3 Applications to finite p-groups

Let p be an odd rational prime, G a finite p-group. In this section we apply the foregoing to $\Lambda = \mathbb{Z}G, \Sigma = QG$ and obtain simplified forms of (I) and (II) of the Introduction. In the process of analysing the terms in (I) and (II), we prove that if Γ is a maximal $\mathbb{Z}$-order containing $\mathbb{Z}G$, then for all $r \geq 1$, $C\ell_{2r-1}(\mathbb{Z}G)$ is a finite p-group, $\frac{K_{2r}(\Gamma)}{\mathrm{Im}\ K_{2r}(\mathbb{Z}G)}$ is a finite p-group, $\frac{K_{2r}(\hat{\Gamma}_p)}{\mathrm{Im}\ K_{2r}(\hat{\mathbb{Z}}_pG)}$ is a p-group, that $\frac{K_{2r+1}(\hat{\Gamma}_p)}{\mathrm{Im}\ K_{2r+1}(\hat{\mathbb{Z}}_pG)}$ and $\frac{K_{2r+1}(\Gamma)}{\mathrm{Im}\ K_{2r+1}(\mathbb{Z}G)}$ are finite groups, and $SK_{2r}(\hat{\mathbb{Z}}_pG) \simeq SK_{2r}(\mathbb{Z}G)/C\ell_{2r}(\mathbb{Z}G)$.

Theorem 3.1 *Let G be a finite p-group, (p an odd prime), Γ a maximal $\mathbb{Z}$-order in QG containing $\mathbb{Z}G$. Then*

(i) For all $r \geq 1$, we have the exact sequences

$$0 \longrightarrow \frac{K_{2r+1}(\Gamma)}{\mathrm{Im}\ K_{2r+1}(\mathbb{Z}G)} \longrightarrow \frac{K_{2r+1}(\hat{\Gamma}_p)}{\mathrm{Im}\ K_{2r+1}(\hat{\mathbb{Z}}_pG)} \longrightarrow C\ell_{2r}(\mathbb{Z}G) \longrightarrow 0$$

and

$$0 \longrightarrow C\ell_{2r}(\mathbb{Z}G) \longrightarrow SK_{2r}(\mathbb{Z}G) \longrightarrow SK_{2r}(\hat{\mathbb{Z}}_pG) \longrightarrow 0$$

where $\frac{K_{2r+1}(\Gamma)}{\mathrm{Im}\ K_{2r+1}(\mathbb{Z}G)}$ and $\frac{K_{2r+1}(\hat{\Gamma}_p)}{\mathrm{Im}\ K_{2r+1}(\hat{\mathbb{Z}}_pG)}$ are finite groups *(ii) For all $r \geq 1$, we have an exact sequence*

$$0 \to \frac{K_{2r}(QG)}{\text{Im } K_{2r}(\mathbb{Z}G)} \to \underset{q \neq p}{\oplus} \; G_{2r-1}(\hat{\mathbb{Z}}_q G / q\hat{\mathbb{Z}}_q G) \oplus \frac{K_{2r}(\hat{Q}_p G)}{\text{Im } K_{2r}(\hat{\mathbb{Z}}_p G)} \to C\ell_{2r-1}(\mathbb{Z}G) \to 0$$

where

$$|G_{2r-1}(\hat{\mathbb{Z}}_q G / q\hat{\mathbb{Z}}_q G)| \equiv |(q)$$

(iii) In the following exact sequence (see 2.12 (i)).

$$0 \longrightarrow \frac{K_{2r}(\hat{,}_p)}{\text{Im } K_{2r}(\hat{\mathbb{Z}}_p G)} \longrightarrow \frac{K_{2r}(\hat{Q}_p G)}{\text{Im } K_{2r}(\hat{\mathbb{Z}}_p G)} \longrightarrow G_{2r-1}(\hat{,}_p / p\hat{,}_p) \longrightarrow 0,$$

$\frac{K_{2r}(\hat{\Gamma}_p)}{\text{Im } K_{2r}(\hat{\mathbb{Z}}_p G)}$ is a p-group and $|G_{2r-1}(\hat{,}_p / p\hat{,}_p)| \equiv 1(p)$. (Note that $\frac{K_{2r}(\hat{Q}_p G)}{\text{Im } K_{2r}(\hat{\mathbb{Z}}_p G)}$ appears in (ii) above.)

(iv) In the following exact sequence (see (II) in the proof of 3.3 (i))

$$0 \longrightarrow \frac{K_{2r}(,\,)}{\text{Im } K_{2r}(\mathbb{Z}G)} \longrightarrow \frac{K_{2r}(QG)}{\text{Im } K_{2r}(\mathbb{Z}G)} \longrightarrow \frac{K_{2r}(QG)}{\text{Im } K_{2r}(,\,)} \longrightarrow 0,$$

$\frac{K_{2r}(\Gamma)}{\text{Im } K_{2r}(\mathbb{Z}G)}$ is a <u>finite</u> p-group and $\frac{K_{2r}(QG)}{\text{Im } K_{2r}(\Gamma)} \simeq \underset{p}{\oplus} \; G_{2r-1}(,\,/p,\,)$ is a torsion group. (Note that the middle term $\frac{K_{2r}(QG)}{\text{Im } K_{2r}(\mathbb{Z}G)}$ appears in (ii) above.)

(v) For all $r \geq 1$, $C\ell_{2r-1}(\mathbb{Z}G)$ is a finite p-group.

Remarks 3.2 Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ such that $\hat{\Sigma}_{\mathbf{p}}$ splits for all prime ideals $\mathbf{p}$ of R. First we analyse the group $\frac{K_r(\Sigma)}{\text{Im}(K_r(\Lambda))}$ and observe that this situation applies notably for $\Lambda = \mathbb{Z}G, \Sigma = QG$ when G is a finite p-group.

Theorem 3.3 *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ such that $\hat{\Sigma}_{\mathbf{p}}$ splits for all prime ideals $\mathbf{p}$ of R. Suppose that , is a maximal order containing Λ.*

Then for all $r \geq 1$, we have

(i)

$$\frac{K_{2r+1}(\Sigma)}{\text{Im}(K_{2r+1}(\Lambda))} \simeq \frac{K_{2r+1}(,\,)}{\text{Im}(K_{2r+1}(\Lambda))} \quad \text{is a finite group}$$

(ii)

$$\begin{aligned} \frac{K_{2r}(\Sigma)}{\text{Im } (K_{2r}(,\,)} &\simeq \underset{\mathbf{p}}{\oplus} \; G_{2r-1}(,\,/\mathbf{p},\,) \\ &\simeq \left(\frac{K_{2r}(\Sigma)}{\text{Im } K_{2r}(\Lambda)}\right) \Big/ \left(\frac{K_{2r}(,\,)}{\text{Im } K_{2r}(\Lambda)}\right) \end{aligned}$$

where for each $\mathbf{p}$, $|G_{2r-1}(,\,/\mathbf{p},\,)| \equiv 1(p)$ for some rational prime p lying below $\mathbf{p}$. Moreover, $\frac{K_{2r}(\Sigma)}{\text{Im } K_{2r}(\Gamma)}$ has no non-zero divisble subgroup. Also, $K_{2r}(,\,)/\text{Im } K_{2r}(\Lambda)$ is a finite group.

Proof. From the commutative diagram

$$\begin{array}{ccc} K_n(\Lambda) & \xrightarrow{\beta} & K_n(,\ ) \\ \searrow^{\alpha} & & \swarrow \gamma \\ & K_n(\Sigma) & \end{array}$$

we obtain an exact sequence

$$\ldots SK_n(,\ ) \longrightarrow \frac{K_n(,\ )}{\mathrm{Im}(K_n(\Lambda))} \longrightarrow \frac{K_n(\Sigma)}{\mathrm{Im}\ K_n(\Lambda)} \longrightarrow \frac{K_n(\Sigma)}{\mathrm{Im}\ K_n(,\ )} \longrightarrow 0 \qquad \text{(I)}$$

Now, $SK_n(,\ ) \simeq \underset{\mathbf{p}}{\oplus} SK_n(\hat{,}\ _{\mathbf{p}})$ (see [8]).

Moreover, $SK_n(\hat{,}\ _{\mathbf{p}}) = 0$ for all $n \geq 1$ by [12] 1.1 and [10].

Hence $SK_n(,\ ) = 0$ for all $n \geq 1$. So, we have a short exact sequence

$$0 \longrightarrow \frac{K_n(,\ )}{\mathrm{Im}\ K_n(\Lambda)} \longrightarrow \frac{K_n(\Sigma)}{\mathrm{Im}\ K_n(\Lambda)} \longrightarrow \frac{K_n(\Sigma)}{\mathrm{Im}\ K_n(,\ )} \longrightarrow 0\ . \qquad \text{(II)}$$

Now consider the localisation sequence

$$\longrightarrow K_{n+1}(,\ ) \longrightarrow K_{n+1}(\Sigma) \longrightarrow \underset{\mathbf{p}}{\oplus}\ G_n(,\ /\mathbf{p},\ ) \longrightarrow K_n(,\ ) \longrightarrow K_n(\Sigma) \qquad \text{(III)}$$

$$\longrightarrow \underset{\mathbf{p}}{\oplus} G_{n-1}(,\ /\mathbf{p},\ ) \longrightarrow SG_{2n-1}(,\ ) \longrightarrow 0$$

Then $G_{2r}(,\ /\mathbf{p},\ ) = 0$ for all $r \geq 1$ since $,\ /\mathbf{p},$ is a finite ring (see [11]. So, it follows from the sequence (III) above that $K_{2r+1}(\Sigma)/\mathrm{Im}(K_{2r+1}(,\ )) = 0$. By substituting in (II) with $n = 2r+1$, we have proved (i). The finiteness assertion follows from [18], 1.5.

We also have an exact sequence

$$\cdots \oplus G_{2r}(,\ /\mathbf{p},\ ) \to K_{2r}(,\ ) \to K_{2r}(\Sigma) \to \underset{\mathbf{p}}{\oplus}\ G_{2r+1}(,\ /\mathbf{p},\ ) \to SG_{2r-1}(,\ ) \to 0$$

where $G_{2r}(,\ /\mathbf{p},\ ) = 0$ and $SG_{2r-1}(,\ ) = \underset{\mathbf{p}}{\oplus}\ SG_{2r-1}(\hat{,}\ _{\mathbf{p}}) = 0$ (see [8], [10], [12]). Hence $\frac{K_{2r}(\Sigma)}{\mathrm{Im}\ K_{2r}(\Gamma)} \simeq \underset{\mathbf{p}}{\oplus}\ G_{2r+1}(,\ /\mathbf{p},\ )$. Now, since $,\ /\mathbf{p},$ is a finite ring, each $G_{2r+1}(,\ /\mathbf{p},\ )$ is finite (see [11]). So, $\frac{K_{2r}(\Sigma)}{\mathrm{Im}\ K_{2r}(\Gamma)}$ has no non-zero divisible subgroups.

Now, by Devissage, $G_{2r+1}(,\ /\mathbf{p},\ ) \simeq K_{2r-1}((,\ /\mathbf{p},\ )/\mathrm{rad}(,\ /\mathbf{p},\ ))$ where $((,\ /\mathbf{p},\ )/\mathrm{rad}(,\ /\mathbf{p},\ ))$ is a finite semi-simple ring and hence a finite product of matrix algebras over finite fields. So, by applying Quillen's result on K-theory of finite fields, we have $|G_{2r-1}(,\ /\mathbf{p},\ )| \equiv 1(p)$ as required.

That $\frac{K_{2r}(\Sigma)}{\mathrm{Im}\ K_{2r}(\Gamma)} \simeq \left(\frac{K_{2r}(\Sigma)}{\mathrm{Im}\ K_{2r}(\Gamma)}\right) \Big/ \left(\frac{K_{2r}(\Gamma)}{\mathrm{Im}\ K_{2r}(\Lambda)}\right)$ follows from (II) above. Finally, it follows from [18], 1.5 that $\frac{K_{2r}(\Gamma)}{\mathrm{Im}\ K_{2r}(\Lambda)}$ is finite.

Proof of 3.1 First observe that for a finite p-group G, $\check{\mathcal{P}} = \{p\}$ (in the notation of 2.5) and so, for any prime $q \neq p, \hat{\mathbb{Z}}_q G$ is a maximal order in $\hat{Q}_q G$ which splits and so, by (III) in the proof of 2.12, $\frac{K_{2r+1}(\hat{Q}_qG)}{\text{Im } K_{2r+1}(\hat{\mathbb{Z}}_qG)} = 0$. Also by 3.3 (i) $\frac{K_{2r+1}(QG)}{\text{Im } K_{2r+1}(\mathbb{Z}G)} \simeq \frac{K_{r+1}(\Gamma)}{\text{Im } K_{2r+1}(\mathbb{Z}G)}$. Hence the sequence (I) of 2.3 becomes (i) For $n = 2r$, $r \geq 1$,

$$0 \longrightarrow \frac{K_{2r+1}(,\ )}{\text{Im } K_{2r+1}(\mathbb{Z}G)} \longrightarrow \frac{K_{2r+1}(\hat{Q}_{\mathbf{p}}G)}{\text{Im } K_{2r+1}(\hat{\mathbb{Z}}_pG)} \longrightarrow C\ell_{2r}(\mathbb{Z}G) \longrightarrow 0 \tag{I}$$

(ii) For $n = 2r - 1$,

$$0 \longrightarrow \frac{K_{2r}(QG)}{\text{Im}K_{2r}(\mathbb{Z}G)} \longrightarrow \underset{q \neq p}{\oplus} G_{2r-1}\left(\frac{\hat{\mathbb{Z}}_qG}{q\hat{\mathbb{Z}}_qG}\right) \oplus \left(\frac{(K_{2r}(\hat{Q}_pG)}{\text{Im}K_{2r}(\hat{\mathbb{Z}}_pG)}\right)$$

$$\longrightarrow C\ell_{2r-1}(\mathbb{Z}G) \longrightarrow 0 \tag{II}$$

where $|G_{2r-1}\left(\frac{\hat{\mathbb{Z}}G}{q\hat{\mathbb{Z}}_qG}\right)| \equiv 1(q)$. Moreover, the exact sequence 2.4 yields

$$0 \longrightarrow C\ell_{2r}(\mathbb{Z}G) \longrightarrow SK_{2r}(\mathbb{Z}G) \longrightarrow SK_{2r}(\hat{\mathbb{Z}}_pG) \longrightarrow 0 \tag{III}$$

since for $q \neq p, SK_{2r}(\hat{\mathbb{Z}}_qG) = 0$ and because $\hat{\mathbb{Z}}_qG$ is a maximal order (see [10] or [13]).

It follows from 2.12 (ii) that

$$\frac{K_{2r+1}(\hat{Q}_pG)}{\text{Im } K_{2r+1}(\hat{\mathbb{Z}}_pG)} \simeq \frac{K_{2r+1}(\hat{,}_p)}{\text{Im } K_{2r+1}(\hat{\mathbb{Z}}_pG)}$$

where we observe that $H_p = 0$ since $SK_n(\hat{,}_p) = 0$ for all $n \geq 1$ because $\hat{Q}_pG$ splits (see [12] 1.1 and [10]). That $\frac{K_{2r+1}(\Gamma)}{\text{Im } K_{2r+1}(\mathbb{Z}G)}$ is finite follows from [18], 1.5. Hence $\frac{K_{2r+1}(\hat{\Gamma}_p)}{\text{Im } K_{2r+1}(\hat{\hat{\mathbb{Z}}}_pG)}$ is finite from (I) above since $C\ell_{2r}(\mathbb{Z}G)$ is finite

(iii) It follows from lemma 2.12 (i) that we have an exact sequence

$$0 \longrightarrow \frac{K_{2r}(\hat{,}_p)}{\text{Im } K_{2r}(\hat{\mathbb{Z}}_pG)} \longrightarrow \frac{K_{2r}(\hat{Q}_pG)}{\text{Im } K_{2r}(\hat{\mathbb{Z}}_pG)} \longrightarrow G_{2r-1}(\hat{,}_p/p\hat{,}_p) \longrightarrow 0$$

and that $G_{2r-1}(\hat{,}_p/p\hat{,}_p) \equiv 1(p)$. We now prove that $\frac{K_{2r}(\hat{\Gamma}_p)}{\text{Im } K_{2r}(\hat{\mathbb{Z}}_pG)}$ is a p-group.

Let $|G| = p^s$, say. Then $\mathbf{a} = p^s, {}_p$ is an ideal of $\hat{\mathbb{Z}}_pG$ and $\hat{,}_p$ and so we have a Cartesian square

$$\begin{array}{ccc} \hat{\mathbb{Z}}_pG & \longrightarrow & \hat{,}_p \\ \downarrow & & \downarrow \\ \hat{\mathbb{Z}}_pG/\mathbf{a} & \longrightarrow & \hat{,}_p/\mathbf{a} \end{array} \tag{IV}$$

which by [3], [23] leads to a long Mayer-Vietoris sequence. So, if for any Abelain group A, we write $A\left(\frac{1}{p}\right)$ for $A \otimes \mathbb{Z}\left(\frac{1}{p}\right)$, then we have the following exact sequence

$$\ldots \longrightarrow K_{n+1}(\hat{,}_p/\mathbf{a})\left(\frac{1}{p}\right) \longrightarrow K_n(\hat{\mathbb{Z}}_pG)\left(\frac{1}{p}\right) \longrightarrow K_n(\hat{,}_p)\left(\frac{1}{p}\right) \oplus K_n(\hat{\mathbb{Z}}_pG/\mathbf{a})\left(\frac{1}{p}\right)$$

$$\longrightarrow K_n(\hat{,}_p/\mathbf{a})\left(\frac{1}{p}\right) \longrightarrow K_{n-1}(\hat{\mathbb{Z}}_pG)\left(\frac{1}{p}\right) \longrightarrow \ldots \tag{V}$$

Now, p^s annihilates $(\hat{\mathbb{Z}}_pG)/\mathbf{a}$ and $\hat{\Gamma}_p/\mathbf{a}$ and so, $(\hat{\mathbb{Z}}_pG)/\mathbf{a}$ and $\hat{\Gamma}_p/\mathbf{a}$ are $\mathbb{Z}/p^s\mathbb{Z}$-algebras. Moreover, $I = \text{rad}\left((\hat{\mathbb{Z}}_pG)/\mathbf{a}\right)$, $J = \text{rad}\left(\hat{\Gamma}_p/\mathbf{a}\right)$ are nilpotent in the finite rings $(\hat{\mathbb{Z}}_pG)/\mathbf{a}, \hat{\Gamma}_p/\mathbf{a}$ respectively and so, by [23], 5.4, we have that for all $n \geq 1$ $K_n(\hat{\mathbb{Z}}_pG)/\mathbf{a}, I)$ and $K_n(\hat{\Gamma}_pG)/\mathbf{a}, J)$ are p-groups. Now, by tensoring the long exact relative sequences (VI), (VII) below by $\mathbb{Z}\left(\frac{1}{p}\right)$

$$\ldots \longrightarrow K_{n+1}((\hat{\mathbb{Z}}_p/\mathbf{a})/I) \longrightarrow K_n((\hat{\mathbb{Z}}_pG)/\mathbf{a}, I) \longrightarrow K_n((\hat{\mathbb{Z}}_pG)/\mathbf{a}) \longrightarrow K_n(\hat{\mathbb{Z}}_pG/\mathbf{a})/I) \longrightarrow \ldots \tag{VI}$$

and

$$\ldots \longrightarrow K_{n+1}((\hat{\Gamma}_p/\mathbf{a})/J) \longrightarrow K_n((\hat{\Gamma}_p/\mathbf{a}, J) \longrightarrow K_n((\hat{\Gamma}_p/\mathbf{a}) \longrightarrow K_n(\hat{\Gamma}_p/\mathbf{a})/J) \longrightarrow \ldots \tag{VII}$$

we have that

$$K_n(\hat{\Gamma}_p/\mathbf{a})\left(\frac{1}{p}\right) \simeq K_n((\hat{\Gamma}_p/\mathbf{a})/J)\left(\frac{1}{p}\right)$$

and

$$K_n((\hat{\mathbb{Z}}_pG)/\mathbf{a})\left(\frac{1}{p}\right) \simeq K_n((\hat{\mathbb{Z}}_pG)/\mathbf{a})/I)\left(\frac{1}{p}\right)$$

Now $(\hat{\mathbb{Z}}_pG/\mathbf{a})/I$ and $(\hat{\Gamma}_p/\mathbf{a})/J$ are finite semi-simple rings and hence direct products of matrix algebras over finite fields. So, by Quillen's result,

$$K_{2r}(\hat{\mathbb{Z}}_pG/\mathbf{a})\left(\frac{1}{p}\right) \simeq K_{2r}((\hat{\mathbb{Z}}_pG/\mathbf{a})/I)\left(\frac{1}{p}\right) = 0 \tag{VIII}$$

and

$$K_{2r}(\hat{\Gamma}_p/\mathbf{a})\left(\frac{1}{p}\right) \simeq K_{2r}((\hat{\Gamma}_p/\mathbf{a})/J)\left(\frac{1}{p}\right) = 0 \tag{IX}$$

Now if $n = 2r$ in the $M - V$ sequence (V) and we substitute (VIII) and (IX) above, then (V) becomes

$$\ldots \longrightarrow K_{2r+1}(\hat{\Gamma}_p/\mathbf{a})\left(\frac{1}{p}\right) \longrightarrow K_{2r}(\hat{\mathbb{Z}}_pG)\left(\frac{1}{p}\right) \longrightarrow K_{2r}(\hat{\Gamma}_p)\left(\frac{1}{p}\right) \longrightarrow 0 \tag{X}$$

i.e. $K_{2r}(\hat{\mathbb{Z}}_pG) \to K_{2r}(\hat{\Gamma}_p)$ is an epimorphism mod p-torsion, i.e. $\frac{K_{2r}(\hat{\Gamma}_p)}{\text{Im }(K_{2r}(\hat{\mathbb{Z}}_pG)}$ is a p-group.

(iv) The proof that $\frac{K_{2r}(\Gamma)}{\text{Im }K_{2r}(\mathbb{Z}G)}$ is a p-group is similar to that for $\frac{K_{2r}(\hat{\Gamma}_p)}{\text{Im }K_{2r}(\hat{\mathbb{Z}}_pG)}$ and we omit some details. If $|G| = p^s$ and we write $\mathbf{b} = p^s\Gamma$, then the Cartesian square

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \mathbb{Z}G/\mathbf{b} & \longrightarrow & \Gamma/\mathbf{b} \end{array}$$

yields a long Mayer-Vietoris sequence

$$\ldots \longrightarrow K_{n+1}(\Gamma/\mathbf{b})\left(\frac{1}{p}\right) \longrightarrow K_n(\mathbb{Z}G)\left(\frac{1}{p}\right) \longrightarrow K_n(\Gamma)\left(\frac{1}{p}\right) \oplus K_n(\mathbb{Z}G/\mathbf{b})\left(\frac{1}{p}\right)$$

$$\longrightarrow K_n(\Gamma/\mathbf{a})\left(\frac{1}{p}\right) \longrightarrow K_{n-1}(\mathbb{Z}G)\left(\frac{1}{p}\right) \longrightarrow \ldots \tag{XI}$$

and when $n = 2r$, we have by similar arguments to those of (iii), that $K_{2r}(\mathbb{Z}G/\mathbf{b})\left(\frac{1}{p}\right) = 0 = K_{2r}(,\ /\mathbf{b})\left(\frac{1}{p}\right)$. Hence the exact sequence (XI) becomes

$$\longrightarrow K_{2r+1}(,\ /\mathbf{b})\left(\frac{1}{p}\right) \longrightarrow K_{2r}(\mathbb{Z}G)\left(\frac{1}{p}\right) \longrightarrow K_{2r}(,\ )\left(\frac{1}{p}\right) \longrightarrow 0$$

and so, $K_{2r}(\mathbb{Z}G) \to K_{2r}(,\ )$ is an epimorphism modulo p-torsion and so $\frac{K_{2r}(\Gamma)}{\text{Im } K_{2r}(\mathbb{Z}G)}$ is a p-group.

We now prove that $\frac{K_{2r}(\Gamma)}{\text{Im } K_{2r}(\mathbb{Z}G)}$ is finite. Observe tht $QG = Q \oplus \left(\overset{r}{\underset{i=1}{\oplus}} M_{n_i}(Q(\omega_i))\right)$ say, where ω_i is a p-power root of unity and $,\ = \mathbb{Z} \oplus \left(\overset{r}{\underset{i=1}{\oplus}} M_{n_i}(\mathbb{Z}[\omega_i]\right)$ (see [4]) where $\mathbb{Z}[\omega_i]$ is the ring of integers in $Q(\omega_i)$. So, for all $n \geq 1$, $K_n(QG) \simeq K_n(Q) \oplus \left(\overset{r}{\underset{i=1}{\oplus}} K_n(Q(\omega_i))\right)$ and $K_n(,\ ) \simeq K_n(\mathbb{Z}) \oplus \left(\overset{r}{\underset{i=1}{\oplus}} K_n(\mathbb{Z}[\omega_i]\right)$. But it is well know that if F is a number field, and O_F the ring of integers of F, then for all $r \geq 1$, $K_{2r}(O_F)$ is finite (see [2]). Hence $K_{2r}(,\ )$ is finite. Hence $\frac{K_{2r}(\Gamma)}{\text{Im } (K_{2r}(\mathbb{Z}G))}$ is finite. Hence $\frac{K_{2r}(\Gamma)}{\text{Im } (K_{2r}(\mathbb{Z}G))}$ is a finite p-group. (Note that it also follows directly from [18], 1.5, that $\frac{K_{2r}(\Gamma)}{\text{Im } K_{2r}(\mathbb{Z}G)}$ is finite.)

(v) We now prove that for all $r \geq 1$, $C\ell_{2r-1}(\mathbb{Z}G)$ is a finite p-group. To do this, it suffices to show that $SK_{2r-1}(\mathbb{Z}G)$ is a finite p-group. This we now set out to do. If we put $n = 2r - 1$ in the $M - V$ sequence (XI), and use the fact that $K_{2r}(\mathbb{Z}/\mathbf{b})\left(\frac{1}{p}\right) = 0 = K_{2r}(,\ /\mathbf{b})\left(\frac{1}{p}\right)$, then the exact sequence (XI) becomes

$$0 \longrightarrow K_{2r-}(\mathbb{Z}G)\left(\frac{1}{p}\right) \longrightarrow K_{2r-1}(,\ )\left(\frac{1}{p}\right) \oplus K_{2r-1}(\mathbb{Z}G/\mathbf{b})\left(\frac{1}{p}\right) \longrightarrow \dots$$

which shows that $K_{2r-1}(\mathbb{Z}G) \to K_{2r-1}(,\ )$ is a monomorphism $\mod p$-torsion: i.e. $\text{Ker}(K_{2r-1}(\mathbb{Z}G) \to K_{2r-1}(,\ ))$ is a p-torsion group. It is also finite since it is finitely generated as a subgroup of $K_{2r-1}(\mathbb{Z}G)$ which is finitely generated (see [16], 2.1). Hence $\text{Ker}(K_{2r-1}(\mathbb{Z}G) \xrightarrow{\beta} K_{2r-1}(,\ ))$ is a finite p-group.

Now, the exact sequence associated to composite $\alpha = \gamma\beta$ in the commutative diagram

$$\begin{array}{ccc} K_{2r-1}(\mathbb{Z}G) & \xrightarrow{\alpha} & K_{2r-1}(QG) \\ \searrow_{\beta} & & \nearrow_{\gamma} \\ & K_{2r-1}(,\ ) & \end{array}$$

is

$$0 \longrightarrow \text{Ker } \beta \longrightarrow SK_{2r-1}(\mathbb{Z}G) \longrightarrow SK_{2r-1}(,\ ) \longrightarrow \dots$$

where Ker β is a finite p-group.

Now, $,\ = \mathbb{Z} \oplus \left(\overset{t}{\underset{i=1}{\oplus}} M_{n_i}(\mathbb{Z}[\omega_i]\right)$ where ω_i is a p-power root of unity (see [4]). So, $SK_n(,\ ) \simeq SK_n(\mathbb{Z}) \oplus \left(\overset{t}{\underset{i=1}{\oplus}} SK_n(\mathbb{Z}[\omega_i]\right)$. But it is a result of Soule [22] that if F is a number field and O_F the ring of integers of F, then $SK_n(O_F) = 0 \ \ \forall \ \ n \geq 1$. Hence $SK_n(,\ ) = 0 \ \ \forall \ \ n \geq 1$. So, $SK_{2r-1}(\mathbb{Z}G) \simeq \text{Ker } \beta$ is a finite p-group and hence $C\ell_{2r-1}(\mathbb{Z}G)$ is also a finite p-group.

4 Some induction techniques for higher class groups of group-rings

4.1 Let $\mathcal{C}$ be a class of finite groups closed under subgroups. For each finite group G, let $\mathcal{C}(G) = \{H \leq G | H \in \mathcal{C}\}$. Then a Mackey functor $\mathcal{M} = (\mathcal{M}^*, M_*)$ (see [5], [6], [15], [19]) (defined from the category of finite groups with monomorphisms to the category of Abelian groups) is called $\mathcal{C}$-generated if for any finite group G, $\underset{H \in \mathcal{C}(G)}{\oplus} \mathcal{M}(H) \stackrel{\mathcal{M}_*}{\longrightarrow} \mathcal{M}(G)$ is onto. $\mathcal{M}$ is called $\mathcal{C}$-computable (with respect to induction) if for any G, $\mathcal{M}_*$ (covariant functor) induces an isomorphism $\mathcal{M}(G) \simeq \varinjlim_{H \in \mathcal{C}(G)} \mathcal{M}(H)$.

$\mathcal{M}$ is called $\mathcal{C}$-detected (or resp. $\mathcal{C}$-computable) with respect to restriction if for all finite groups G the homomorphism $\mathcal{M}(G) \to \varprojlim_{H \in \mathcal{C}(G)} \mathcal{M}(H)$ induced by $\mathcal{M}^*$ (a contravariant functor) is a monomorphism (resp. isomorphism).

If $H \leq G$ and $i : H \to G$ the inclusion map, it is usual to write $\mathrm{ind}_H^G = \mathcal{M}_*(i) : \mathcal{M}(H) \to \mathcal{M}(G)$ and $\mathrm{res}_H^G = \mathcal{M}^*(i) : \mathcal{M}(G) \to \mathcal{M}(H)$ for the induced homomorphisms.

The next result is crucial for the applications.

Theorem 4.2 [5], [19] *Let $\mathcal{M}$ be a Mackey functor that is also a module over the Green functor $\mathcal{G}$. Suppose that $\mathcal{C}$ is a class of finite groups such that $\mathcal{G}$ is $\mathcal{C}$-generated. Then $\mathcal{M}$ is $\mathcal{C}$-computable for both induction and restriction.*

Remarks 4.3 Let R be a Dedekind domain with quotient field F. In [6], [7], [13], it was proved that the higher K-functors $K_n(R-), SK_n(R-), G_n(R-), SG_n(R-)$, etc. for all $n \geq 0$ are Mackey functors on the category of finite groups and that they are also modules over the Green functors $G_0(R-)$. It was also shown that these K-theoretic functors are hyper-elementary computable (see [5], [6], [13]). Hence $C\ell_n(R-) := \mathrm{Ker}(SK_n(R-) \to \underset{\mathbf{p}}{\oplus} SK_n(\hat{R}_{\mathbf{p}}-)$ are also Mackey functors since $(SK_n(R-) \to \underset{\mathbf{p}}{\oplus} SK_n(\hat{R}_{\mathbf{p}}-)$ is a morphism of Mackey functors that are modules over the Green functors $G_0(R-)$.

On the other hand, it follows easily from [19], 1.18, that if R is a Dedekind domain with quotient field F, then the functors $K_n(R-), C\ell_n(R-)$ are additive functors from the category of R-orders with bimodule morphisms to the category $\mathcal{A}b$ of Abelian groups and so, $K_n(R-), C\ell_n(R-)$ are Mackey functors that are modules over the Green functors $G_0(R-)$ and are also hyper-elementary computable.

In this section, we obtain further results on $C\ell_n(RG)$ based on these induction techniques that have worked at lower levels, see [19], [15], [5].

4.4 A Mackey functor $\mathcal{M} = (\mathcal{M}^*, \mathcal{M}_*)$ ($\mathcal{M}^*$ contravariant, M_* covariant) is said to be p-local if $M(G) = M^*(G) = M_*(G)$ is a $\mathbb{Z}_{(p)}$-module for all finite groups G. Let $H \leq G$, define $\varphi_H : \Omega(G) \to \mathbb{Z}$ by $\varphi_H(S) = |S^H|$ = number of elements in S^H where S is a G-set and

$S^H = \{s \in S | gs = s \text{ for all } g \in H\}$. Let Conj$(G)$ be the set of conjugacy classes of G. Then we have a homomorphism $\varphi = \Pi\varphi_H : \Omega(G) \to \prod_{G \in \text{conj}(G)} \mathbb{Z}$ which is injective with finite cokernel (see [15] or [5]).

Note that any Mackey functor is in a canonical way an $\Omega(-)$- module and any Green functor is a $\Omega(-)$-algebra [15], [19]. Moreover, any p-local Mackey functor is an $\Omega(-)_{(p)}$-module. Hence all the higher K-functors $K_n(R-)_{(p)}$ $SK_n(R-)_{(p)}$ $C\ell_n(R-)_{(p)}$ etc. are $\Omega(-)_{(p)}$-modules, where for any functor M we write $M_{(p)}$ for $\mathbb{Z}_{(p)} \otimes M$.

It is also well known that an element $x \in \Omega(G)$ (or in $\Omega(G)_{(p)}$) is an idempotent if and only if $\varphi_H(x) \in \{0, 1\}$ for all $H \leq G$ (see [15], [5] or [19]).

Theorem 4.5 [19] *Let p be a rational prime and G a finite group, C, a cyclic subgroup of G or order prime to p. Then there exist idempotents $e_C(G) \in \Omega(G)_{(p)}$ such that for all* $H \leq G, \varphi_H(e_C(G)) = \begin{cases} 1 & \text{if for some } C' \text{ conjugate} \\ & \text{to } C, C' \lhd H \text{ and } H/C' \\ & \text{is a } p-\text{group} \\ 0 & \text{otherwise} \end{cases}$

Definition 4.6 *Let p be a prime. A p-hyper-elementary group is a finite group of the form $C_n \rtimes \pi$ where C_n is a cyclic group of order n and π is a p-group. Let K be a field of characteristic zero. A p-hyper-elementary group is said to be $p-K$-elementary of $Im[\pi \xrightarrow{\text{conj}} \text{Aut}(C_n) \simeq (\mathbb{Z}/n)^*] \subseteq$ $\text{Gal}(K(\zeta_n)/K)$ where ζ_n is the primitive n^{th} root of 1 and the Galois group $\text{Gal}(K(\zeta_n)/K)$ is regarded as a subgroup of $\text{Aut}(C_n)$ via the action on $\langle\zeta_n\rangle \simeq C_n$. A finite group is K-elementary if it is $p-K$-elementary for some p. Note that a group is hyper-elementary iff it is Q-elementary since $\text{Gal}(Q(\zeta_n)/Q) = \text{Aut}(C_n)$. The group $C_n \times \pi$ is $\mathbb{C}$-elementary if it is a direct product.*

Definition 4.7 *Let G be a finite group and F a field of characteristic zero. Then two elements $g, h \in G$ are said to be F-conjugate if h is conjugate to g^a for some a in $\text{Gal}(F\zeta_n/F)$ where n is the order of g. For example, g and h are Q-conjugate iff $\langle g\rangle$ and $\langle h\rangle$ are conjugate subgroups of G.*

Also, if $C = \langle g\rangle$ is a cyclic subgroup of G such that n=order of $g = |C|$, define $N_G^F(C) = N_G^F(g) = \{x \in G | xgx^{-1} = g^a$ for some $a \in \text{Gal}(F\zeta_n/F)$. Write $\mathcal{P}(G)$ for the st of p-subgroups of G.

We now record the following important result

Theorem 4.8 [19] *Let p be a fixed prime, F a field of characteristic zero, $G = C_n \rtimes \pi (p \nmid n, \pi$ a p-group) a p-hyper-elementary group. Then $FC_n \simeq \Pi_{i=1}^m F_i$ where $F_i = F(\zeta_{n_i})$ for some n_i dividing n. Moreover, G is $p-F$-elementary if and only if the conjugation action of π on FC_n leaves each F_i invariant. In this case $FG = F(C_n \rtimes \pi) \simeq \Pi_{i=1}^m F_i(\pi)^t$ where $F_i(\pi)^t$ is the twisted group-ring with twisting $t : \pi \to \text{Gal}(F_i/F)$ induced by conjugation action of π on F_i.*

If further, R is a Dedekind domain with field of fraction F and if $R_i \subset F_i$ is the integral closure of R, then $\Pi_{i=1}^m R_i[\pi]^t$ is an R-order in FG and $RG = R[C_n \times \pi] \subseteq \Pi_{i=1}^m R_i[\pi]^t \subseteq \frac{1}{n} RG$.

Remarks 4.9 If in 4.8, $F = Q$, then $\text{Gal}(Q\zeta_n/Q) = (\mathbb{Z}/n)^*$, $\mathbb{Z}\zeta_n$ is the ring of integers in $Q\zeta_n, QC_n = \prod_{d|n} Q\zeta_d$ and the maximal order in QC_n is $\prod_{d|n} \mathbb{Z}\zeta_d$ (see [19], [4]). So, if $G = C_n \rtimes \pi, p \nmid n$, π a p-group, we have an inclusion of $\mathbb{Z}$-orders $\mathbb{Z}G = \mathbb{Z}(C_n \rtimes \pi) \subseteq \prod_{d|n} \mathbb{Z}\zeta_d[\pi]^t$ (of index prime to p) and so by Corollary 2.7 and Theorem 2.9

$$C\ell_n(\mathbb{Z}G)_{(p)} \simeq \prod_{d|n} C\ell_n(\mathbb{Z}\zeta_d[\pi]^t)_{(p)}$$

since $p \notin \check{\mathcal{P}}$.

Remarks 4.10 Let R be a Dedekind domain with quotient field F, $M = C\ell_n(R-)$ $n \geq 1$, p a rational prime and $Cy(G)$ the set of conjugacy class representatives of cyclic subgroups $C \subset G$ of order prime to p. For each $C \in Cy(G)$, let $e_C \in \Omega(G)_{(p)}$ be the idempotent defined in 3.4. Put $M_C(G) = e_C(G)M(G) \subseteq M(G)$. Then it follows from [19], 11.5, that

(i) For any finte group G, $M(G) = \bigoplus_{C \in Cy(G)} M_C(G)$

(ii) For any finte group G and each $C \in Cy(G)$

$$M_C(G) \simeq \varinjlim_{\pi \in \mathcal{P}(N(C))} M_C(C \rtimes \pi) \simeq \varprojlim_{\pi \in \mathcal{P}(N(C))} M_C(C \rtimes \pi)$$

where the limits are taken w.r.t. M_*, M^* applied to inclusions and conjugation by elements of $N_G(C) =: N(C)$

(iii) Let $G = C_n \rtimes \pi$ where π is a p-group and $p \nmid n$. For any $H = C_m \rtimes \pi \subseteq G, (m|n)$, $\text{Res}_M^G \circ \text{Ind}_H^G$ is an automorphism of $M(H)$ and for each k dividing m, write M_k for M_C where $C \leq G$ is the subgroup of order k and set $M_k(G) = 0$ if $k \nmid m$. Then we have isomorphisms ${}^k\text{ind}_H^G : M_k(H) \xrightarrow{\simeq} M_k(G)$ and ${}^k\text{Res}_H^G : M_k(G) \simeq M_k(H)$. Moreover,

$$M_n(G) = \text{Ker}(\oplus \text{Res} : M(G) = M(C_n \rtimes \pi) \to \oplus p|nM(C_{n/p} \rtimes \pi)] \ .$$

The following theorem 4.11 is the target result for this section. The proof is an adaptation of that in [19], 11.8, in the context of $C\ell_1$ and some details are omitted.

Theorem 4.11 *Let p be an odd rational prime, G any finite group. let $g_1, \ldots, g_r$ be a set of conjugacy class representatives of elements of order prime to p. Let $s_i :=$ order of $g_i = |\langle g_i \rangle|$. Then for all $n \geq 1$, $C\ell_n(\mathbb{Z}G)_{(p)} \simeq \bigoplus_{i=1}^{r} \varinjlim_{\pi \in \mathcal{P}(N_G(\langle g_i \rangle))} C\ell_n(\mathbb{Z}\zeta_{s_I}(\pi)^t)_{(p)}$.*

Proof. Let $\mathcal{M} = \mathbb{Z}_{(p)} \otimes C\ell_n(\mathbb{Z}-) := C\ell(\mathbb{Z}-)_{(p)}$, for $n \geq 0$. Note that $\mathcal{M}$ is a Mackey functor either through [6], [7], [13] or because it is a functor on the category of $\mathbb{Z}$-orders with bimodule morphisms (see 4.2 and [19]). Let G be a fixed finite group and $Cy(G)$ a set of conjugacy class representatives of cyclic subgroups $C \subseteq G$ of order prime to p. Note that two elements g, h are Q-conjugate iff they generate conjugate subgroups (see 4.7 or [19]). Hence, by 4.10 (i) $\mathcal{M}(G) = \underset{C \in Cy(G)}{\oplus} \mathcal{M}_C(G)$. If for a fixed C, $s = |C|$, we have by 4.10 (ii) that

$$\mathcal{M}_C(G) = \underset{\pi \in \mathcal{P}(N_C)}{\varinjlim} \mathcal{M}_s(C \rtimes \pi) \tag{II}$$

(in the notation of 4.10 (iii)).

Now, by [19], 11.2, $\mathcal{M}$ is computable with respect to $p - Q$-elementary subgroups. So, for any $\pi \in \mathcal{P}(N(C))$,

$$\mathcal{M}_S(C \rtimes \pi) = \varinjlim \{\mathcal{M}_S(C \rtimes \rho) | \rho \in \pi \cap N_G^Q(C)\}$$

Now, by 4.8, $\mathbb{Z}(C_s \rtimes \pi) \simeq \prod_{d|s} \mathbb{Z}\zeta_d(\pi)^t \leq \frac{1}{s}\ \mathbb{Z}(C_s \rtimes \pi)$ and $\mathbb{Z}\zeta_s(\pi)^t \subseteq \prod_{d|s} \mathbb{Z}\zeta_d(\pi)^t \leq \frac{1}{s}\ \mathbb{Z}\zeta_s[\pi]^t$. Also by Remarks 4.9,

$$C\ell_n(\mathbb{Z}(C_s \rtimes \pi)_{(p)} \simeq \prod_{d|s} C\ell_n(\mathbb{Z}\zeta_d(\pi)^t_{(p)}$$

Now, since there are r Q-conjugacy class representatives of elements of order prime to p, we have

$$C\ell_n(\mathbb{Z}G)_{(p)} \simeq \bigoplus_{i=1}^{r} \underset{\pi \in \mathcal{P}(N_G(\langle g_i \rangle))}{\varinjlim} C\ell_n(\mathbb{Z}\zeta_{s_i}(\pi)^t)_{(p)}$$

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