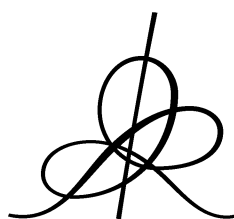


NON-RADIALLY SYMMETRIC SOLUTIONS TO THE GINZBURG-LANDAU EQUATION

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NON-RADIALLY SYMMETRIC SOLUTIONS TO THE GINZBURG-LANDAU EQUATION *

by

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Abstract. We consider the question of existence of non-radial solutions to Ginzburg-Landau equation. We present results indicating that such solutions do exist. We look for such solutions as saddle points of the renormalized Ginzburg-Landau free energy functional (the latter was introduced in reference [OS1]). There are two main points in our analysis: we look for solutions having certain point symmetries and we characterize saddle point solutions in terms of critical points of certain intervortex energy function which we introduce. The latter critical points correspond to forceless vortex configuration.

1. Introduction

The Ginzburg-Landau equation, which among other things, describes, to a certain approximation, stationary states of superfluids and solitary waves in plasmas, has recently attracted considerable attention (partly generated by the publication of the research monograph [BBH]). This equation is simple to write down (we consider the purer case of the entire plane \mathbb{R}^2 , rather than a domain in \mathbb{R}^2 as in [BBH]):

$$-\Delta\psi + (|\psi|^2 - 1)\psi = 0 , \tag{1.1}$$

where $\psi: \mathbb{R}^2 \rightarrow \mathbb{C}$, with the boundary condition

$$|\psi| \rightarrow 1 \quad \text{as } |x| \rightarrow \infty , \tag{1.2}$$

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but not easy to analyze. In fact, so far only radially symmetric solutions, i.e. solutions of the form $\psi_n(x) = f_n(r)e^{in\theta}$, where r and θ are polar coordinates for $x \in \mathbb{R}^2$, are known for (1.1)–(1.2) (see [GP, H, HH, CEQ, FP, OS1, CK, M1,2, LL]). Solutions ψ_n are called the n -vortices. Note that $n = \deg\psi_n$. Here $\deg\psi$, the *degree* (or *vorticity*) of ψ (satisfying (1.2)), is the total index (winding number) of ψ , considered as a vector field on \mathbb{R}^2 , at ∞ , i.e.

$$\deg \psi := \frac{1}{2\pi} \int_{|x|=R} d(\arg \psi)$$

for R sufficiently large.

Existence and properties of the vortex solutions were established only recently. What we know now is

- (i) Existence and uniqueness (modulo symmetry transformations and in the class of radially symmetric functions) ([HH, CEQ, FP, OS1])
- (ii) Stability for $|n| \leq 1$ and instability for $|n| > 1$ ([OS1], earlier results on stability for the disc are due to [LL, M1])
- (iii) Uniqueness of $\psi_{\pm 1}$ (again, modulo symmetry transformation) in the class of functions, ψ , with $\deg\psi = \pm 1$ and obeying $\int (|\psi|^2 - 1)^2 < \infty$ ([M2]).

Thus the next question is are these non-radially symmetric solutions?

In this paper we present results indicating that such solutions do exist. There are two key ingredients in our analysis. Firstly, we characterized non-radially symmetric solutions as critical points of the intervortex energy function described below (see also [OS2]). Secondly, we look for solutions having certain point symmetries. The latter fact reduces the number of free parameters describing such solutions to one (the size of the corresponding polygon of vortices).

Solution breaking rotational symmetry were found to exist in the case of the Ginzburg-

Landau equation in the ball $B_R = \{x \in \mathbb{R}^2 \mid |x| \leq R\}$ with the boundary condition $\psi|_{\partial B_R} = e^{in\theta}$ and $|n| \geq 2$ (see [BBH, Thm IX.1]). However, in the case of the ball there is an external mechanism leading to the symmetry breaking: the boundary condition. The latter repels vortices forcing their confinement. On the other hand the energy is lowered by breaking up multiple vortices into $+1$ - (or -1 -) vortices and merging vortices of opposite signs. Thus for R not so small the lowest energy is reached by a configuration of $|n|$ vortices of vorticities ± 1 depending on the sign of n which, obviously, is not rotationally symmetric.

The paper is organized as follows. In Sections 2–3 we review some material from [OS1]: the variational formulation of the problem and some specific properties of the vortex solutions. In Section 4 we define the intervortex energy and discuss its properties. In particular we discuss the correlation term in the (upper bound on) expansion of the intervortex energy for large intervortex separations and a definition of G -symmetric vortex energies, where G is a subgroup of the symmetry group of (1.1)

In Section 5 we consider point symmetries (C_{Nv}), present one of our main results, Theorem 5.1, on existence of critical points for C_{Nv} -symmetric intervortex energies and derive some general relations for those energies. In Section 6 we prove Theorem 5.1 and present a discussion of some other cases.

Finally, we have five appendices where all the hard analytical and numerical work is concentrated. In these appendices we compute various asymptotic expansions beyond the leading order. We feel that these appendices are of interest on their own as they address rather subtle computational issues.

2. Renormalized Ginzburg-Landau energy

It is a straightforward observation that Eqn (1.1) is the equation for critical points of the following functional

$$\mathcal{E}(\psi) = \frac{1}{2} \int \left(|\nabla \psi|^2 + \frac{1}{2} (|\psi|^2 - 1)^2 \right). \quad (2.1)$$

Indeed, if we define the variational derivative, $\partial_\psi \mathcal{E}(\psi)$, of \mathcal{E} by

$$\operatorname{Re} \int \xi \partial_\psi \mathcal{E}(\psi) = \left. \frac{\partial}{\partial \lambda} \mathcal{E}(\psi_\lambda) \right|_{\lambda=0} \quad (2.2)$$

for any path ψ_λ s.t. $\psi_0 = \psi$ and $\left. \frac{\partial}{\partial \lambda} \psi_\lambda \right|_{\lambda=0} = \xi$, then the l.h.s. of Eqn (1.1) is equal to $\overline{\partial_\psi \mathcal{E}(\psi)} = \partial_{\bar{\psi}} \mathcal{E}(\psi)$ for $\mathcal{E}(\psi)$ given in (2.1).

(2.1) is the celebrated Ginzburg-Landau (free) energy. However, there is a problem with it in our context. It is shown in [OS1] that if ψ is an arbitrary C^1 vector field on \mathbb{R}^2 s.t. $|\psi| \rightarrow 1$ as $|x| \rightarrow \infty$ uniformly in $\hat{x} = \frac{x}{|x|}$ and $\deg \psi \neq 0$, then $\mathcal{E}(\psi) = \infty$.

We renormalize the Ginzburg-Landau energy functional as follows (see [OS1]). Let $\chi(x)$ be a smooth positive function on \mathbb{R}^2 vanishing at the origin and converging to one at infinity. Define

$$\mathcal{E}_{\text{ren}}(\psi) = \frac{1}{2} \int \left(|\nabla \psi|^2 - \frac{(\deg \psi)^2}{r^2} \chi + F(|\psi|^2) \right) d^2 x \quad (2.3)$$

where

$$F(u) = \frac{1}{2} (u - 1)^2. \quad (2.4)$$

Properties of the renormalized energy functional, $\mathcal{E}_{\text{ren}}(\psi)$, are investigated in [OS1].

In this paper we take

$$\chi(x) = \begin{cases} 1 & \text{for } |x| \geq R + R^{-1}, \\ 0 & \text{for } |x| \leq R \end{cases} \quad (2.5)$$

for R very large compared to all length scales appearing below.

3. Vortices

It is shown in [HH, CEQ, FP, OS1] that for any any n , Eqn (1.1) has a solution, unique modulo symmetry transformations, of the form

$$\psi_n(x) = f_n(r)e^{in\theta} , \quad (3.1)$$

where $1 > f_n \geq 0$ and is monotonically increasing from $f_n(0) = 0$ to 1 as r increases to ∞ . For $n = 0$, $f_n(r) = 1$. For $|n| > 0$, $f_n(r)$ does not admit an explicit expression. These are the n -vortices mentioned in the introduction. Of course, each solution ψ_n generates a one-parameter for $n = 0$ and a three-parameter for $|n| > 0$ family of solutions of (1.1). The latter are obtained by applying symmetry transformations to ψ_n .

The function $f_n(r)$ in (3.1) satisfies the ordinary differential equation

$$-\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f_n}{\partial r} \right) + \frac{n^2}{r^2} f_n - (1 - f_n^2) f_n = 0 . \quad (3.2)$$

The (self) energy of the n -vortex is given by $E_{n,R} := \mathcal{E}_{\text{ren}}(\psi_n)$. To compute $E_{n,R}$ we use that if ψ is a solution to (1.1), then, due to the integration by part formula $\int |\nabla\psi|^2 = -\int \bar{\psi} \Delta\psi$, we have

$$\mathcal{E}_{\text{ren}}(\psi) = \frac{1}{2} \int \left(1 - |\psi|^2 - \frac{1}{2} (1 - |\psi|^2)^2 - \frac{(\text{deg}\psi)^2}{r^2} \chi \right) . \quad (3.3)$$

Using this formula for $\psi = \psi_n$ and using the asymptotic expression (this can be easily derived from (3.2), in the general case see [S, OS4])

$$f_n(r) = 1 - \frac{n^2}{2r^2} + O\left(\frac{1}{r^4}\right) \quad (3.4)$$

for $r \gg 1$, we obtain

$$E_{n,R} = \pi n^2 \ln\left(\frac{R}{|n|}\right) + c(|n|) + O\left(\frac{1}{R^2}\right) . \quad (3.5)$$

The constant $c(n)$ can be computed numerically (which is not quite trivial, see Appendix 1) which yields

$$c(1) = 0.376 \pi, \quad c(2) = 0.535 \pi, \quad c(3) = 0.577 \pi, \quad c(5) = 0.615 \pi. \quad (3.6)$$

The asymptotics of $c(n)$ for $n \gg 1$ is found analytically in Appendix 2.

4. Intervortex energy

In this section we introduce and discuss a key concept of the intervortex energy (see also [OS3,5]). We begin with some definitions.

By a vortex configuration, \underline{c} , we will understand a pair $(\underline{a}, \underline{n})$, where $\underline{a} = (a_1, \dots, a_K)$, $a_j \in \mathbb{R}^2$, and $\underline{n} = (n_1, \dots, n_K)$, $n_j \in \mathbb{Z}$, for some $K \geq 1$ (positions of the vortex centers and their vorticities). Consider once differentiable functions $\psi: \mathbb{R}^2 \rightarrow \mathbb{C}$ satisfying $|\psi| \rightarrow 1$ as $|x| \rightarrow \infty$. We say that the vortex configuration of ψ is $\underline{c} = (\underline{a}, \underline{n})$, $\text{conf } \psi = \underline{c}$, if ψ has zeros (only) at a_1, \dots, a_K with local indices n_1, \dots, n_K , respectively, i.e.

$$\int_{\gamma_j} d(\arg \psi) = 2\pi n_j \quad (4.1)$$

for any contour γ_j containing a_j , but not the other zeros of ψ , and for $j = 1, \dots, K$. (Strictly speaking we have to specify the phase-factor, or rotation angle, for each vortex; but these will play no role in our considerations and are not displayed or mentioned in what follows.) Now we define

$$E_R(\underline{c}) = \inf \{ \mathcal{E}_{\text{ren}}(\psi) \mid \text{conf } \psi = \underline{c} \}. \quad (4.2)$$

We expect that $E_R(\underline{c}) > -\infty$. An argument supporting this statement is somewhat lengthy and will be presented elsewhere (see [OS5]). Of course, for bounded domains, this inequality is trivial. We call $E_R(\underline{c})$ the *energy of the vortex configuration* \underline{c} . It will

play a central rôle in our analysis. Note also that $E(\underline{c})$ serves as a Hamiltonian for the vortex dynamics in the adiabatic approximation (see [OS2]).

In what follows we keep the vortex indices \underline{n} fixed and write $E_R(\underline{a})$ for $E_R(\underline{c})$. It is clear intuitively that a minimizer in (4.2) exists if and only if $\nabla E_R(\underline{a}) = 0$ (the force acting on the vortex centers is zero). However, to establish this fact is not so easy.

Theorem 4.1. *If there is a minimizer for variational problem (4.2), then this minimizer satisfies Ginzburg-Landau equation (1.1).*

Proof. Let ψ be a minimizer for (4.2). Since for any differentiable function $\xi: \mathbb{R}^2 \rightarrow \mathbb{C}$ vanishing together with its gradient sufficiently fast at ∞ and vanishing at the points a_1, \dots, a_m we have

$$\begin{aligned} 0 &= \left. \frac{\partial}{\partial \lambda} \mathcal{E}_{\text{ren}}(\psi + \lambda \xi) \right|_{\lambda=0} \\ &= \text{Re} \int \bar{\xi} (-\Delta \psi + (|\psi|^2 - 1)\psi) , \end{aligned}$$

we conclude that ψ satisfies (1.1) for $x \neq a_1, \dots, a_m$. On the other hand, since $\psi \in H_1^{\text{loc}}(\mathbb{R}^2)$, we have that $-\Delta \psi + (|\psi|^2 - 1)\psi \in H_{-1}^{\text{loc}}(\mathbb{R}^2)$. Hence $-\Delta \psi + (|\psi|^2 - 1)\psi = 0$ on \mathbb{R}^2 . \square

Conjecture 4.2. $\nabla E_R(\underline{a}_0) = 0$ for some \underline{a}_0 (remember \underline{n} is fixed) if and only if there is a minimizer for problem (4.2) at the configuration \underline{a}_0 and consequently, due to Theorem 4.1, if and only if Ginzburg-Landau equation (1.1) has a solution with the configuration \underline{a}_0 .

The goal of this paper is to find forceless vortex configurations, i.e. configurations \underline{c} s.t.

$$\nabla E_R(\underline{a}) = 0 . \tag{4.3}$$

To this end we study the intervortex energy $E_R(\underline{a})$ for very small and very large intervortex separations.

Let $d_{\underline{a}} = \min_{i \neq j} |a_i - a_j|$ for $\underline{a} = (a_1, \dots, a_K)$. For $d_{\underline{a}}$ large we prove in Section 7 the following upper bound

$$E_R(\underline{a}) \leq E_R^{(0)} - A(\underline{a}) + O(d_{\underline{a}}^{-8/3}) + O(R^{-2}) , \quad (4.4)$$

where $E_R^{(0)} = \sum_{i=1}^K E_{n_i, R} + H\left(\frac{\underline{a}}{R}\right)$, and $A(\underline{a})$ is a homogeneous function of degree -2 , provided \underline{a} satisfies $\nabla H(\underline{a}) = 0$. Here, recall, $E_{n, R} = \mathcal{E}_{\text{ren}}(\psi_n)$ is the self-energy of the n -vortex (see (3.5)) and $H(\underline{a})$ is the energy of the vortex pair interactions,

$$H(\underline{a}) = -\pi \sum_{i \neq j} n_i n_j \ln |a_{ij}| , \quad (4.5)$$

with $a_{ij} = a_i - a_j$.

The correlation term $A(\underline{a})$ is of importance for us here. We have an explicit expression for it, see Eqns (A3.4)-(A3.5), and compute it explicitly in the cases of interest. We conjecture that $A(\underline{a}) > 0$ always.

Observe that upper bound (4.4) with the remainder $O(d_{\underline{a}}^{-1})$ instead of $-A(\underline{a}) + O(d_{\underline{a}}^{-8/3})$ is obtained by choosing the Hartree-type function $\psi^{(0)}(x) = \prod_{i=1}^K \psi_{n_i}(x - a_i)$ describing “independent” vortices. For asymptotically forceless configurations, i.e. the ones with $\nabla H(\underline{a}) = 0$, this estimate can be somewhat improved, but in order to move even to the remainder estimate $O(d_{\underline{a}}^{-2} \ln d_{\underline{a}})$ in the latter case, one has to refine upon this function and include the leading correlations.

Remark 4.3. As $d_{\underline{a}} \rightarrow \infty$, the following important asymptotic expression was demonstrated in [OS5]

$$E_R(\underline{a}) = \sum_{i=1}^K E_{n_i, R} + H\left(\frac{\underline{a}}{R}\right) + \text{Rem} , \quad (4.6)$$

where $\text{Rem} = O(d_{\underline{a}}^{-2} \ln d_{\underline{a}})$ in general and $= O(d_{\underline{a}}^{-2})$ if $\nabla H(\underline{a}) = 0$, as $d_{\underline{a}} \rightarrow \infty$.

As mentioned in the introduction, our second idea is to consider solutions to (1.1) invariant under point groups transformation. Consequently, we introduce intervortex en-

ergy functions invariant under such groups. Consider a subgroup G of the total symmetry group

$$G_{\text{sym}} = O(2) \times T(2) \times U(1)$$

($T(n)$ is the group of translations of \mathbb{R}^n) of Ginzburg-Landau equation (1). For a G -invariant vortex configuration $\underline{c} = (\underline{a}, \underline{n})$ (i.e. invariant under the spatial part of G) we define G -invariant vortex interaction energy $E_{R,G}(\underline{a})$ as

$$E_{R,G}(\underline{a}) = \inf\{\mathcal{E}_{\text{ren}}(\psi) \mid \text{conf}\psi = \underline{c}, \psi \text{ is } G\text{-invariant}\}$$

(as before we fix \underline{n} and omit it from the relation).

Theorem 4.1 and Conjecture 4.2 extend obviously to the G -symmetric situation. In particular we have the following conjecture:

If \underline{a}_0 is a critical point of $E_{R,G}(\underline{a})$ (i.e. $\nabla E_{R,G}(\underline{a}_0) = 0$), then (1.1) has a G -invariant solution.

Our goal in what follows is, for appropriate groups G , namely, point groups C_{Nv} (see the next section), to find critical points of the G -invariant intervortex energy $E_{R,G}(\underline{a})$.

5. Point symmetries

We look for solutions of Eqn (1.1) having symmetry groups, C_{Nv} . These groups consist of rotations around the origin by angles a multiple of $\frac{2\pi}{N}$ and reflection(s) in one (and therefore N) line(s) through the origin. Such solutions are determined by fixing vortex configurations having the desired symmetry group. We consider vortex configurations consisting of N m -vortices uniformly spaced on a circle of radius a and a single $(-k)$ -vortex in the center of the circle, which is placed at the origin. Several such configurations and their symmetry lines are shown in Fig. 1.

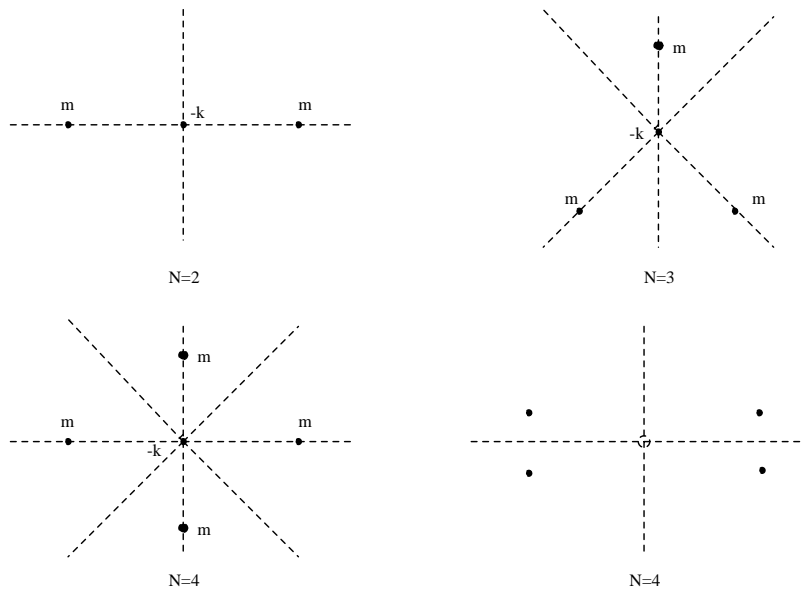


Fig. 1. Symmetric configurations and their reflection lines

Such configurations have the symmetry group C_{Nv} . On the other hand, the symmetry group C_{Nv} determines such a configuration uniquely up to the vortex values m and k and the size a .

As was pointed out at the end of the previous section we rely on the argument that C_{Nv} -symmetric solutions are in one-to-one correspondence with critical points of the C_{Nv} -symmetric intervortex energy

$$E_R(\underline{c}) \equiv E_{R,C_{Nv}}(\underline{c})$$

(here and in what follows we consider only C_{Nv} -symmetric intervortex energies and we often omit the subindex C_{Nv}). Thus our goal is to find critical points of $E_R(\underline{c})$. One of the central results of this paper is the following

Theorem 5.1. *There exist critical points of $E_{R,C_{Nv}}(\underline{c})$ among the configurations, \underline{c} , described above for the following values of the parameters*

$$(N, m, k) = (2, 2, 1) \text{ and } (4, 2, 3)$$

(see Fig. 1, a critical value of the parameter a is not specified, but its existence is established).

This theorem is proven in Section 6. In the rest of this section we establish general properties of the energy $E_{R,C_{Nv}}(\underline{c})$ and find a necessary condition on the parameters N , m and k .

Observe that if \underline{c} is a configuration described above, then (again we do not display the parameters \underline{n})

$$\nabla_{a_j} E_R(\underline{a}) = \hat{a}_j \partial_{|a_j|} E_R(\underline{a}) \quad \text{and} \quad \nabla_{a_j} H(\underline{a}) = \hat{a}_j \partial_{|a_j|} H(\underline{a}) \quad \forall j, \quad (5.1)$$

where $\hat{a} = a/|a|$. Thus in this case it suffices to investigate the energy $E_R(\underline{a})$ as a function of one variable, scale parameter a .

Note that if $m \geq 2$, then there is a continuum of configurations, labeled by a parameter $\alpha > 0$, with the same symmetry group, say C_{Nv} , as a given configuration, which have the given configuration as a limit as $\alpha \rightarrow 0$. For instance, for $m = 2$, each m -vortex can be split into a pair of 1-vortices with all pairs lying either on the circle or on the lines joining their parent m -vortices to the origin, at equal distance, α , to those m -vortices (see Fig. 2)

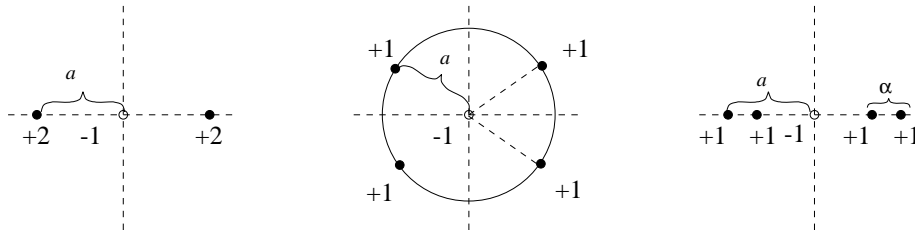


Fig. 2.

By symmetry, the energy of the resulting configurations has a critical point at $\alpha = 0$. A simple analysis of the break-up of a 2-vortex shows that this critical point is a local maximum. Indeed, e.g., for $m = 2$, it was shown in [OS1] that the linearization of Eqn

(1.1) (= the Hessian of the energy functional) around the 2-vortex solutions $\psi_2 = f_2(r)e^{2i\theta}$ has exactly one negative mode (= an eigenfunction corresponding to a negative eigenvalue) of the form $\xi = e^{4i\varphi}\xi_4(r) + \xi_0(r)$, where $\xi_k(r)$ are some real functions. Then the function $\psi_2 + \lambda\xi$ for $|\lambda|$ sufficiently small lowers the energy of ψ_2 . On the other hand this function has two simple zeros (i.e. of vorticities +1) in a vicinity of $x = 0$. Indeed, in the complex notation $z = x_1 + ix_2 \leftrightarrow x = (x_1, x_2)$, $\psi_2(z) = bz^2 + O(z^3)$, while $\xi(z) = c + O(z)$, for some positive numbers b and c , in a neighbourhood of $z = 0$. Hence $\psi_2(z) + \lambda\xi(z) = bz^2 + \lambda c + O(z^3) + O(\lambda z)$ and therefore has two simple zeros $z_{\pm} = \pm\sqrt{\frac{\lambda c}{b}} + O(\lambda^{\frac{3}{4}})$ in a neighbourhood of $z = 0$. This shows in particular that splitting of a 2-vortex lowers the energy.

Proposition 5.2. *Let a configuration \underline{a}_0 , as described above, be asymptotically forceless, i.e. $\nabla H(\underline{a}_0) = 0$. Then*

$$k = \frac{1}{2}(N-1)m . \quad (5.2)$$

Proof. By (4.1), the equation $\nabla H(\underline{a}_0) = 0$ for the configuration described is equivalent to the equation

$$\frac{\partial}{\partial a} H(\underline{a}_0) = 0 . \quad (5.3)$$

Since

$$H(\underline{a}) = H\left(\frac{\underline{a}}{a}\right) - \pi \sum_{i \neq j} n_i n_j \ln a , \quad (5.4)$$

the latter equation implies that $\sum_{i \neq j} n_i n_j = 0$, which, due to the relation

$$\sum_{i \neq j} n_i n_j = -2Nmk + N(N-1)m^2 , \quad (5.5)$$

is equivalent to (5.2). \square

Note that Eqn (5.3) implies that if $\nabla H(\underline{a}_0) = 0$ then $\nabla H(\underline{a}) = 0$ for all \underline{a} 's of the form $\underline{a} = s\underline{a}_0$, $s > 0$. The latter fact implies another proof of (5.2). Indeed, $H\left(\frac{\underline{a}}{R}\right)$ behaves

for large R as $\text{const} \cdot \ln R + \text{const}$. Hence for the asymptotically force-free configuration (i.e. the one with $\nabla H(\underline{a}) = 0$) the constant in front of $\ln R$ is independent of the scale parameter a . This asymptotic scale invariance implies that the leading term,

$$\pi(Nm - k)^2 \ln R ,$$

for the configuration with $a = 0$, i.e. when all the vortices collapse onto the center of the circle, is equal to the leading term,

$$\pi(Nm^2 + k^2) \ln R ,$$

for the configuration with a very large a so that the vortices in such a configuration can be treated as virtually independent (see (4.4)). Hence

$$(Nm - k)^2 = Nm^2 + k^2 ,$$

which implies (5.2).

Observe that Eqn (5.2) is equivalent to the relation

$$H\left(\frac{a}{R}\right) = H\left(\frac{a}{a}\right) = H(\underline{a}) , \quad \text{independent of } a . \quad (5.6)$$

Indeed, this follows from Eqns (5.4) and (5.5).

Relation (5.2) between k and m will be assumed from now on.

Now for the configuration above we introduce the energy differences

$$\Delta E(\underline{a}) := E_R(\underline{a}) - \pi(Nm - k)^2 \ln R \quad (5.7)$$

(remember that $Nm - k$ is the total vorticity of the configuration in question). Recall that $E_{n,R}$ denotes the energy of a single vortex of vorticity n , i.e. $E_{n,R} = \mathcal{E}_{\text{ren}}(\psi_n)$. Denote the energy difference for this vortex by ΔE_n :

$$E_{n,R} = \pi n^2 \ln R + \Delta E_n . \quad (5.8)$$

Clearly

$$E_R(\underline{0}) = E_{Nm-k,R} \quad \text{and} \quad \Delta E(\underline{0}) = \Delta E_{Nm-k} . \quad (5.9)$$

This together with (3.5) implies (modulo $O(R^{-2})$) that

$$\Delta E(\underline{0}) = -\pi(Nm - k)^2 \ln(Nm - k) + c(Nm - k) . \quad (5.10)$$

On the other hand, for the intervortex distances very large, Eqns (5.7), (5.6), (4.6) and (3.5) imply (modulo $O(R^{-2}) + o(a^{-2})$)

$$\Delta E(\underline{a}) \leq -\pi(Nm^2 \ln m + k^2 \ln k) + Nc(m) + c(k) + H(\underline{a}) - Ca^{-2} , \quad (5.11)$$

where $C = A(\underline{a}/a)$. Compute $H(\underline{a})$ for the configuration at hand. Since the distances between the vortices on the circle are $2a \sin \frac{\pi}{N}$, $2a \sin \frac{2\pi}{N}$, \dots , $2a \sin \frac{(N-1)\pi}{N}$, we find

$$H(\underline{a}) = -\pi m^2 N \sum_{k=1}^{N-1} \ln \left(2 \sin \frac{k\pi}{N} \right) . \quad (5.12)$$

This equation together with Eqn (5.11) yields, modulo $O(R^{-2}) + o(a^{-2})$, that for large intervortex distances

$$\begin{aligned} \Delta E(\underline{a}) &\leq -\pi(Nm^2 \ln m + k^2 \ln k) + Nc(m) + c(k) \\ &\quad - \pi m^2 N \sum_{k=1}^{N-1} \ln \left(2 \sin \frac{k\pi}{N} \right) - Ca^{-2} . \end{aligned} \quad (5.13)$$

In the next section we establish existence of points \underline{a}_0 s.t. $\nabla E(\underline{a}_0) = 0$ for given configurations, by comparing $\Delta E(\underline{0})$ and $\Delta E(\underline{a})$, for large intervortex distances (a).

6. The simplest cases. Proof Theorem 5.1

In this section we consider some special, in fact the simplest, cases of the vortex configurations introduced in Section 5. Recall that the latter consists of a vortex of vorticity

$-k$ sitting at the origin and N vortices, each of vorticity m , distributed equidistantly on the circle of radius a with the center at the origin. Such a configuration will be fixed by the symmetry group C_{Nv} , so that the only remaining free parameter is the radius of the circle a . Thus, we denote, with a slight abuse of notation, $\Delta E(a) = \Delta E(\underline{a})$.

Proof of Theorem 5.1. The correlation coefficient, C , in Eqn (5.13), is computed for the configurations of interest in Appendix 3:

$$C = 8\pi, \quad 20\pi \quad \text{for} \quad (N, m, k) = (2, 2, 1), (4, 2, 3). \quad (6.1)$$

(We suspect that for general (N, m, k) , $k = \frac{1}{2}(N-1)m$, C is of the form $\frac{\pi}{4} \cdot (\text{integer})$.)

Thus

$$\Delta E(a) \quad \text{monotonically increases to} \quad \Delta E(\infty) \quad \text{as} \quad a \rightarrow \infty. \quad (6.2)$$

Moreover due to (3.6), we have

$$\Delta E(\infty) < \Delta E(0), \quad (6.3)$$

again for the configurations $(N, m, k) = (2, 2, 1), (4, 2, 3)$ (explicit computations are given below). Hence $\Delta E(a)$ has at least one minimum for those configurations as claimed.

Computation of (6.3).

(a) The case $N = 2, m = 2$ and $k = 1$ (recall, $E_R(a) \equiv E_R(\underline{a})$, etc.) We have

$$\Delta E(0) \equiv \Delta E_3(0) = c(3) - 9\pi \ln 3 = -9.31\pi. \quad (6.4)$$

On the other hand, Eqn (5.11) implies that for a very large

$$\begin{aligned} \Delta E(a) &\leq c(1) + (2c(2) - 8\pi \ln 2) - 8\pi \ln 2 - Ca^{-2} + O\left(\frac{\ln a}{a^4}\right) \\ &= -9.64\pi - Ca^{-2} + O\left(\frac{\ln a}{a^4}\right). \end{aligned} \quad (6.5)$$

(b) The case $N = 4, m = 2$ and $k = 3$ (see Fig. 1). In this case

$$\Delta E(0) = \Delta E_5(0) = c(5) - 25\pi \ln 5 = -39.62\pi. \quad (6.6)$$

On the other hand, Eqn (5.11) implies that for large a we have the following asymptotic behaviour

$$\begin{aligned} \Delta E(a) &\leq (4c(2) - 16\pi \ln 2) + (c(3) - 9\pi \ln 3) \\ &\quad - 32\pi \ln 2 - Ca^{-2} + O\left(\frac{\ln a}{a^4}\right) \\ &= -40.44\pi - Ca^{-2} + O\left(\frac{\ln a}{a^4}\right). \end{aligned} \tag{6.7}$$

Thus (6.3) is shown. \square

Remarks. a. Examine the case of $m = 1$, i.e. the vortices on the circle are simple. In this case $k = \frac{1}{2}(N-1)$. Thus in the simplest case $N = 3$ and $k = 1$ we take the ($m = 1$)-vortices equally spaced (Fig. 3).

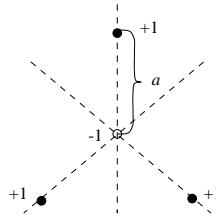


Fig. 3.

Eqns (4.9), (4.12) and (3.6) yield that in this case $\Delta E(0) < \Delta E(\infty)$ (in fact, $\Delta E(0) = \Delta E_2(0) = -2.238\pi$ and $\Delta E(\infty) = -1.792\pi$). Numerical computations show (see Appendices 3 and 4) that $\Delta E'(\infty) > 0$ and $\Delta E'(0) > 0$ (in fact, for $a \gg 1$, $\Delta E(a) = 4c(1) - 3\pi \ln 3 - Ca^{-2} = -1.792\pi - Ca^{-2}$ with $C > 0$). Thus we cannot conclude existence of a critical point for $E_R(a)$ in this case. However, a more careful numerical analysis indicates that, probably, there exist two extremal points of $E_R(a)$, one minimum and another maximum, for $\frac{1}{\sqrt{2}} \leq a \leq 2$. Similar configurations for N large (and odd) are analyzed in Appendix 5.

b. The case $N = 2$, $m = 2$ and $k = 1$, is the limiting case of $N = 4$, $m = 1$ and $k = 1$ (see Fig. 2). All three configurations have the same symmetry group, C_{2v} : rotation by π and reflections in the vertical and horizontal axes passing through the vortex -1 . After

the symmetry group is fixed, the second and third configurations have two free parameters: the scale parameter a and the angle/distance, α , between two of its neighbouring 1-vortices (see Fig. 3). As $\alpha \rightarrow 0$, the second and third configurations are continuously transformed into the first one.

7. Upper bound on the intervortex energy

In this section we prove inequality (4.4) for the energy, $E_R(\underline{a})$, of vortex configurations. Namely we prove

Theorem 7.1. *We have the estimate*

$$E_R(\underline{a}) \leq E_R^{(0)} + \text{Rem} + O(\max |a_j|^2/R^2), \quad (7.1)$$

where $E_R^{(0)} = \sum_{k=1}^k E_{n_i, R} + H\left(\frac{a}{R}\right)$, and

$$\text{Rem} = \begin{cases} O(d_a^{-2}) & \text{if } \nabla H(\underline{a}) = 0 \\ O(d_a^{-2} \ln d_a) & \text{otherwise} \end{cases} \quad (7.2)$$

Moreover, if $\nabla H(\underline{a}) = 0$, then estimate (7.2) can be improved as

$$\text{Rem} = -A(\underline{a}) + O(d_a^{-\frac{8}{3}}) + O\left(\frac{1}{R^2}\right), \quad (7.3)$$

where $A(\underline{a})$, the correlation term, is a homogeneous degree -2 function, explicitly given by the following conditionally convergent integral

$$A(\underline{a}) = \frac{1}{4} \int \left[|\nabla \varphi_0|^4 - \sum_j |\nabla \varphi_j|^4 \right] \quad (7.5)$$

(here $\nabla H(\underline{a}) = 0$ is assumed) with

$$\varphi_0 = \sum_j \varphi_j \text{ and } \varphi_j(x) = n_j \theta(x - a_j),$$

$$\theta(x) = \text{the polar angle of } x \in \mathbb{R}^2.$$

Proof. We prove the upper bound (7.1) by using the variational inequality

$$E_R(\underline{a}) \leq \mathcal{E}_R(\psi) , \quad (7.6)$$

valid for any function ψ having the given vortex configuration \underline{a} , and by showing that for an appropriate ψ , $\mathcal{E}_R(\psi)$ is of the form of the r.h.s. of (7.6). Namely we show that

$$\mathcal{E}_{\text{ren}}(\psi) = E_R^{(0)} + \text{Rem} , \quad (7.7)$$

where Rem is given by either (7.2) or (7.3), as appropriate. Then (7.1) follows from (7.6) and (7.7).

Before proceeding to a proof of these estimates, we show that the integral on the r.h.s. of (7.4) is conditionally convergent in the forceless case $\nabla H(\underline{a}) = 0$. Since the integrand has singularities at the points a_1, \dots, a_K , it suffices to show that the integrals over the discs, $D(a_k, \varepsilon)$, centered at a_k and of a radius $\varepsilon > 0$, converge. Consider the integral over the disc $D(a_k, \varepsilon)$. Let

$$\varphi_{(k)}(x) = \sum_{j \neq k} \varphi_j(x) . \quad (7.8)$$

Since the function $\varphi_{(k)}(x)$ is harmonic in $D(a_k, \varepsilon)$, it has an expansion around the point a_k of the form

$$\varphi_{(k)}(x) = \sum_{m=0}^{\infty} c_m r_k^m \cos m(\theta_k - \theta^{(m)}) , \quad (7.9)$$

where r_k and θ_k are the polar coordinates of $x_k = x - a_k$ and c_m and $\theta^{(m)}$ are some constants.

In the forceless case,

$$\nabla \varphi_{(k)}(a_k) = -\frac{1}{2\pi n_k} J \nabla_{a_k} H(\underline{a}) = 0 \quad (7.10)$$

and therefore

$$\nabla \varphi_{(k)}(x) = c_k (x_k \cos 2\theta_k - x_k^\perp \sin 2\theta_k) + O\left(\frac{r_k^2}{d_{\underline{a}}^3}\right) , \quad (7.11)$$

where $c_k = O(1/d_{\underline{a}}^2)$ is a constant, $r_k = |x_k|$ and $x^\perp = (-x_2, x_1)$. Now, writing

$$\int_{D(a_k, \varepsilon)} (|\nabla\varphi|^4 - |\nabla\varphi_k|^4) = \int_{D(a_k, \varepsilon)} (2|\nabla\varphi_k|^2\alpha_k + \alpha_k^2) , \quad (7.12)$$

where

$$\alpha_k := 2\nabla\varphi_k \cdot \nabla\varphi_{(k)} + |\nabla\varphi_{(k)}|^2 \quad (7.13)$$

and using (7.11), we see that the singular part of the integral above is

$$\begin{aligned} & 4 \int_{D(a_k, \varepsilon)} |\nabla\varphi_k|^2 \nabla\varphi_k \cdot \nabla\varphi_{(k)} = \\ & = 4 \int_{D(a_k, \varepsilon)} \frac{n_k^2}{r_k^2} (-c_k \sin 2\theta_k + O(r_k)) = \int_{D(a_k, \varepsilon)} O\left(\frac{1}{r_k}\right) < \infty . \end{aligned} \quad (7.14)$$

Thus the integral on the r.h.s. of (7.4) is conditionally convergent, in the sense that it is well defined as a limit of the similar integrals with small discs around the points a_1, \dots, a_K excised, as the radii of those discs tend to 0.

Now we proceed to establishing estimate (7.6). We begin with proving estimate (7.1) with remainder (7.2). Let $\psi_i(x) = \psi^{(n_i)}(x_i)$, where $x_i = x - a_i$, and let $f_i \equiv |\psi_i|$. Consider a class of functions ψ of the form $\psi = f e^{i\varphi_0}$ with a function f satisfying

$$f = f_i + O\left(\frac{1}{r \cdot d_{\underline{a}}^n}\right) \quad \text{if } r_j \ll d_{\underline{a}} , \quad \forall i \quad (7.15)$$

where $n = 2$ if $\nabla H(\underline{a})$ and $n = 1$ otherwise and $r_i = |x - a_i|$, and

$$f = 1 + O\left(\frac{1}{d(x, \underline{a})^2}\right) \quad \text{if } d(x, \underline{a}) \gg 1 , \quad (7.16)$$

where

$$d(x, \underline{a}) = \min_j |x - a_j| ,$$

with the corresponding estimates of their first derivatives.

We construct a function satisfying (7.15) and (7.16). Let $D(z, \rho)$ denote the disc of radius ρ centered at a point z . Let $\{\chi_j\}_1^K$ be a smooth partition of unity, i.e. $\sum_{l=1}^K \chi_l = 1$, having the following properties,

$$B(a_j, \frac{1}{3}d_{\underline{a}}) \subset \text{supp}\chi_j \quad \forall j$$

and

$$\nabla^n \chi_j = O(d_{\underline{a}}^{-n}), n = 0, 1, 2.$$

Then the following function satisfies (7.15) and (7.16): $f = \sum f_j \chi_j$. Indeed, (7.6) is obvious, while (7.7) follows from the relation

$$f_j = 1 + O(r_j^{-1}) \tag{7.17}$$

We prove the following

Lemma 7.1. *Assume ψ satisfies (7.15)–(7.16). Then*

$$\mathcal{E}_R(\psi) = E_R^{(0)} + \text{Rem} + O\left(\frac{1}{R^2}\right), \tag{7.18}$$

where, recall, $E_R^{(0)}$ is given in Theorem 7.1 and Rem is given by (7.2).

Proof. Let $D_j = D(a_j, r_0)$, the disc with the center at a_j and of the radius $r_0 = \frac{1}{3}d_{\underline{a}}$.

We decompose the energy functional as

$$\mathcal{E}_R(\psi) = \sum_j \int_{D_j} e(\psi) + \int_{D_R \setminus \cup D_j} e(\psi), \tag{7.19}$$

where $e(\psi)$ is the energy density,

$$e(\psi) = \frac{1}{2}|\nabla\psi|^2 + \frac{1}{4}(|\psi|^2 - 1)^2. \tag{7.20}$$

Let $e_1(\varphi) = \frac{1}{2}|\nabla\varphi|^2$ and $\langle f(\psi) \rangle = f(\psi) - \sum_k f(\psi_k)$. Eqn (4.6) implies

$$\int_{D_R \setminus \cup D_k} e(\psi) = \int_{D_R \setminus \cup D_k} e_1(\varphi_0) + \int_{D_R \setminus \cup D_k} O(d(x, \underline{a})^{-4}). \tag{7.21}$$

Next, the estimates (7.17) and

$$\nabla|\psi_i| = O(r_j^{-3}) \quad (7.22)$$

give

$$\int_{D_R \setminus \cup D_k} e_1(\varphi_i) = \int_{D_R \setminus \cup D_k} e(\psi_i) + O(r_0^{-2}) . \quad (7.23)$$

This together with Eqn (7.12) yields

$$\int_{D_R \setminus \cup D_k} \langle e(\psi) \rangle = \frac{1}{2} \sum_{i \neq j} \int_{D_R \setminus \cup D_k} \nabla\varphi_i \nabla\varphi_j + O(r_0^{-2}) . \quad (7.24)$$

Next, in the region D_i we have $\psi = e^{i\varphi_0} f_i$, where, recall, $f_i \equiv |\psi_i|$. Expansion (7.9)

implies that

$$\int_{D_i} \nabla\varphi_i \cdot \nabla\varphi_{(i)} = 0 . \quad (7.25)$$

Using this relation we obtain

$$\int_{D_i} e(\psi) = \int_{D_i} e(\psi_i) + \int_{D_i} e_1(\varphi_{(i)}) + R ,$$

where

$$R = \int_{D_i} (f_i^2 - 1)\alpha_i .$$

Expanding

$$\nabla\varphi_{(i)} = \nabla\varphi_{(i)}(a_i) + O\left(\frac{r_i}{d_{\underline{a}}^2}\right) \quad (7.26)$$

and using that $|\nabla\varphi_{(i)}(x)|^2 = O(d(x, \underline{a})^{-2})$, $\nabla\varphi_i(x) = O(r_i^{-1})$ and $\int_{D_i} (1 - f_i^2)\nabla\varphi_i = 0$,

we obtain that

$$R = O\left(\frac{\ln r_0}{d_{\underline{a}}^2}\right) .$$

In the forceless case we can improve this estimate using relation (7.9) again to show that, as in (7.14),

$$\begin{aligned} & \int_{D_i} (f_i^2 - 1) \nabla \varphi_i \cdot \nabla \varphi_{(i)} \\ &= \int_{D_i} (f_i^2 - 1) \left(-c_i \sin 2\theta_i + O\left(\frac{r_i}{d_{\underline{a}}^3}\right) \right) \\ &= \int_{D_i} (f_i^2 - 1) O\left(\frac{r_i}{d_{\underline{a}}^3}\right) = O\left(\frac{r_0}{d_{\underline{a}}^3}\right) . \end{aligned}$$

This gives

$$R = O\left(\frac{r_0}{d_{\underline{a}}^3}\right) \quad \text{if } \nabla \varphi_i(a_i) = 0 .$$

Finally, we observe that due to (7.15)

$$\begin{aligned} \frac{1}{2} \int_{D_k} |\nabla \varphi_{(k)}|^2 &= \sum_{j \neq k} \int_{D_k} e_1(\psi_j) + I_{D_k} \\ &= \sum_{j \neq k} \int e(\psi_j) + I_{D_k} + O(r_0^{-2}) , \end{aligned}$$

where $I_D := \frac{1}{2} \sum_{i \neq j} \int_{D_k} \nabla \varphi_i \cdot \nabla \varphi_j$

Collecting the estimates above, we arrive at

$$\int_{D_k} \langle e(\psi) \rangle = I_{D_k} + O\left(\frac{\ln r_0}{d_{\underline{a}}^2}\right) + O\left(\frac{1}{r_0^2}\right) , \quad (7.27)$$

which together with (7.11) and (7.16) yields

$$\mathcal{E}_R(\psi) = E + \text{Rem} \quad (7.28)$$

where Rem is given in (7.2) and $E = \int \left(g - \frac{(\deg \psi)^2}{n^2} \chi \right)$ with $g = \sum_j e(\psi_j) + \frac{1}{2} \sum_{i \neq j} \nabla \varphi_i \cdot \nabla \varphi_j$.

Now, by the definition of the cut-off function χ ($\chi \geq 0$, $\chi = 1$ for $|x| \geq R$) we have

$$E \leq \int_{B(0,R)} g + \int_{B(0,R)^c} \left(g - \frac{n}{2r^2} \right) \quad (7.29)$$

where $n = \deg \psi$. First we compute the first integral on the r.h.s..

By the definition $E_{n,R}$ and since $a_i \ll R$ we have

$$\int_{D_R} e(\psi_i) = \int_{D_{R+a_i}} e(\psi^{(n_i)}) = E_{n_i,R} + O\left(\frac{1}{R^2}\right). \quad (7.30)$$

Now we show that

$$I_{D_R} \equiv \frac{1}{2} \sum_{i \neq j} \int_{D_R} \nabla \varphi_i \nabla \varphi_j = - \sum_{i \neq j} \pi n_i n_j \ln \left(\frac{|a_{ij}|}{R} \right). \quad (7.31)$$

We compute

$$\int_{D_R} \nabla \varphi_i \nabla \varphi_j = n_i n_j \int_0^{2\pi} \int_0^R \frac{r - a \cos \theta}{r^2 + a^2 - 2ar \cos \theta} dr d\theta, \quad (7.32)$$

where $a = |a_{ij}|$. Furthermore, changing the variable of integration as $\theta \rightarrow z = e^{i\theta}$ and computing the residue, we find

$$\begin{aligned} \int_0^{2\pi} \frac{r - a \cos \theta}{r^2 + a^2 - 2ar \cos \theta} d\theta &= \frac{\pi}{r} - \frac{r^2 - a^2}{2iar^2} \oint_{|z|=1} \frac{dz}{\left(z - \frac{r}{a}\right) \left(z - \frac{a}{r}\right)} \\ &= \frac{\pi}{r} + \frac{\pi}{r} \frac{r^2 - a^2}{|r^2 - a^2|} = \frac{2\pi}{r} \begin{cases} 1 & \text{if } r > a \\ 0 & \text{if } r < a \end{cases}. \end{aligned}$$

The last two equations yield (7.24). Observe also that up to a multiplicative constant expression (7.24) can be found from the symmetry considerations: the invariance of the integral on the l.h.s. under translations ($a_i \rightarrow a_i + h$ and $a_j \rightarrow a_j + h \forall h \in \mathbb{R}^2$) and rotations ($a_i \rightarrow ga_i$ and $a_j \rightarrow ga_j \forall g \in O(2)$) imply that it depends only on $|a_{ij}|$. Its scaling properties under the dilations ($a_i \rightarrow \lambda a_i$ and $a_j \rightarrow \lambda a_j \forall \lambda \in \mathbb{R}$) imply that it is a multiple of $\ln \left(\frac{|a_{ij}|}{R} \right)$.

Eqns (7.30) and (7.31) imply

$$\int_{B(0,R)} g = \sum E_{n_i,R} + H(\underline{a}/R) + O(1/R^2). \quad (7.33)$$

Next we estimate the second integral on the r.h.s. of (7.29). By Eqn (7.17) and (7.22) we have

$$g = \frac{1}{2} |\nabla \varphi_0|^2 + O(d(x, \underline{a})^{-4}) .$$

Furthermore, expanding the terms $\nabla \theta(x - a_j)$ in $\nabla \varphi_0(x) = \sum n_j \nabla \theta(x - a_j)$ around the point x we obtain

$$\nabla \varphi_0(x) = n \nabla \theta(x) - \theta''(x) \sum n_j a_j + O\left(\frac{\sum n_j a_j^2}{d(x, \underline{a})^3}\right) \quad (7.34)$$

where $\theta''(x)$ is the Hessian of $\theta(x)$. Choosing the origin so that $\sum n_j a_j = 0$ eliminates the second term on the r.h.s.. (Otherwise we could have used that by an explicit computation we have

$$\theta''(x) \nabla \theta(x) = -\frac{x}{r^4} ,$$

the integral of which over the exterior of the ball $B(0, R)$ vanishes.) Hence

$$\begin{aligned} \int_{B(0, R)^c} \left(g - \frac{n^2}{2r^2}\right) &= \int_{B(0, R)^c} O\left(\frac{\sum n_j a_j^2}{d(x, \underline{a})^4}\right) \\ &= O\left(\frac{\sum n_j a_j^2}{R^2}\right) . \end{aligned} \quad (7.35)$$

Estimates (7.28), (7.29), (7.33) and (7.35) imply (7.7) with Rem given in (7.2). \square

Remark 7.3. The statement of Lemma 7.2 remains true for a wider class of functions defined by replacing (7.7) by the following condition

$$\begin{aligned} f = f_i + O\left(\frac{1}{r \cdot d_{\underline{a}}^n}\right) \text{ and } \int_0^{2\pi} \text{Re}(e^{-i\varphi_0} \psi - f_i) d\theta &= O\left(\frac{1}{d_{\underline{a}}^{n+1}}\right) , \\ \text{if } |x - a_i| &\ll d_{\underline{a}} , \end{aligned} \quad (7.36)$$

with the corresponding estimates of their first derivatives, where $n = 2$ if $\nabla H(\underline{a}) = 0$ and $n = 1$ otherwise.

To prove this we write ψ in the region D_i as $\psi = e^{i\varphi_0}(f_i + \xi)$, where $f_i \equiv |\psi_i|$. Using relation (7.25) and using that

$$\int_{D_j} f_j \nabla \varphi_j \cdot \nabla \operatorname{Im} \xi = n_j \int_{D_j} f_j \frac{\partial}{\partial \theta} \operatorname{Im} \xi = 0, \quad (7.37)$$

we obtain

$$\int_{D_i} e(\psi) = \int_{D_i} e(\psi_i) + \int_{D_i} e_1(\varphi_{(i)}) + R + R', \quad (7.38)$$

where R is given above and

$$\begin{aligned} R' &= \int_{D_i} \left\{ (|\nabla \varphi_0|^2 + f_i^2 - 1) f_i \operatorname{Re} \xi + f_i^2 (\operatorname{Re} \xi)^2 \right. \\ &\quad + \frac{1}{2} |\nabla \varphi_0|^2 |\xi|^2 + \frac{1}{2} |\nabla \xi|^2 + 2 \nabla f_i \cdot \nabla \operatorname{Re} \xi + f_i \nabla \varphi_{(i)} \cdot \nabla \operatorname{Im} \xi \\ &\quad \left. + \operatorname{Im}(\xi \nabla \varphi_0 \cdot \nabla \xi) + \frac{1}{2} (f_i^2 - 1 + 2 f_i \operatorname{Re} \xi) |\xi|^2 + \frac{1}{4} |\xi|^4 \right\}. \end{aligned} \quad (7.39)$$

Using that, due to (7.36), $\xi = O\left(\frac{1}{r \cdot d_{\underline{a}}}\right)$ and $\int_0^{2\pi} \operatorname{Re} \xi d\theta = O\left(\frac{1}{d_{\underline{a}}^2}\right)$ in D_j and using that $|\nabla \varphi_i|^2 + f_i^2 - 1 = O(r_i^{-4})$, we find

$$R' = O\left(\frac{\ln r_0}{d_{\underline{a}}^2}\right). \quad (7.40)$$

Now we proceed to proving estimate (7.4) with Rem given by (7.3). First we describe a class of test functions for which we prove this estimate: $\psi = e^{i\varphi_0} f$ with

$$f = \begin{cases} f_j - \frac{1}{2} f_j^{-1} \alpha_j \eta_j & \text{in } D(a_j, \frac{1}{3} d_{\underline{a}}) \quad \forall j \\ 1 - \frac{1}{2} |\nabla \varphi_0|^2 + O(d(x, \underline{a})^{-4}) & \text{in } \left(\bigcup_j D(a_j, \frac{1}{4} d_{\underline{a}}) \right)^c \end{cases} \quad (7.41)$$

where we used definition (7.13) and where η_j are smooth cut-off functions depending only on $r_j = |x_j|$ (i.e. radially symmetric in the x_j -variables) satisfying

$$B(a_j, \frac{1}{2} d_{\underline{a}}) \setminus B(a_j, 2d_{\underline{a}}^\gamma) \subset \operatorname{supp} \eta_j \subset B(a_j, \frac{1}{2} d_{\underline{a}}) \setminus B(a_j, d_{\underline{a}}^\gamma) \quad (7.43)$$

and

$$\nabla^n \eta_j = O(d_{\underline{a}}^{-\gamma n}), \quad n = 0, 1, 2, \quad (7.44)$$

for $\gamma = \frac{1}{3}$ (not optimal). (The f_j^{-1} 's in (7.41) play no important role and are chosen purely with a view of simplifying some expressions below.)

The following function satisfies (7.41) and (7.42)

$$f = \sum f_j \chi_j - \sum \frac{1}{2} f_j^{-1} \alpha_j \eta_j . \quad (7.45)$$

To prove this we use the expansion

$$f_j = 1 - \frac{1}{2} |\nabla \varphi_j|^2 + O(r_j^{-4}) \quad (7.46)$$

and the estimate

$$\alpha_j = O(d_{\underline{a}}^{-2}) \text{ in } D(a_j, d_{\underline{a}}) , \quad (7.47)$$

which is shown by expanding the function $\nabla \varphi_{(j)}(x)$ around a_j and using that $\nabla \varphi_{(j)}(a_j) = -\frac{1}{2\pi n_j} \nabla_{a_j} H(\underline{a}) = 0$ and that $\nabla \varphi_j(x) = O(r_j^{-1})$.

Our next task is to prove the following

Lemma 7.4. *Let \underline{a} be forceless in the sense that $\nabla H(\underline{a}) = 0$. Then estimate (7.7) with (7.3) holds for any function ψ satisfying (7.21)–(7.22).*

Proof. The proof follows the lines of the proof of Lemma 7.2 but with some subtle modifications which we elaborate upon now.

First of all instead of $e_1(\psi) = \frac{1}{2} |\nabla \varphi|^2$ used in the proof of Lemma 7.2 we use the density

$$e_2(\varphi) = \frac{1}{2} |\nabla \varphi|^2 - \frac{1}{4} |\nabla \varphi|^4 , \quad (7.48)$$

which is a better approximation to the density $e(\psi)$, and instead of (7.17) we use (7.27).

In particular we have

$$e(\psi_j) = e_2(\varphi_j) + O(r_j^{-6}) . \quad (7.49)$$

Denot $f_j := 1 - f_j^2 - |\nabla\varphi_j|^2$. For any k and for $u_k = e^{i\varphi_0}(f_k + \xi)$, where ξ is a real function, we have the following identity

$$\langle e(u_k) \rangle = \frac{1}{2} \sum_{i \neq j} \nabla\varphi_i \cdot \nabla\varphi_j - A(\varphi) + B_k(\xi) + R_k, \quad (7.50)$$

where

$$B_k(\xi) := -\frac{1}{2}g_k(\alpha_k + 2f_k\xi) + \frac{1}{4}\alpha_k^2 + \alpha_k f_k \xi + f_k^2 \xi^2 \quad (7.51)$$

and

$$\begin{aligned} R = & \sum_{j \neq k} (e_2(\varphi_j) - e(\psi_j)) - \frac{1}{2}(g_k - \alpha_k)\xi^2 + f_k \xi_k + \frac{1}{4}\xi_k^4 \\ & + \frac{1}{2}(2\nabla f_k \cdot \nabla \xi + |\nabla \xi|^2) \end{aligned} \quad (7.52)$$

Now we take $\xi = -\frac{1}{2}f_k^{-1}\alpha_k\eta_k$. Then

$$e(\psi) = e(u_k) \quad \text{on } D(a_k, \frac{1}{3}d_{\underline{a}}). \quad (7.53)$$

Due to (7.28) and the corresponding estimate for the derivatives of α_j and due to (7.25), (7.27) and (7.29) we have

$$R_k = O(d_{\underline{a}}^{-4\gamma-2}) \quad (7.54)$$

Note now that the form of (7.21) is chosen so that

$$B_k(\xi) = 0 \quad \text{on } B(a_k, \frac{1}{2}d_{\underline{a}}) \setminus B(a_k, d_{\underline{a}}^\gamma) \subset \{\eta_k = 1\}.$$

Now we estimate $B_k(\xi)$ on the entire disc $D(a_k, \frac{1}{3}d_{\underline{a}})$. Expanding the function $\nabla\varphi_{(k)}(x)$ around the point a_k and using that $\nabla\varphi_{(k)}(a_k) = -\frac{1}{2\pi n_k} J \nabla_{a_k} H(\underline{a}) = 0$, we find

$$\alpha_k(x) = 2\nabla\varphi_k(x) \cdot \varphi''_{(k)}(a_k)x_k + O(r_k d_{\underline{a}}^{-3}), \quad (7.55)$$

where, recall, $x_k = -a_k$, and φ'' is the Hessian (the matrix of second derivatives) of a function φ . Using this expression in estimating $B_k(\xi)$ we find

$$B_k(\xi) = -g_k \nabla\varphi_k(x) \cdot \varphi''_{(k)}(a_k)x_k \bar{\eta}_k + O(r^{-3}d_{\underline{a}}^{-3} + d_{\underline{a}}^{-4})\bar{\eta}_k \quad \text{on } D(a_k, \frac{1}{3}d_{\underline{a}}), \quad (7.56)$$

where $\bar{\eta}_k = 1 - \eta_k$. The first term on the r.h.s. of this expression is singular at $x_k = x - a_k = 0$, but the integral of it is conditionally convergent and equals 0. Indeed, since the function $\varphi_{(k)}(x)$ is harmonic in $D(a_k, \frac{1}{3}d_{\underline{a}})$ we have that (cf (7.11))

$$\varphi_{(k)}''(a_k)x_k = c(x_k \cos 2\theta_k - x_k^\perp \sin 2\theta_k) , \quad (7.57)$$

where $c = O(d_{\underline{a}}^{-2})$, $x^\perp = (-x_2, x_1)$ and θ_k is the polar angle of x_k (see Eqn (7.9)). Since g_k and $\bar{\eta}_k$ depend only on r_k (we write $(g_k \bar{\eta}_k)(r_k)$ for $g_k(x)\bar{\eta}_k(x)$), we have

$$\int (g_k \bar{\eta}_k)(r_k) \nabla \varphi_k(x) \varphi_{(k)}''(a_k)x_k = -c \int (g_k \bar{\eta}_k)(r_k) \sin 2\theta_k = 0 \quad (7.58)$$

(strictly speaking we have first to excise a small disc around $x_k = 0$ and then take the radius of this disc to zero).

Eqns (7.32), (7.33), (7.35) and (7.37) imply that

$$\begin{aligned} \int_{D(a_k, \frac{1}{3}d_{\underline{a}})} \langle e(\psi) \rangle &= \int_{D(a_k, \frac{1}{3}d_{\underline{a}})} \left(\frac{1}{2} \sum_{i \neq j} \nabla \varphi_i \cdot \nabla \varphi_j - A(\varphi) \right) \\ &+ O(d_{\underline{a}}^{-3} + d_{\underline{a}}^{-2-4\gamma} + d_{\underline{a}}^{-4+2\gamma}) . \end{aligned} \quad (7.59)$$

Finally, we derive the estimate

$$\langle e(\psi) \rangle = \frac{1}{2} \sum_{i \neq j} \nabla \varphi_i \nabla \varphi_j - A(\varphi) + O(d(x, \underline{a})^{-6}) \quad (7.60)$$

on $\left(\bigcup_k D(a_k, \frac{1}{4}d_{\underline{a}}) \right)^c$. Indeed, Eqn (7.42) implies that

$$e(\psi) = e_2(\varphi_0) + O(d(x, \underline{a})^{-6}) , \quad (7.61)$$

which together with (7.49) implies (7.60).

Now, Eqns (7.59) and (7.60) with $\gamma = 1/3$ imply

$$\mathcal{E}_R(\psi) = E - A(\underline{a}) + O(d_{\underline{a}}^{8/3}) , \quad (7.62)$$

where the term E is defined after Eqn (7.28) and $A(\underline{a}) = \int A(\varphi)$. Eqns (7.29), (7.33), (7.35) and (7.61) imply (7.7) with Rem given by (7.3). \square

Lemmas 7.2 and 7.4 and inequality (7.6) imply Theorem 7.1. \square

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Appendix 1. Computation of $c(n)$

In this appendix we compute the constants $c(n)$ in expression (3.5) for the self-energy, $E_{n,R}$, of the n -vortex (see Eqn (3.6)). To this end we derive a convenient formula for $E_{n,R}$. Multiplying Eqn (3.2) by $r^2 f'_n(r)$, where $f'(r) = \frac{\partial f(r)}{\partial r}$, integrating the result over r , observing that the first two integrands are full derivatives and integrating the last term by parts, we obtain the quantization relation (see [BMR])

$$\int_0^\infty (1 - f_n^2)^2 r dr = n^2 .$$

This equation together with Eqn (3.3) yields an expression for $E_{n,R}$:

$$E_{n,R} = -\frac{\pi}{2} \pi n^2 + \pi \int_0^\infty \left(1 - f_n^2 - \frac{n^2}{r^2} \chi\right) r dr .$$

However we prefer to use a different representation of $E_{n,R}$ which is obtained from above if we write $1 - f_n^2 = (1 - f_n^2) f_n^2 + (1 + f_n^2)^2$ and use the quantization formula above again:

$$E_{n,R} = -\frac{\pi}{2} \pi n^2 + \pi \int_0^\infty \left[(1 - f_n^2) f_n^2 - \frac{n^2}{r^2} \chi\right] r dr . \quad (A1.1)$$

First of all in order to avoid a numerical evaluation of the integral in (A1.1) over an infinite range, we use for large r an expansion of $f_n(r)$ in $\frac{1}{r}$. However, $f_n(r)$ is not

analytic at $r = \infty$, it has an essential singularity at this point. Hence the resulting series is asymptotic. We truncate this series at the order $O\left(\frac{1}{r^6}\right)$. To compensate for this truncation we add to the resulting polynomial in $\frac{1}{r}$ a multiple of the decaying solution $e^{-\sqrt{2}r}/\sqrt{r}$ of the linearization of Eqn (3.2) around 1. We should linearize Eqn (3.2) around the resulting polynomial, but the powers of $\frac{1}{r^2}$ lead to similar powers multiplying $e^{-\sqrt{2}r}/\sqrt{r}$, thus it suffices to linearize around 1. The result is

$$f_n(r) = \left\{ 1 - \frac{n^2}{2r^2} - \frac{n^2(1+n^2/8)}{r^4} - \frac{1}{r^6} \left(\frac{n^4}{2} + \frac{n^2+16}{2} \left(n^2 + \frac{n^4}{8} \right) \right) - \dots \right\} - c \frac{e^{-\sqrt{2}r}}{\sqrt{r}} (1 + \dots), \quad (\text{A1.2})$$

where c is a constant to be determined by a matching procedure. Plugging this expression into Eqn (A1.1), we obtain

$$E_{n,R} - \pi n^2 \ln R = \frac{\pi n^2}{2} + \pi \int_0^{r_0} f_n^2(1 - f_n^2) r dr - \pi n^2 \left(\ln r_0 + \frac{n^2 - 2}{2r_0^2} + \frac{n^2 - 16}{4r_0^4} \right) + O(r_0^{-6}) \quad (\text{A1.3})$$

for any $r_0 > 0$. We pick $6 \leq r_0 \leq 10$. This relation together with Eqn (3.5) implies that

$$\frac{1}{\pi} c(n) = \frac{n^2}{2} + \int_0^{r_0} f_n^2(1 - f_n^2) r dr - n^2 \left(\ln \frac{r_0}{|n|} + \frac{n^2 - 2}{2r_0^2} + \frac{n^2 - 16}{4r_0^4} \right) + O(r_0^{-6}). \quad (\text{A1.4})$$

For a numerical solution of Eqn (3.2) we take the interval $(0.3, r_0)$. Since Eqn (3.2) linearized around the function 1 has the solutions

$$\frac{1}{\sqrt{r}} e^{\pm\sqrt{2}r}, \quad (\text{A1.5})$$

one should do the numerical iteration procedure starting from the upper limit, r_0 . In this way the dangerous, exponentially growing solution would not effect our procedure.

In the range $0 < r \leq 0.3$ we use the fact that, as Eqn (3.2) shows, the function $f_n(r)$ is analytic in a disc $|r| < O(1)$, so it can be presented by a convergent series:

$$f_n(r) = \alpha r^n \left\{ 1 - \frac{r^2}{4(n+1)} + \frac{r^4}{8(n+2)} \left(\frac{1}{4(n+1)} + \alpha^2 \delta_{n,1} \right) + \frac{r^6}{12(n+3)} \left[\alpha^2 \left(\delta_{n,2} - \frac{3}{4(n+1)} \delta_{n,1} \right) - \frac{1}{8(n+2)} \left(\frac{1}{4(n+1)} + \alpha^2 \delta_{n,1} \right) \right] + \dots \right\} \quad (\text{A1.6})$$

for some number $\alpha > 0$. Here $\delta_{n,k}$ is the Kronecker symbol, $\delta_{n,k} = 1$ for $n = k$ and $= 0$ for $n \neq k$. (We expect that the pole closest to the origin lies on the imaginary axis.)

To finish off the computation of $c(n)$ we have to find the value of the parameters α and c . This is done by matching the solution (A1.2) for small r with that, Eqn (A1.6), for large r . Specifically, using Eqn (A1.2), we compute $f_n(r_0)$ and $f'_n(r_0)$ for various values of the parameter c . Using these values as initial conditions, we integrate Eqn (3.2) backward to $r = 0.3$, which yields $f_{\text{right}}(0.3)$ and $f'_{\text{right}}(0.3)$. On the other hand using Eqn (A1.6), we compute $f_{\text{left}}(0.3)$ and $f'_{\text{left}}(0.3)$ for various values of the parameter α . Now we match $f_{\text{right}}(0.3)$ and $f'_{\text{right}}(0.3)$ with $f_{\text{left}}(0.3)$ and $f'_{\text{left}}(0.3)$ by minimizing $[(f_{\text{right}}(0.3) - f_{\text{left}}(0.3))^2 + (f'_{\text{right}}(0.3) - f'_{\text{left}}(0.3))^2]^{1/2}$. This yields the values of the parameters c and α . After this we compute $c(n)$, using formulae (A1.4) and (A1.6).

Appendix 2. Large n asymptotics of the vortex (self) energy

In this appendix we find the large n asymptotics of the constant $c(n)$ in expression (3.6) for the (self) energy of the n -vortex. To this end we use the following large n asymptotics for the function $f_n(r)$ defined in (3.2):

$$f_n(r) = \begin{cases} \sqrt{1 - \frac{n^2}{r^2}} & \text{if } r - n \gg \left(\frac{n}{2}\right)^{\frac{1}{3}} \\ \left(\frac{2}{n}\right)^{\frac{1}{3}} g(z) & \text{if } |r - n| \ll n \end{cases}, \quad (\text{A2.1})$$

where the variable z is defined by the relation

$$r = n + \left(\frac{n}{2}\right)^{\frac{1}{3}} z \quad (\text{A2.2})$$

and the function $g(z)$ is a solution to the equation

$$g'' + zg - g^3 = 0 . \quad (\text{A2.3})$$

The function $g(z)$ has the following asymptotics:

$$\begin{aligned} g(z) &= z^{\frac{1}{2}} && \text{if } z \gg 1 \\ &= \text{const } \phi(z) && \text{if } z \ll -1 , \end{aligned} \quad (\text{A2.4})$$

where $\phi(z)$ is the Eiry function. In particular, we have

$$g(z) = \frac{0.39}{(-z)^{\frac{1}{4}}} e^{-\frac{2}{3}(-z)^{3/2}} \quad \text{for } z \ll -1 . \quad (\text{A2.5})$$

Plugging expression (A2.1)-(A2.2) into Eqn (A1.1) and using (A2.4) and (A2.5), we find that

$$c(n) = \alpha n^{\frac{2}{3}} \pi + c + O(n^{-\frac{2}{3}}) , \quad (\text{A2.6})$$

where c is some constant and

$$\alpha = 2^{\frac{1}{3}} \int_{-\infty}^{\infty} (g^2(z) - z\theta(z)) dz , \quad (\text{A2.7})$$

$\theta(z) = 1$ for $z \geq 0$ and $= 0$ for $z < 0$. Multiplying Eqn (A2.3) by $g'(z)$ and taking the integral of the result, we find that $\alpha = 0$ and therefore

$$c(n) = c + O(n^{-\frac{2}{3}}) \quad (\text{A2.8})$$

as $n \rightarrow \infty$. A rough numerical computation yields the following value for the constant c :

$$c \approx 0.7\pi . \quad (\text{A2.9})$$

Appendix 3. Computation of correlation coefficients

In this appendix we compute the correlation function

$$A = A(\underline{a}) = \frac{1}{4} \int \left[|\nabla\varphi_0|^4 - \sum_j |\nabla\varphi_j|^4 \right] \quad (\text{A3.1})$$

with

$$\varphi_0 = \sum_j \varphi_j \text{ and } \varphi_j(x) = n_j \theta(x - a_j), \quad (\text{A3.2})$$

$$\theta(x) = \text{the polar angle of } x \in \mathbb{R}^2, \quad (\text{A3.3})$$

(see Eqn (4.4)) for configurations of $K = N + 1$ vortices with N vortices of vorticity m lying in the circle of radius a and one vortex of vorticity $-k$, at the center of this circle, s.t. $\nabla H(\underline{a}) = 0$.

Write $\underline{a} = a \cdot \underline{b}$ where \underline{b} is a fixed configuration with N vortices on a unit circle and one, in the center. Changing the variable of integration in (A3.4) as $x = ay$, we find

$$A(\underline{a}) = C a^{-2} \quad (\text{A3.4})$$

where C depends on \underline{b} only. Our task now is to find the sign of C for the configurations of interest. We denote $A = A(\underline{a})$.

1. $N = 2$, $m = 2$ and $k = 1$. Here there are two double vortices on the circle and one single vortex of opposite vorticity at the center (see Fig 1). Below we shall use the dimensionless variable

$$\rho = \frac{|x|}{a}. \quad (\text{A3.5})$$

For the configuration under consideration we have

$$A = \frac{1}{4a^2} \int_0^\infty \rho d\rho \int_0^{2\pi} d\theta \left\{ \frac{48}{\alpha} - \frac{16 \cos(2\theta)}{\alpha \rho^2} + \frac{64 \cos^2(2\theta)}{\alpha^2} - \frac{64}{\alpha^2} (1 + 2\rho^2 + 2\rho^2 \cos(2\theta)) \right\}, \quad (\text{A3.6})$$

where

$$\alpha = \rho^4 + 1 + 2\rho^2 \cos(2\theta) . \quad (\text{A3.7})$$

(In general, for a_j , $j = 1, \dots, N$, distributed equidistantly on the circle of radius a , $\alpha = \prod_{j=1}^N (x - a_j)^2 / a^{2N}$.) First we take an integral over θ . To this end we change the variable of integration as $\theta \rightarrow z = \exp(2i\theta)$, i.e. we write the internal integral in (A3.8) as an integral over a unit circle. A simple calculation gives

$$\int_0^{2\pi} \frac{d\theta}{\alpha^2} = \frac{2\pi(1 + \rho^4)}{|1 - \rho^4|^3} , \quad \int_0^{2\pi} \frac{d\theta}{\alpha^2} \cos(2\theta) = -\frac{4\pi\rho^2}{|1 - \rho^4|^3} , \quad (\text{A3.8})$$

$$\int_0^{2\pi} \frac{d\theta}{\alpha} = \frac{2\pi\rho^2}{|1 - \rho^4|} , \quad \int_0^{2\pi} \frac{d\theta}{\alpha} \cos(2\theta) = -\frac{2\pi}{|1 - \rho^4|} \min\left\{\rho^2, \frac{1}{\rho^2}\right\} , \quad (\text{A3.9})$$

$$\int_0^{2\pi} \frac{d\theta}{\alpha^2} \cos^2(2\theta) = \frac{\pi}{|1 - \rho^4|^3} \cdot \begin{cases} 1 + 4\rho^4 - \rho^8 & \text{for } \rho < 1 \\ (\rho^8 + 4\rho^4 - 1)/\rho^4 & \text{for } \rho > 1. \end{cases} \quad (\text{A3.10})$$

Inserting expressions (A3.7)–(A3.10) into Eqn (A3.7), we obtain

$$A = \frac{4\pi}{a^2} \left\{ 2 \int_0^1 dx \frac{1-x}{(1+x)^3} + \int_1^\infty \frac{dx}{(1+x)^3} \left(3x + 1 + \frac{3}{x} + \frac{1}{x^2} \right) \right\} .$$

This gives

$$A = \frac{8\pi}{a^2} . \quad (\text{A3.11})$$

Hence in the configuration under consideration the energy $E_R(\underline{a})$ is given by

$$\frac{1}{\pi} E_R(\underline{a}) - 9 \ln R = -9.64 - \frac{8}{a^2} + O\left(\frac{\ln a}{a^4}\right) . \quad (\text{A3.12})$$

2. $N = 3$, $m = 1$ and $k = 1$. Similarly to Eqn (A3.8) we obtain

$$A = \frac{1}{4a^2} \int_0^\infty d\rho \rho \int_0^{2\pi} d\theta \cdot \left\{ \frac{6}{\alpha} (1 + 2\rho^2) - \frac{12 \sin(3\theta)}{\rho\alpha} - \frac{9(1 + \rho^2)}{\alpha^2} (1 + \rho^2 + 2\rho^4) + \frac{36\rho^2 \sin^2(3\theta)}{\alpha^2} - \frac{36\rho^5 \sin(3\theta)}{\alpha^2} \right\} , \quad (\text{A3.13})$$

where $\alpha = \rho^6 + 1 + 2r^3 \sin(3\theta)$. The integrals in Eqn (A3.13) can be taken explicitly. To do this we set $z = \exp(3i\theta)$.

$$\int_0^{2\pi} \frac{d\theta}{\alpha} = \frac{2\pi}{|1 - \rho^6|}, \quad \int_0^{2\pi} \frac{d\theta}{\alpha} \sin(3\theta) = -\frac{2\pi}{|1 - \rho^6|} \min\left(\rho^3, \frac{1}{\rho^3}\right) \quad (\text{A3.14})$$

$$\int_0^{2\pi} \frac{d\theta}{\alpha^2} = \frac{2\pi(1 + \rho^6)}{|1 - \rho^6|^3}, \quad \int_0^{2\pi} \frac{d\theta}{\alpha^2} \sin(3\theta) = -\frac{4\pi\rho^3}{|1 - \rho^6|^3} \quad (\text{A3.15})$$

and

$$\int_0^{2\pi} \frac{d\theta}{\alpha^2} \sin^2(3\theta) = \frac{\pi}{|1 - \rho^6|^3} \begin{cases} 1 + 4\rho^6 - \rho^{12} & \text{for } \rho < 1 \\ (\rho^{12} + 4\rho^6 - 1)/\rho^6 & \text{for } \rho > 1. \end{cases} \quad (\text{A3.16})$$

Inserting expressions (A3.14)–(A3.16) into Eqn (A3.13), we obtain

$$A = \frac{3\pi}{4a^2} \left\{ \int_0^1 dx \frac{5x + 9x^2 - 1 - 2x^3 - 2x^4}{(1 + x + x^2)^3} + \int_1^\infty dx \left(\frac{4}{1 + x + x^2} - \frac{9}{(1 + x + x^2)^2} + \frac{10x + 18}{(1 + x + x^2)^3} + \frac{6x + 2}{x^2(1 + x + x^2)^3} \right) \right\}. \quad (\text{A3.17})$$

A simple calculation of integrals in Eqn (A3.16) gives explicit answers for A :

$$A = \frac{2\pi}{a^2}. \quad (\text{A3.18})$$

Hence the energy for such configurations is given by

$$\frac{1}{\pi} E_R(a) - 4 \ln R = -1.792 - \frac{2}{a^2}. \quad (\text{A3.19})$$

3. $N = 4$, $m = 2$ and $k = 3$. Here, there are four double vortices in the corners of a rectangular and a (-3)-vortex in the centre. For this configuration we have

$$A = \frac{16}{a^2} \int_0^{2\pi} d\theta \int_0^\infty \frac{d\rho \cdot \rho}{\alpha} \cdot \left\{ \frac{4\rho^{12}}{\alpha} + \frac{36\rho^4 \cos^2(4\theta)}{\alpha} + 4.5\rho^4 + 13.5 \cos(4\theta) + \frac{24\rho^8}{\alpha} \cos(4\theta) - \frac{1}{\alpha} [(\rho^2 + 1)^6 - 2\rho^2(\rho^2 + 1)^2(\rho^4 + 1) + 4\rho^6] - 2\rho^4 \cos(4\theta)(3(\rho^2 + 1)^2 - 2\rho^2)/\alpha \right\}, \quad (\text{A3.20})$$

where

$$\alpha = \rho^8 + 1 - 2\rho^4 \cos(4\theta) .$$

The change of variables $2\theta \rightarrow \tilde{\theta} + \frac{\pi}{2}$, $\rho^8 \rightarrow \tilde{\rho}^4$ reduces the integrals over θ in Eqn (A3.20)

to the one given in Eqns (A3.8)–(A3.10). As a result we obtain

$$A = \frac{16\pi}{a^2} \left\{ \int_0^1 dx \left[\frac{1-3x}{1+x+x^2+x^3} + \frac{2(5x^5+23x^4+18x^3+6x^2-3x-1)}{(1+x+x^2+x^3)^3} \right. \right. \\ \left. \left. + \int_1^\infty dx \left(\frac{7.5}{1+x^2} - \frac{1.5}{x^2(1+x^2)} - \frac{4(1+x+x^2)}{x^2(1+x+x^2+x^3)} \right. \right. \right. \\ \left. \left. \left. - \frac{2}{(1+x+x^2+x^3)^3} \left(x^5 + 11x^4 - 2x^3 - 22x^2 - 31x - 21 - \frac{12}{x} - \frac{4}{x^2} \right) \right) \right] \right\} \quad (\text{A3.21})$$

A direct calculation of the integrals in Eqn (A3.11) gives the following answer

$$A = \frac{80\pi}{a^2} \quad (\text{A3.22})$$

and therefore the energy of the configuration in question is

$$\frac{1}{\pi} E_R(a) - 25 \ln R = -40.44 - \frac{80}{a^2} . \quad (\text{A3.23})$$

Note, that for all the configurations under consideration the correlation term, A , is given by

$$A = \frac{\pi}{4a^2} \cdot M$$

where M is an integer, i.e. the quantity given by an integral in A is quantized. Moreover, the “quantization” takes place separately for the integrals over regions $r < 1$ and $r > 1$.

We conjecture that this property is general and holds for any forceless configuration.

Appendix 4. Inequality $E'_R(0) > 0$

In this appendix we show that $E_R(a) - E_R(0) > 0$ for the configuration consisting of N 1-vortices equidistributed on the circle of radius a and one $(-\frac{N-1}{2})$ -vortex in the center and for a sufficiently small. We assume that N is odd but otherwise arbitrary.

For $a = 0$ the configuration in question collapses into a single $\frac{N+1}{2}$ -vortex, $\psi_{\frac{N+1}{2}}$, sitting at the origin. Let L be the Hessian of $\mathcal{E}_{\text{ren}}(\psi)$ at $\psi = \psi_{\frac{N+1}{2}}$. It was shown in [OS1] that the subspaces

$$\left\{ u_1(r)e^{im\theta} + u_2(r)e^{i(2\frac{N+1}{2}-m)\theta} \mid u_k \in L^2(rdr), k = 1, 2 \right\}, \quad (\text{A4.1})$$

$m = \frac{N+1}{2}, \frac{N+1}{2} + 1, \dots$, which are orthogonal to each other and span the entire Hilbert space $L^2(\mathbb{R}^2)$, are invariant under action of the operator L . Moreover, it was shown that L in the sectors with $m \geq 3\frac{N-1}{2} - 1$ is non-negative and 0 is not its eigenvalue (actually, the statement in [OS1] is formulated for $m \geq 3\frac{N-1}{2}$ but the proof works also for $m = 3\frac{N-1}{2} - 1$), while in the sectors $\frac{N+1}{2} + 2 \leq m \leq 2\frac{N+1}{2}$, the operator L has negative eigenvalues. Now observe that the sectors with $\frac{N+1}{2} \leq m \leq 3\frac{N-1}{2} - 2$ do not have C_{Nv} symmetry and consequently forbidden in our case. Thus on the subspace invariant under action of the group C_{Nv} , $L \geq 0$ and 0 is now its eigenvalue. The latter implies that

$$E_R(a) - E_R(0) > 0 \quad (\text{A4.2})$$

for any odd N and for a sufficiently small.

Appendix 5. Large N asymptotics

In this appendix we find asymptotic behaviour of the energy of the circular, asymptotically forceless configurations, i.e. the ones with $\nabla H(\underline{a}) = 0$, for large values of N . More precisely, the configurations we consider consist of N 1-vortices equally spaced on the circle of radius a and with the center at the origin and one $(-k)$ -vortex at the center. Recall that the condition $\nabla H(\underline{a}) = 0$ is equivalent to the relation $k = -\frac{1}{2}(N - 1)$. We assume in addition that N is odd and $a \gg N$.

According to Eqn (5.10) and since $\sin \frac{\pi k}{N} = \sin \frac{\pi(N-k)}{N}$, the energy of the above configuration is

$$E_R(\underline{a}) = \pi \left(\frac{N+1}{2} \right)^2 \ln R - \pi \left(\frac{N-1}{2} \right)^2 \ln \left(\frac{N-1}{2} \right) + Nc(1) - 2\pi N \sum_{k=1}^{\frac{N-1}{2}} \ln \left(2 \sin \frac{\pi k}{N} \right), \quad (\text{A5.1})$$

where, recall, we use the notation $E_R(a) = E_R(\underline{a})$. For $a = 0$ (the ‘‘initial state’’) the energy is given by Eqn (3.5):

$$E_R(0) = \pi \left(\frac{N+1}{2} \right)^2 \ln R - \pi \left(\frac{N+1}{2} \right) \ln \left(\frac{N+1}{2} \right). \quad (\text{A5.2})$$

In order to calculate the sum in Eqn (A5.1) we use the Euler expansion

$$\sum_{k=M}^L f(k) = \int_{M-\frac{1}{2}}^{L+\frac{1}{2}} f(x) dx - \frac{1}{24} \left(f' \left(L + \frac{1}{2} \right) - f' \left(M + \frac{1}{2} \right) \right) \quad (\text{A5.3})$$

and two equalities

$$\int_0^{\pi/2} \ln(2 \sin x) dx = 0, \quad (\text{A5.4})$$

we obtain

$$\begin{aligned} \sum_{k=1}^{\frac{N-1}{2}} \ln \left(2 \sin \frac{\pi k}{N} \right) &= \int_{\frac{1}{2}}^{\frac{N}{2}} \ln \left(2 \sin \frac{\pi x}{N} \right) dx + \frac{1}{24} \cdot \frac{\pi}{N} \cot \frac{\pi}{2N} \\ &= -\frac{N}{\pi} \int_0^{\pi/2N} \ln(2 \sin y) dy + \frac{\pi}{24N} \cot \frac{\pi}{2N}. \end{aligned}$$

For $N \gg 1$, this yields modulo terms $O(1)$ in N

$$\sum_{k=1}^{\frac{N-1}{2}} \ln \left(2 \sin \frac{\pi k}{N} \right) = \frac{1}{2} \ln N. \quad (\text{A5.5})$$

As a result we have for the energy difference

$$\begin{aligned} E_R(a) - E_R(0) &= N \left[C(1) + \left(\frac{1}{2} - \ln 2 \right) \pi \right] \\ &= 0.183\pi N. \end{aligned} \quad (\text{A5.6})$$

Thus for single vortices on the circle the energy for a large is greater than the energy for $a = 0$.

References

- [BBH1] F. Bethuel, H. Brezis, and F. Hélein, Asymptotics for the minimization of a Ginzburg-Landau functional, *Calc. Variat. and PDE* **1** (1993), 123–148.
- [BBH2] F. Bethuel, H. Brezis, and F. Hélein, *Ginzburg-Landau Vortices*, Birkhäuser, Basel, 1994.
- [BMR] H. Brezis, F. Merle, and T. Rivière, Quantization effects for $-\Delta u = u(1 - |u|^2)$ in \mathbb{R}^2 , *Arch. Rational Mech. Anal.* **126** (1994), 35–58.
- [CEQ] Y. Chen, C. Elliot and T. Qui (1994), Shooting method for vortex solutions of a complex-valued Ginzburg-Landau equation, *Proc. Royal Soc. Edinburgh* **124A**, 1068–1075.
- [CK] S. Chanillo and M. Kiesling, Symmetry of solutions of Ginzburg-Landau equations, *Compt. Rendu. Acad. Sci. Paris*, t. 327, Série I (1995), 1023–1026.
- [FP] P. Fife and L.A. Peletier (1996), On the location of defects in stationary solutions of the Ginzburg-Landau equations on R^2 , *Quart. Appl. Math.* **54**, 85–104.
- [GT] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd Edition, Springer-Verlag, New York, 1983.
- [GP] V.L. Ginzburg and L. Pitaevskii, On the theory of superfluidity, *Sov. Phys. JETP* **7** (1958), 585.
- [H] P. Hagan, Spiral waves in reaction diffusion equations, *SIAM J. Applied Math.* **42** (1982), 762–786.
- [HH] M. Hervé and R. Hervé (1994), Étude qualitative des solutions réelles d’une équation différentielle liée a l’équation de Ginzburg-Landau, *Ann. Inst. Henri Poincaré, Analyse non linéaire* **11**, 427–440.
- [JT] A. Jaffe and C. Taubes, *Vortices and Monopoles*, Birkhäuser, 1980.
- [LL] E. Lieb and M. Loss, Symmetry of the Ginzburg-Landau minimizer in a disc, *Math. Res. Lett.* **1** (1994), 701–715.
- [M1] P. Mironescu, On the stability of radial solutions of the Ginzburg-Landau equation, *Journal of Functional Analysis* **130** (1995), 334–344.
- [M2] P. Mironescu, Les minimiseurs locaux pour l’équation de Ginzburg-Landau sont à symétrie radiale, *C.R. Acad. Sci.*, Paris.
- [OS1] Yu.N. Ovchinnikov and I.M. Sigal, Ginzburg-Landau equation I. General discussion, in *P.D.E.’s and Their Applications* (L. Seco et al., eds.), CRM Proceedings and Research Notes **12** (1997), 199–220.

- [OS2] Yu.N. Ovchinnikov and I.M. Sigal, The Ginzburg-Landau equation III. Vortex dynamics, *Nonlinearity* **11** (1998), 1277–1294.
- [OS3] Yu.N. Ovchinnikov and I.M. Sigal, Dynamics of localized structures, *Physica A* **261** (1998), 143–158.
- [OS4] Yu.N. Ovchinnikov and I.M. Sigal, Spatial asymptotics of solutions of the Ginzburg-Landau and related equations, *Reviews Math. Phys.* (2000), to appear.
- [OS5] Yu.N. Ovchinnikov and I.M. Sigal, The energy of Ginzburg-Landau vortices, preprint, Toronto, 2000.
- [S] I. Shafrir, Remarks on solutions of $-\Delta u = (1 - |u|^2)u$ in \mathbb{R}^2 , *C.R. Acad. Sci. Paris*, t. 318, Série I (1994), 327–331.
- [Str] M. Struwe, *Variational Methods*, Springer-Verlag, 1990.

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