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Primordial magnetic fields from metric perturbations

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ABSTRACT

We study the amplification of electromagnetic vacuum fluctuations induced by the evolution of scalar metric perturbations at the end of inflation. Such perturbations break the conformal invariance of Maxwell equations in Friedmann-Robertson-Walker backgrounds and allow the growth of magnetic fields on super-Hubble scales. We relate the strength of the fields generated by this mechanism with the power spectrum of scalar perturbations and estimate the amplification on galactic scales for different values of the spectral index. Finally we discuss the possible effects of finite conductivity during reheating.

1 Introduction

The existence of cosmic magnetic fields with large coherence lengths (> 10 kpc) and typical strength of 10^{-6} G, still remains an open problem in astrophysics [1]. A partial explanation, widely considered in the literature, is based on the amplification of seed fields by means of the so called galactic dynamo mechanism. In this mechanism, the differential rotation of the galaxy is able to transfer energy into the magnetic field, but nevertheless it still requires a pre-exisiting field to be amplified. The present bounds on the necessary seed fields to comply with observations are in the range $B_{seed} \gtrsim 10^{-17} - 10^{-22}$ G ($h = 0.65 - 0.5$) at decoupling time, coherent on a comoving scale of $\lambda_G \sim 10$ kpc, for a flat universe without cosmological constant. For a flat universe with nonvanishing cosmological constant, the limits can be relaxed up to $B_{seed} \gtrsim 10^{-25} - 10^{-30}$ G ($h = 0.65 - 0.5$) at decoupling for $\Omega_\Lambda = 0.7$ and $\Omega_m = 0.3$ [2]. The observations of micro-Gauss magnetic fields in two high-redshift objects (see [1, 2] and references therein) could, if correct, impose more stringent conditions on the seeds fields or even on the dynamo mechanism itself.

The cosmological origin of the seed fields is one of the most interesting possibilities, although some other mechanisms at the astrophysical level, such as the Biermann battery process, have also been considered [3, 4]. In the cosmological case, in which we will be mainly interested in this work, it is natural to expect [5] that the same mechanism that gave rise to the large-scale galactic structure, i.e. amplification of quantum fluctuations during inflation, was also responsible for the generation of the primordial magnetic fields. However, it was soon noticed [5] that the gravitational amplification does not operate in the case of electromagnetic (EM) fields. This is because of the conformal triviality of Maxwell equations in Friedmann-Robertson-Walker (FRW) backgrounds, i.e. conformally invariant equations in a conformally flat space-time. In order to avoid this difficulty, several production mechanisms have been proposed in which Maxwell equations are modified in different ways. Thus for example, the addition of mass terms to the photon or higher-curvature terms in the Lagrangian was studied in [5]. The contribution of the conformal anomaly was included in [6]. In the context of string cosmology, the effects of a dynamical dilaton field were taken into account in [7]. Other examples include non-minimal gravitational-electromagnetic coupling [8], inflaton coupling to EM Lagrangian [9], spontaneous breaking of Lorentz invariance [10] or backreaction of minimally coupled charged scalars [11, 12, 13]. Some of them are able to generate fields of the required strength to seed the galactic dynamo or even to account for the observations without further amplification.

In this paper we explore the alternative possibility, i.e. we avoid conformal triviality by considering deviations from the FRW metric (see [14] for a suggestion along these lines). This approach is rather natural since we know that galaxies formed from small metric inhomogeneities present at large scales and, in addition, it does not require any modification of Maxwell electromagnetism. In the inflationary cosmology, metric perturbations are generated when quantum fluctuations become super-Hubble sized and thereafter evolve as classical fluctuations, reentering the horizon during radiation or matter dominated eras [15]. The same mechanism would operate on large-scale EM fluctuations. However, if conformal invariance is not broken, each positive or negative frequency EM mode will evolve independently, without mixing. This implies that photons cannot be created and therefore magnetic fields are not amplified. However, in the presence of an inhomogeneous background, we will show that the mode-mode coupling between EM and metric perturbations generates the mixing. This in turn will allow us to relate the strength of the magnetic field created by this mechanism and the particular form of the metric perturbations described by the corresponding power spectrum. Those photons produced in the inflation-radiation transition with very long wavelengths can be seen as static electric or magnetic fields. Because of the high conductivity of the Universe in the radiation era, the electric components are rapidly damped whereas, thanks to magnetic flux conservation, the magnetic fields will remain frozen in the plasma and their subsequent evolution will be trivial, Ba^2 =const [5, 9]. The paper is organized as follows. In section 2 we obtain the Maxwell equations in the presence of an inhomogeneous background and calculate the occupation number of the photons produced. In section 3 we apply these results to calculate the corresponding magnetic field generated at galactic scales. Section 4 is devoted to the analysis of the effects of finite conductivity in those results and finally, section 5 includes the main conclusions of the paper.

2 Maxwell equations and photon production

Although there are previous works on the production of scalar and fermionic particles in inhomogeneous backgrounds [16, 17], in this paper we will need to extend the analysis to the case of gauge fields. Let us then consider Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 0, \tag{1}$$

in a background metric that can be splitted as $g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$, where

$$g^0_{\mu\nu} dx^\mu dx^\nu = a^2(\eta)(d\eta^2 - \delta_{ij} dx^i dx^j) \tag{2}$$

is the flat FRW metric in conformal time and

$$h_{\mu\nu} dx^\mu dx^\nu = 2a^2(\eta)\Phi(d\eta^2 + \delta_{ij} dx^i dx^j) \tag{3}$$

is the most general form of the linearized scalar metric perturbation in the longitudinal gauge and where it has been assumed that the spatial part of the energy-momentum tensor is diagonal, as indeed happens in the inflationary or perfect fluid cosmologies [15]. In this expression $\Phi(\eta, \vec{x})$ is the gauge invariant gravitational potential. The equation (1) can be written as:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{g} g^{\mu\alpha} g^{\nu\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) \right) = 0, \tag{4}$$

which leads in this background to the following linearized equations

$$\frac{\partial}{\partial x^i} \left((1 - 2\Phi)(\partial_i A_0 - \partial_0 A_i) \right) = 0, \tag{5}$$

for $\nu = 0$ and

$$\frac{\partial}{\partial\eta}\left((1-2\Phi)(\partial_i A_0 - \partial_0 A_i)\right)$$
$$+\frac{\partial}{\partial x^j}\left((1+2\Phi)(\partial_j A_i - \partial_i A_j)\right) = 0, \tag{6}$$

for $\nu = i$. In addition, we will use the Coulomb gauge condition $\vec{\nabla}\cdot\vec{A} = 0$.

In order to study the amplification of vacuum fluctuations, let us consider a particular solution of the above equations that we will denote by $A_\mu^{\vec{k},\lambda}(x)$ such that asymptotically in the past it behaves as a positive frequency plane wave with momentum $\vec{k}$ and polarization λ, i.e,

$$A_\mu^{\vec{k},\lambda}(x) \stackrel{\eta\rightarrow-\infty}{\longrightarrow} A_\mu^{(0)\vec{k},\lambda}(x) = \frac{1}{\sqrt{2kV}}\epsilon_\mu(\vec{k},\lambda)e^{i(\vec{k}\vec{x}-k\eta)} \tag{7}$$

where $k^2 = \vec{k}^2$. For the two physical polarization states we have, $\vec{\epsilon}(\vec{k},\lambda)\cdot\vec{k} = 0$ and $\epsilon_0(\vec{k},\lambda) = 0$. We will work in a finite box with comoving volume V and we will take the continuum limit at the end of the calculation. We are assuming that metric perturbations vanish before inflation starts, so that we can define an appropriate initial conformal vacuum state. Because of the presence of the inhomogeneous background, in the asymptotic future, this solution will behave as a linear superposition of positive and negative frequency modes with different momenta and different polarizations, i.e.,

$$A_\mu^{\vec{k},\lambda}(x) \stackrel{\eta\rightarrow\infty}{\longrightarrow} \sum_{\lambda'}\sum_q\left(\alpha_{kq\lambda\lambda'}\frac{\epsilon_\mu(\vec{q},\lambda')}{\sqrt{2qV}}e^{i(\vec{q}\vec{x}-q\eta)} + \beta_{kq\lambda\lambda'}\frac{\epsilon_\mu^*(\vec{q},\lambda')}{\sqrt{2qV}}e^{-i(\vec{q}\vec{x}-q\eta)}\right) \tag{8}$$

It is possible to obtain an expression for the Bogolyubov coefficients $\alpha_{kq\lambda\lambda'}$ and $\beta_{kq\lambda\lambda'}$ to first order in the metric perturbations. With that purpose, we look for solutions of the equations of motion in the form:

$$A_\mu^{\vec{k},\lambda}(x) = A_\mu^{(0)\vec{k},\lambda}(x) + A_\mu^{(1)\vec{k},\lambda}(x) + ... \tag{9}$$

where $A_\mu^{(0)\vec{k},\lambda}(x)$ is the solution in the absence of perturbations given by (7). Introducing this expansion in (5) and Fourier transforming, we obtain for the temporal component of the EM field to first order in the perturbations:

$$A_0^{(1)\vec{k},\lambda}(\vec{q},\eta) = -\sqrt{\frac{2k}{V}}\frac{\vec{q}\cdot\vec{\epsilon}(\vec{k},\lambda)}{q^2}\Phi(\vec{k}+\vec{q},\eta)e^{-ik\eta} \tag{10}$$

where, as usual, $\Phi(\vec{q},\eta) = (2\pi)^{-3/2}\int d^3x e^{i\vec{q}\vec{x}}\Phi(\vec{x},\eta)$. The zeroth order equation implies $A_0^{(0)\vec{k},\lambda}(\vec{q},\eta) = 0$. The spatial equations (6) can be written to first order as:

$$2\Phi' A_i^{(0)\prime} + \partial_i A_0^{(1)\prime} - A_i^{(1)\prime\prime} + 2\vec{\nabla}\Phi\cdot\vec{\nabla}A_i^{(0)} - 2\vec{\nabla}\Phi\cdot\partial_i\vec{A}^{(0)} + \vec{\nabla}^2 A_i^{(1)} + 4\Phi\vec{\nabla}^2 A_i^{(0)} = 0 \tag{11}$$

Inserting again expansion (9), these equations can be rewritten in Fourier space as:

$$\frac{d^2}{d\eta^2}A_i^{(1)\vec{k},\lambda}(\vec{q},\eta) + q^2 A_i^{(1)\vec{k},\lambda}(\vec{q},\eta) - J_i^{\vec{k},\lambda}(\vec{q},\eta) = 0 \tag{12}$$

where:

$$J_i^{\vec{k},\lambda}(\vec{q},\eta) = -\sqrt{\frac{2k}{V}}\left(\left(i\Phi'(\vec{k}+\vec{q},\eta)+\frac{k^2-\vec{k}\cdot\vec{q}}{k}\Phi(\vec{k}+\vec{q},\eta)\right)\epsilon_i(\vec{k},\lambda)e^{-ik\eta}\right.$$
$$\left. + (\vec{\epsilon}(\vec{k},\lambda)\cdot\vec{q})\ \Phi(\vec{k}+\vec{q},\eta)\frac{k_i}{k}e^{-ik\eta} - i\frac{\vec{\epsilon}(\vec{k},\lambda)\cdot\vec{q}}{q^2}\frac{d}{d\eta}\left(\Phi(\vec{k}+\vec{q},\eta)e^{-ik\eta}\right)q_i\right) \quad (13)$$

Solving these equations we find, up to first order in the perturbations:

$$A_i^{\vec{k},\lambda}(\vec{q},\eta) = \frac{\epsilon_i(\vec{k},\lambda)}{\sqrt{2kV}}\delta(\vec{q}-\vec{k})e^{-ik\eta} + \frac{1}{q}\int_{\eta_0}^{\eta} J_i^{\vec{k},\lambda}(\vec{q},\eta')\sin(q(\eta-\eta'))d\eta' \quad (14)$$

where η_0 denotes the starting time of inflation. Comparing this expression with (8), it is straightforward to obtain the Bogolyubov coefficients $\beta_{kq\lambda\lambda'}$, they are given by:

$$\beta_{kq\lambda\lambda'} = \frac{-i}{\sqrt{2qV}}\int_{\eta_0}^{\eta_1} \vec{\epsilon}\,(\vec{q},\lambda')\cdot\vec{J}^{\,\vec{k},\lambda}(\vec{q},\eta)e^{-iq\eta}d\eta \quad (15)$$

where η_1 denotes the present time. The total number of photons created with comoving wavenumber $k_G = 2\pi/\lambda_G$, corresponding to the relevant coherence length, is therefore given by [18]:

$$N_{k_G} = \sum_{\lambda,\lambda'}\sum_{k}|\beta_{kk_G\lambda\lambda'}|^2 \quad (16)$$

We will concentrate only in the effect of super-Hubble scalar perturbations whose evolution is relatively simple [15]:

$$\Phi(\vec{k},\eta) = C_k\frac{1}{a}\frac{d}{d\eta}\left(\frac{1}{a}\int a^2 d\eta\right) + D_k\frac{a'}{a^3}, \quad (17)$$

the second term decreases during inflation and can soon be neglected. Thus, it will be useful to rewrite the perturbation as: $\Phi(\vec{k},\eta) = C_k\mathcal{F}(\eta)$. During inflation or preheating, these perturbations evolve in time, whereas they are practically constant during radiation or matter eras. We will neglect the effects of the perturbations once they reenter the horizon. This is a good approximation for modes reentering right after the end of inflation since they are rapidly damped. In addition, we will show that those modes are the more relevant ones in the calculation.

The power spectrum corresponding to (17) is given by:

$$\mathcal{P}_\Phi(k) = \frac{k^3|C_k|^2}{2\pi^2 V} = A_S^2\left(\frac{k}{k_C}\right)^{n-1} \quad (18)$$

where for simplicity we have taken a power-law behaviour with spectral index n and we have set the normalization at the COBE scale $\lambda_C \simeq 3000$ Mpc with $A_S \simeq 5\cdot 10^{-5}$. In the case of a blue spectrum, with positive tilt ($n>1$), perturbations will grow at small scales and it is necessary to introduce a cut-off k_{max} in order to avoid excessive primordial black hole production [19]. Accordingly, only below the cut-off the perturbative method will be reliable. For negative tilt or scale-invariant spectrum there will be also a small scale cut-off related to the minimum size of the horizon $k_{max} \lesssim a_I H_I$, where the I subscript denotes the end of inflation.

We can obtain an explicit expression for the total number of photons (16) in terms of the power spectrum. Taking the continuum limit $\sum_k \to (2\pi)^{-3/2} V \int d^3k$, we get:

$$\begin{aligned} N_{k_G} &= \sum_{\lambda,\lambda'} V \int \frac{d^3k}{(2\pi)^{3/2}} |\beta_{k k_G \lambda\lambda'}|^2 \\ &= \sum_{\lambda,\lambda'} V \int \frac{d^3k}{(2\pi)^{3/2}} \frac{|C_{|k+k_G|}|^2}{2k_G V^2} \left| \int d\eta \left(\sqrt{2k} \left(\left(i\mathcal{F}\,' + \frac{k^2 - \vec{k}\cdot\vec{k}_G}{k}\mathcal{F} \right) (\vec{\epsilon}(\vec{k},\lambda)\cdot\vec{\epsilon}(\vec{k}_G,\lambda')) \right.\right.\right. \\ &+ \left.\left.\left. (\vec{\epsilon}(\vec{k},\lambda)\cdot\vec{k}_G)\,(\vec{\epsilon}(\vec{k}_G,\lambda')\cdot\vec{k})\,\frac{\mathcal{F}}{k} \right) e^{-i(k_G+k)\eta} \right) \right|^2 \end{aligned} \tag{19}$$

Notice that the last term in (13) does not contribute to $\beta_{kq\lambda\lambda'}$ because of the transversality condition of the polarization vectors. The integration in d^3k is dominated by the upper limit, i.e. $k \gg k_G$ and accordingly we can ignore the effect of the terms proportional to $\vec{k}_G$. In addition, for those modes k which are outside the Hubble radius at the end of inflation, we have $k\eta \ll 1$. With these simplifications we obtain:

$$N_{k_G} \simeq \sum_{\lambda,\lambda'} \int \frac{dk\, d\Omega}{(2\pi)^{3/2}} \frac{|C_k|^2 k^2}{2k_G V} \left| \int d\eta \left(\sqrt{2k} \left((i\mathcal{F}\,' + k\mathcal{F})\,(\vec{\epsilon}(\vec{k},\lambda)\cdot\vec{\epsilon}(\vec{k}_G,\lambda')) \right) \right) \right|^2 \tag{20}$$

Performing the integration in the angular variables and using the definition of the power spectrum in (18), we obtain:

$$N_{k_G} \simeq \frac{4(2\pi)^{3/2}}{3k_G} \int dk A_S^2 \left(\frac{k}{k_C}\right)^{n-1} \left| \int d\eta\, (i\mathcal{F}\,' + k\mathcal{F}) \right|^2 \tag{21}$$

Finally, we will estimate the time integral. The behaviour of scales that reenter the horizon during the radiation dominated era is oscillatory with a decaying amplitude [15], therefore, there is no long-time contribution to the integral that could spoil the perturbative method. Thus, for simplicity we will assume that the function $\mathcal{F}$ vanishes for $\eta \geq 1/k$, and accordingly we estimate, $|\int d\eta\, (i\mathcal{F}\,' + k\mathcal{F})|^2 \sim \mathcal{O}(1)$. Our final expression for the occupation number is:

$$N_{k_G} \simeq \frac{4(2\pi)^{3/2} A_S^2}{3k_G (k_C)^{n-1}} \int_{k_C}^{k_{max}} dk k^{n-1} \simeq \frac{4(2\pi)^{3/2} A_S^2}{3\,n} \frac{k_{max}^n}{k_G\, k_C^{n-1}} \tag{22}$$

3 Magnetic field generation

The energy density stored in a magnetic field mode B_k with wavenumber k is given by:

$$\rho_B(\omega) = \omega \frac{d\rho_B}{d\omega} = \frac{|B_k|^2}{2}, \tag{23}$$